COMPUTATIONS FOR COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA III: GROUPS OF RANK SEVEN AND EIGHT

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Abstract. In this paper we extend the computations in parts I and II of this series of papers and complete the proof of a conjecture of Lehrer and Solomon expressing the character of a finite Coxeter group W acting on the p^{th} graded component of its Orlik-Solomon algebra as a sum of characters induced from linear characters of centralizers of elements of W for groups of rank seven and eight. For classical Coxeter groups, these characters are given using a formula that is expected to hold in all ranks.

1. Introduction

Suppose that W is a finite Coxeter group and that V is the complexified reflection representation of W. Let $\mathcal{A}$ be the set of reflecting hyperplanes of W in V and let

$$M = V \setminus \bigcup_{H\in\mathcal{A}} H$$

denote the complement of these hyperplanes in V. The reflection length of an element $w \in W$ is the least integer p such that w may be written as a product of p reflections. Clearly, conjugate elements have the same reflection length. Lehrer and Solomon [16, 1.6] conjectured that there is a $\mathbb{C}W$-module isomorphism

$$H^p(M,\mathbb{C}) \cong \bigoplus_{c} \operatorname{Ind}_{C_W(c)}^{W} \chi_c \qquad p = 0, \ldots, \operatorname{rank}(W) \tag{1.1}$$

where c runs over a set of representatives of the conjugacy classes of W with reflection length equal p and χ_c is a suitable linear character of the centralizer $C_W(c)$ of c in W. Lehrer and Solomon proved (1.1) for symmetric groups. In [8] we proposed an inductive approach to the Lehrer-Solomon conjecture that establishes a direct connection between the character of the Orlik-Solomon algebra of W and the regular character of W. This inductive approach has been used to prove (1.1) for dihedral groups [8], symmetric groups [7], and Coxeter groups with rank at most six [2, 3].

In this paper we extend the computations in [2, 3] to finite Coxeter groups of rank seven and eight and complete the proof of the conjectures in [8] that relate the Orlik-Solomon and regular characters of these groups. As a consequence, the conjectures in [8], as well as the Lehrer-Solomon conjecture, are shown to hold for all finite Coxeter groups of rank at most eight. In particular, these conjectures hold for all exceptional finite Coxeter groups.

As described in more detail below, the proof for the Coxeter groups of type E_7 and E_8 uses the techniques developed in [2, 3], except that we use Fleischmann and Janiszczak's computation [10] of the Möbius functions of fixed point sets in the intersection lattice of

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$\mathcal{A}$ to compute the character of the top component of the Orlik-Solomon algebra. We take this opportunity to correct several minor misprints in the table for E_8 in [10]. For Coxeter groups of classical types in this paper we give explicit formulas for the characters χ_c in all ranks and then use the methods developed in [2, 3] to verify that (1.1) holds for rank less than or equal eight. The formulas for the characters χ_c given below are similar to the formulas in [16].

The rest of this paper is organized as follows. In §2 we establish notation, give a concise review of the constructions and conjectures in [2, 3, 8], and state the main theorem to be proved in this paper (Theorem 1); in §3 we give explicit constructions of the linear characters that we expect to satisfy the conclusion of Theorem 1 for classical groups, and we verify that these characters do indeed satisfy the conclusion of Theorem 1 for groups of type B_n and D_n for $n \leqslant 8$; finally, in §4 we give specific linear characters that satisfy the conclusion of Theorem 1 for the exceptional groups of type E_7 and E_8, thus completing the proof of the theorem.

2. Statement of the main theorem

We begin by summarizing the constructions in [2, 3, 8]. The reader is referred to these sources for more details and proofs. Let (W, S) be a finite Coxeter system and let $A(W)$ be the Orlik-Solomon algebra of W. The inductive strategy for proving (1.1) proposed in [8] is to decompose the left regular $\mathbb{C}W$-module and the Orlik-Solomon algebra of W (considered as a left $\mathbb{C}W$-module) into direct sums, and then relate the characters of the individual summands. The decomposition of $\mathbb{C}W$ is given by a set of orthogonal idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ constructed by Bergeron, Bergeron, Howlett and Taylor [1]. Here Λ denotes the set of *shapes* of W, that is, the set of subsets of S modulo the equivalence relation given by conjugacy in W. Alternately, Λ indexes the conjugacy classes of parabolic subgroups of W. For each subset L of S, Bergeron et al. construct a quasi-idempotent e_L in the descent algebra of W and define $e_\lambda = \sum_{L\in\lambda} e_L$. Denoting the regular character of W by ρ and the character of $\mathbb{C}We_\lambda$ by ρ_λ, it follows that

$$\rho = \sum_{\lambda\in\Lambda} \rho_\lambda. \tag{2.1}$$

The set of shapes Λ also indexes the orbits of W on the lattice of $\mathcal{A}$ and a construction of Lehrer and Solomon [7, §2] yields a decomposition $A(W) = \bigoplus_{\lambda\in\Lambda} A_\lambda$. Denoting the Orlik-Solomon character of W by ω and the character of A_λ by ω_λ, we have

$$\omega = \sum_{\lambda\in\Lambda} \omega_\lambda. \tag{2.2}$$

Suppose that $L \subseteq S$. Then (W_L, L) is a Coxeter system and we may consider the Orlik-Solomon algebra $A(W_L)$ of W_L. For $J \subseteq L$ we denote by e_J^L the quasi-idempotents in $\mathbb{C}W_L$ constructed in [1]. The homogeneous component of $A(W_L)$ of highest degree is called the *top component* of $A(W_L)$. By analogy, the submodule $\mathbb{C}W_L e_L^L$ of $\mathbb{C}W_L$ is called the *top component* of $\mathbb{C}W_L$. The top components of $A(W_L)$ and $\mathbb{C}W_L$ are $N_W(W_L)$-stable subspaces of $A(W)$ and $\mathbb{C}W$, respectively. We denote their characters by $\widetilde{\omega_L}$ and $\widetilde{\rho_L}$. If λ is in Λ and L is in λ, then

$$\omega_\lambda = \operatorname{Ind}_{N_W(W_L)}^W \widetilde{\omega_L} \quad \text{and} \quad \rho_\lambda = \operatorname{Ind}_{N_W(W_L)}^W \widetilde{\rho_L} \tag{2.3}$$

(see [7, Proposition 4.8]). In order to state our main result we need to recall two more definitions. First, an element in W_L is *cuspidal* if it does not lie in any proper parabolic subgroup of W_L. Second, for n in $N_W(W_L)$, let $\alpha_L(n)$ be the determinant of the restriction of n to the space of fixed points of W_L in V. Note that α_L is a linear character of $N_W(W_L)$.

Theorem 1. *Suppose that (W, S) is a finite Coxeter system of rank at most eight. Let $L \subseteq S$ and suppose that $\mathcal{C}_L$ is a set of representatives of the cuspidal conjugacy classes of W_L. Then for each $w \in \mathcal{C}_L$ there exists a linear character φ_w of $C_W(w)$ such that*

$$\widetilde{\rho_L} = \sum_{w\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w = \alpha_L \epsilon \widetilde{\omega_L},$$

where ϵ is the sign character of W.

Set $\alpha_w = \alpha_L|_{C_W(w)}$ when w is cuspidal in W_L. Then it follows from (2.3) and the theorem that for λ in Λ we have

$$\rho_\lambda = \operatorname{Ind}_{N_W(W_L)}^{W} \widetilde{\rho_L} = \operatorname{Ind}_{N_W(W_L)}^{W} \Big(\sum_{w\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w \Big) = \sum_{w\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{W} \varphi_w$$

and

$$\omega_\lambda = \operatorname{Ind}_{N_W(W_L)}^{W} \widetilde{\omega_L} = \operatorname{Ind}_{N_W(W_L)}^{W} \Big(\sum_{w\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{N_W(W_L)} (\alpha_w \epsilon \varphi_w) \Big) = \sum_{w\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{W} (\alpha_w \epsilon \varphi_w). \tag{2.4}$$

Then the Lehrer-Solomon conjecture for finite Coxeter groups with rank at most eight follows immediately from (2.4) and (2.3) because

$$H^p(M, \mathbb{C}) \cong \bigoplus_{\operatorname{rank}(\lambda)=p} A_\lambda,$$

where $\operatorname{rank}(\lambda) = |L|$ for any L in λ (see [7, §2]).

Using the notation above, set $\chi_w = \alpha_w \epsilon \varphi_w$.

Corollary 2. *Suppose that W is a finite Coxeter group of rank at most eight. Then there is a $\mathbb{C}W$-module isomorphism*

$$H^p(M, \mathbb{C}) \cong \bigoplus_{w} \operatorname{Ind}_{C_W(w)}^{W} (\chi_w) \qquad p = 0, \dots, \operatorname{rank}(W)$$

where w runs over a set of representatives of the conjugacy classes of W with reflection length equal p and χ_w is a linear character of $C_W(w)$.

Using (2.1), (2.2), and (2.3), the theorem yields the following corollary, which relates the Orlik-Solomon and the regular characters of W.

Corollary 3. *Suppose that W is a finite Coxeter group of rank at most eight and that $\mathcal{R}$ is a set of conjugacy class representatives of W. Then for each $w \in \mathcal{R}$ there exists a linear character φ_w of $C_W(w)$ such that*

$$\rho = \sum_{w\in\mathcal{R}} \operatorname{Ind}_{C_W(w)}^{W} \varphi_w \quad \text{and} \quad \omega = \epsilon \sum_{w\in\mathcal{R}} \operatorname{Ind}_{C_W(w)}^{W} (\alpha_w \varphi_w),$$

where ϵ is the sign character of W.

As noted above, Theorem 1 has been proved for symmetric groups, dihedral groups, and Coxeter groups of rank at most six in earlier work. It is also shown in [7] that if the conclusion of Theorem 1 holds for Coxeter groups W and W' then it holds for $W \times W'$. Therefore, it suffices to consider only irreducible Coxeter groups. In this paper we complete the proof of the theorem by showing that the conclusion of Theorem 1 holds for Coxeter groups of type B_7, B_8, D_7, D_8, E_7, and E_8.

To prove Theorem 1 for the groups just listed, we follow the approach described in [8, §4] using the GAP computer algebra system system [19] with the CHEVIE [12] and ZigZag [18] packages.

STEP 1. Compute $\widetilde{\rho_L}$, $\widetilde{\omega_L}$, and verify that $\widetilde{\rho_L}$ and $\alpha_L \epsilon \widetilde{\omega_L}$ are equal.

STEP 2. Find linear characters φ_w such that

$$\widetilde{\rho_L} = \sum_{w \in \mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w.$$

The character of W_L afforded by $\mathbb{C}W_L e_L^L$ is denoted by ρ_L. Then $\widetilde{\rho_L}$ is an extension of ρ_L to $N_W(W_L)$. Similarly, the character of W_L on $A(W_L)$ is denoted by ω_L, and $\widetilde{\omega_L}$ is the extension of ω_L to $N_W(W_L)$. The computation of the top component characters $\rho_S = \widetilde{\rho_S}$ is described in [2, §3.1]. This computation has been implemented in the `ECharacters` function in the ZigZag package. The `ECharacters` function takes a finite Coxeter group as its argument and returns the list of the characters ρ_λ. Then $\rho_{\{S\}} = \widetilde{\rho_S}$. For L a proper subset of S, the computation of $\widetilde{\rho_L}$, given ρ_L, is described in [3, §2].

The computation of the characters $\widetilde{\omega_L}$, for L a proper subset of S, is described in [2, §3] and [3, §2]. The method used in these references is computationally too expensive to compute the character ω_S for the groups of rank eight. In this paper we take an alternate approach.

Orlik and Solomon [17] proved that the graded character of an element w in W on $A(W)$ may be computed using the Möbius function of the poset of fixed points of w in the lattice of $\mathcal{A}$. Precisely, let $\mathcal{L}$ be the intersection lattice of $\mathcal{A}$ and for w in W, let $\mathcal{L}^w$ denote the subposet of w-stable subspaces in $\mathcal{L}$. Then

$$\sum_{i=0}^{n} \operatorname{Trace}\left(w, H^i(M, \mathbb{C})\right) t^i = \sum_{X \in \mathcal{L}^w} \mu_w(X) (-t)^{n - \dim X}, \tag{2.5}$$

where μ_w is the Möbius function of $\mathcal{L}^w$, n is the rank of W, and t is an indeterminate. Denote the polynomial in (2.5) by $P_w(t)$. Then $\omega_S(w)$ is the coefficient of t^n in $P_w(t)$.

The polynomials $P_w(t)$ have been computed by Lehrer for Coxeter groups of types A and B [14, 15], and Fleischmann and Janiszczak in all cases [9, 10]. When W has type E_7 or E_8 we used the polynomials calculated by Fleischmann and Janiszczak to find ω_S. For the groups of type B_7, D_7, B_8, and D_8 we calculated the polynomials $P_w(t)$ as described below.

The first step is to calculate $\mathcal{L}$. The subspaces in $\mathcal{L}$ are parameterized by the set of pairs (τ, L) where $L \subseteq S$ runs through a fixed set of representatives of the shapes of W and $\tau \in W$ is a coset representative of $N_W(W_L)$ in W. The pair (τ, L) corresponds with the subspace τX_L of V, where $X_L = \bigcap_{s \in L} \operatorname{Fix}(s)$.

The subspace corresponding to (τ, L) is contained in the subspace corresponding to (σ, K) if and only if $\tau X_L \subseteq \sigma X_K$. This in turn holds if and only if $\sigma^{-1}\tau X_L \subseteq X_K$ which holds if and only if $W_K \subseteq W_L^{\tau^{-1}\sigma}$. This last condition can be checked by calculating a minimal length representative z of the (W_L, W_K)-double coset of $\tau^{-1}\sigma$. Then $W_K \subseteq W_L^{\tau^{-1}\sigma}$ if and only if $K \subseteq L^z$. We also remark that it suffices to assume that $\sigma = 1$, for the spaces contained in (σ, K) are precisely the spaces $(\sigma\tau, L)$ for which (τ, L) is contained in $(1, K)$.

Finally, to determine which subspaces τX_L are in the subposet $\mathcal{L}^w$, we observe that $w\tau X_L = \tau X_L$ if and only if $\tau^{-1}w\tau X_L = X_L$. This in turn holds if and only if $\tau^{-1}w\tau \in N_W(W_L)$. It remains to calculate $P_w(t)$ using the formula above, after calculating all the values of μ by recursion. Note that if w_0 is central in W then $\mathcal{L}^w = \mathcal{L}^{w_0 w}$. In this situation, only one of $P_w(t)$ or $P_{w_0 w}(t)$ needs to be calculated whenever w and $w_0 w$ lie in different conjugacy classes.

We remark that the polynomials calculated using the method above in types B_7 and B_8 agree with those given by Lehrer [14]. We also note that in the table for E_8 in [10] the values for the classes called $2A_2$ and $2D_4$ are missing a minus sign. Also, the second appearance of $A_1 + E_6(a_2)$ should read $A_2 + E_6(a_2)$ and the correct polynomial for the class $E_8(a_3)$ is $(t-1)(t+1)(t^2+1)(13t^4-1)$.

Step 1 is completed by comparing $\widetilde{\rho_L}$ and $\alpha_L \epsilon \widetilde{\omega_L}$ once both have been computed. It remains to complete Step 2: find linear characters φ_w such that

$$\widetilde{\rho_L} = \sum_{w \in \mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w.$$

This is accomplished in the next section for the groups of type B_n and D_n, and in the final section for the groups of type E_7 and E_8.

3. Classical groups

In this section we take W to be of classical type. For each subset L of S and each cuspidal element w in W_L we construct a linear character φ_w of $C_W(w)$. We then verify that when the rank of W is at most eight, these characters satisfy the conclusion of Theorem 1. For symmetric groups, the characters φ_w are the ones constructed in [7], where it is shown that they satisfy the conclusion of Theorem 1. We expect that the characters φ_w constructed in this section satisfy the conclusion of Theorem 1 for all ranks.

Note that because $\widetilde{\rho_L}$ only depends on the conjugacy class of W_L, or equivalently, the shape of L, we need only consider a suitably chosen representative in each conjugacy class of parabolic subgroups of W, and because $\operatorname{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w$ depends only in the conjugacy class of w, we need only consider one suitably chosen representative in each cuspidal conjugacy class of W_L.

Our construction follows the same general pattern as the construction in [7], which goes back at least to [16]. It is most naturally phrased in terms of permutations and signed permutations. We begin by reviewing the construction for symmetric groups.

3.1. **The characters φ_λ^A.** A *composition* is a non-empty tuple $\lambda = (\lambda_1, \dots, \lambda_a)$ of positive integers and a *partition* is a composition with the property that $\lambda_i \geqslant \lambda_{i+1}$ for $1 \leqslant i \leqslant a-1$. The numbers λ_i are called the *parts* of λ, the sum of the parts of λ is denoted by $|\lambda|$, and the number of parts of λ is denoted by $l(\lambda)$. If $|\lambda| = n$, then λ is called a composition, or a partition, of n. If λ is a partition of n, we write $\lambda \vdash n$. By convention, the empty tuple is a composition, and a partition, of zero.

For a positive integer n, set $[n] = \{1, 2, \dots, n\}$ and let S_n denote the group of permutations of $[n]$. If s_i denotes the transposition that switches i and $i+1$, then $S = \{s_1, s_2, \dots, s_{n-1}\}$ is a Coxeter generating set for S_n such that (S_n, S) is a Coxeter system of type A_{n-1}. The product of symmetric groups $S_{\lambda_1} \times \cdots \times S_{\lambda_a}$ is considered as a subgroup, S_λ, of the symmetric group S_n via the obvious embedding, where S_{λ_1} acts on $\{1, \dots, \lambda_1\}$, S_{λ_2} acts on $\{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}$, and so on. Subgroups of S_n of the form S_λ are called *Young subgroups.* If L is a subset of S, then $\langle L \rangle = S_\lambda$ for a unique composition λ of n. Moreover, the rule $L \mapsto \langle L \rangle$ defines a bijection between the set of subsets of S and the set of Young subgroups of S_n. Two Young subgroups S_λ and $S_{\lambda'}$ are conjugate if and only if λ and λ' determine the same partition of n. In this way, the set of shapes of S_n is parametrized by partitions of n.

The set of conjugacy classes in S_n is also parametrized by partitions of n. Suppose $\lambda = (\lambda_1, \cdots, \lambda_a)$ is a partition of n. For $1 \leqslant i \leqslant a$ define c_i in S_n by

$$c_i(v) = \begin{cases} v+1 & \text{if } v = u+1, \dots, u+\lambda_i - 1, \\ u+1 & \text{if } v = u+\lambda_i, \\ v & \text{otherwise}, \end{cases} \qquad \text{where} \qquad u = \sum_{k=1}^{i-1} \lambda_k.$$

(Here and in the formulas below, we use the convention that an empty sum is 0.) Then c_i is a λ_i-cycle in the direct factor S_{λ_i} of S_λ. Define $w_\lambda = c_1 c_2 \cdots c_a$. Then

- w_λ is a representative of the unique cuspidal conjugacy class in S_λ and
- $\{ w_\lambda \mid \lambda \vdash n \}$ is a complete set of conjugacy class representatives in S_n.

For each i such that $\lambda_i = \lambda_{i+1}$ define x_i in S_n by

$$x_i(v) = \begin{cases} v+\lambda_i & \text{if } v = u+1, \dots, u+\lambda_i, \\ v-\lambda_i & \text{if } v = u+\lambda_i+1, \dots, u+2\lambda_i, \\ v & \text{otherwise}, \end{cases} \qquad \text{where} \qquad u = \sum_{k=1}^{i-1} \lambda_k.$$

Then conjugation by x_i permutes $\{c_1, \dots, c_a\}$ by exchanging the cycles c_i and c_{i+1} and hence x_i centralizes w_λ. It is well-known that $C_{S_\lambda}(w_\lambda)$ is the abelian group generated by the cycles $c_1, \dots, c_a$ and that $C_{S_n}(w_\lambda)$ is generated by $C_{S_\lambda}(w_\lambda)$ together with the involutions x_i, for each i such that $\lambda_i = \lambda_{i+1}$. The abelianization of $C_{S_n}(w_\lambda)$ is generated by the images of the first c_i and x_i for every cycle length. Whenever i is a part of λ define $\lambda(i) = |\{ k \mid \lambda_k = i \}|$ and $\bar{\imath} = \min\{ k \mid \lambda_k = i \}$.

Lemma 4. *Let* $X_\lambda = \{ c_{\bar{\imath}} \mid i \in \lambda \} \amalg \{ x_{\bar{\imath}} \mid \lambda(i) > 1 \}$. *Suppose* $\psi \colon X_\lambda \to \mathbb{C}^\times$ *satisfies*

(1) $\psi(c_i)$ *is a* λ_i^{th} *root of unity for all* $c_i \in X_\lambda$, *and*
(2) $\psi(x_i)^2 = 1$ *for all* $x_i \in X_\lambda$.

Then ψ *has a unique extension to a linear character of* $C_{S_n}(w_\lambda)$. *Moreover, every linear character of* $C_{S_n}(w_\lambda)$ *arises in this way.*

Proof. See the proof of Lemma 6 below. □

For $k \geqslant 1$ denote the k^{th} root of unity $e^{2\pi i/k}$ by ζ_k. For a partition $\lambda = (\lambda_1, \dots, \lambda_a)$ of n, let φ_λ^A be the character of $C_{S_n}(w_\lambda)$ defined (as in the preceding lemma) by

- $\varphi_\lambda^A(c_i) = \zeta_{|c_i|}$ for all $c_i \in X_\lambda$ and
- $\varphi_\lambda^A(x_i) = 1$ for all $x_i \in X_\lambda$.

The next theorem is proved in [7].

Theorem 5. *Suppose λ is a partition of n and let $\widetilde{\rho_\lambda}$ be the top component character of the parabolic subgroup S_λ of S_n. Then*

$$\widetilde{\rho_\lambda} = \mathrm{Ind}_{C_{S_n}(w_\lambda)}^{N_{S_n}(S_\lambda)} \varphi_\lambda^A.$$

3.2. **The characters φ_μ^B.** A *signed partition* is a composition

$$\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+),$$

where $\mu_1^- \leqslant \cdots \leqslant \mu_a^-$ and $\mu_1^+ \geqslant \cdots \geqslant \mu_b^+$. Then

$$\mu^- = (\mu_a^-, \dots, \mu_1^-) \qquad \text{and} \qquad \mu^+ = (\mu_1^+, \dots, \mu_b^+)$$

are partitions. We use this labeling convention for compatibility with [11] and the GAP functions described below, where the conjugacy class representatives are chosen to have minimal length. If $|\mu^-| + |\mu^+| = n$, then μ is a signed partition of n and we write $\mu \Vdash n$.

A *signed permutation* of n is a permutation $w\colon \pm[n] \to \pm[n]$ such that $w(-i) = -w(i)$ for $1 \leqslant i \leqslant n$. Signed permutations will be identified with their restrictions to functions $[n] \to \pm[n]$ without comment. Let W_n denote the group of all signed permutations of n. Let S denote the set of transpositions $s_i = (i\ i+1)$ for $1 \leqslant i \leqslant n-1$, together with the signed permutation t defined by $t(1) = -1$ and $t(i) = i$ for $2 \leqslant i \leqslant n$. Then (W_n, S) is a Coxeter system of type B_n.

For a signed partition $\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+)$ of n define W_μ to be the subgroup

$$W_{\mu_1^-} \times \cdots \times W_{\mu_a^-} \times S_{\mu_1^+} \times \cdots \times S_{\mu_b^+}$$

of W_n. Here, a similar identification to that for the embedding of $S_\lambda = S_{\lambda_1^+} \times \cdots \times S_{\lambda_a}$ in S_n is used. Thus, $W_{\mu_1^-}$ acts on $\{1, \dots, \mu_1^-\}$, $W_{\mu_2^-}$ acts on $\{\mu_1^- + 1, \dots, \mu_1^- + \mu_2^-\}$, ..., $S_{\mu_1^+}$ acts on $\{|\mu^-| + 1, \dots, |\mu^-| + \mu_1^+\}$, and so on.

The conjugacy classes of parabolic subgroups of W_n, and hence the set of shapes of W_n, are parametrized by the set of partitions of m for $0 \leqslant m \leqslant n$. Suppose $0 \leqslant m \leqslant n$ and λ is a partition of m. Define μ_λ to be the signed partition $(\,(n-m), \lambda)$ of n and define $W_\lambda = W_{\mu_\lambda}$. Then the subgroups W_λ for λ a partition of m with $0 \leqslant m \leqslant n$ form a set of representatives for the conjugacy classes of parabolic subgroups of W_n.

The set of signed partitions of n indexes the conjugacy classes in W_n as follows. Let $\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+)$ be a signed partition of n. For $1 \leqslant i \leqslant a$, define c_i in W_n by

$$c_i(v) = \begin{cases} v+1 & \text{if } v = u+1, \dots, u+\mu_i^- - 1, \\ -(u+1) & \text{if } v = u+\mu_i^-, \\ v & \text{if } v \in [n] \setminus \{u+1, \dots, u+\mu_i^-\}, \end{cases} \qquad \text{where} \quad u = \sum_{k=1}^{i-1} \mu_k^-.$$

Then c_i is a negative μ_i^--cycle in the direct factor $W_{\mu_i^-}$ of W_μ. Similarly, for $1 \leqslant j \leqslant b$ define d_j in W_n by

$$d_j(v) = \begin{cases} v+1 & \text{if } v = u+1, \dots, u+\mu_j^+ - 1, \\ u+1 & \text{if } v = u+\mu_j^+, \\ v & \text{if } v \in [n] \setminus \{u+1, \dots, u+\mu_j^+\}, \end{cases} \qquad \text{where } u = |\mu^-| + \sum_{k=1}^{j-1} \mu_k^+$$

Then d_j is a positive μ_j^+-cycle in the direct factor $S_{\mu_j^+}$ of W_μ. Finally, define

$$w_\mu = c_1 \cdots c_a d_1 \cdots d_b.$$

Then $\{ w_\mu \mid \mu \Vdash n \}$ is a set of conjugacy class representatives in W_n.

For a signed partition μ of n, define $\overline{\mu}$ to be the signed partition $((|\mu^-|), \mu^+)$ of n. Then for a partition λ of m with $0 \leqslant m \leqslant n$, $\{ w_\mu \mid \overline{\mu} = \mu_\lambda \}$ is a set of representatives of the cuspidal conjugacy classes in the parabolic subgroup W_λ.

Suppose λ is a partition of m with $0 \leqslant m \leqslant n$ and μ is a signed partition such that $\overline{\mu} = \mu_\lambda$. For each i such that $\mu_i^- = \mu_{i+1}^-$ define x_i in W_n by

$$x_i(\nu) = \begin{cases} \nu + \mu_i^- & \text{if } \nu = u+1, \dots, u+\mu_i^-, \\ \nu - \mu_i^- & \text{if } \nu = u+\mu_i^-+1, \dots, u+2\mu_i^-, \\ \nu & \text{if } \nu \in [n] \setminus \{u+1, \dots, u+2\mu_i^-\}, \end{cases} \quad \text{where} \quad u = \sum_{k=1}^{i-1} \mu_k^-.$$

Then conjugation by x_i permutes $\{c_1, \dots, c_a, d_1, \dots, d_b\}$ by exchanging the negative cycles c_i and c_{i+1} and hence x_i centralizes w_μ. Next, for each j such that $\mu_j^+ = \mu_{j+1}^+$ define y_j in W_n by

$$y_j(\nu) = \begin{cases} \nu + \mu_j^+ & \text{if } \nu = u+1, \dots, u+\mu_j^+, \\ \nu - \mu_j^+ & \text{if } \nu = u+\mu_j^++1, \dots, u+2\mu_j^+, \\ \nu & \text{if } \nu \in [n] \setminus \{u+1, \dots, u+2\mu_j^+\}, \end{cases} \quad \text{where } u = m + \sum_{k=1}^{j-1} \mu_k^+.$$

Then conjugation by y_j permutes $\{c_1, \dots, c_a, d_1, \dots, d_b\}$ by exchanging the positive cycles d_j and d_{j+1} and hence y_j centralizes w_μ. Finally, for $1 \leqslant j \leqslant b$ define r_j in W_n by

$$r_j(\nu) = \begin{cases} -\nu & \text{if } \nu = u+1, \dots, u+\mu_j^+, \\ \nu & \text{if } \nu \in [n] \setminus \{u+1, \dots, u+\mu_j^+\}, \end{cases} \quad \text{where} \quad u = m + \sum_{k=1}^{j-1} \mu_k^+.$$

Then r_j centralizes c_i for $1 \leqslant i \leqslant a$ and d_k for $1 \leqslant k \leqslant b$ and hence centralizes w_μ.

It is not hard to see that $C_{W_\lambda}(w_\mu)$ is generated by the elements c_i, for $1 \leqslant i \leqslant a$; d_j, for $1 \leqslant j \leqslant b$; and x_i, for $1 \leqslant i \leqslant a$ such that $\mu_i^- = \mu_{i+1}^-$. By [13, Proposition 4.4], the elements r_j, for $1 \leqslant j \leqslant b$, and y_j, for $1 \leqslant j \leqslant b$ with $\mu_j^+ = \mu_{j+1}^+$, generate a complement to $C_{W_\lambda}(w_\mu)$ in $C_{W_n}(w_\mu)$.

With the preceding notation, define the following. Whenever i is a part of μ^- define $\mu^-(i) = \big|\{ k \mid \mu_k^- = i \}\big|$ and $\bar{\imath} = \min\{ k \mid \mu_k^- = i \}$. Similarly, whenever j is a part of μ^+ define $\mu^+(j) = \big|\{ k \mid \mu_k^+ = j \}\big|$ and $\bar{\jmath} = \min\{ k \mid \mu_k^+ = j \}$.

Lemma 6. *Let*

$$X_\mu = \{ c_{\bar{\imath}} \mid i \in \mu^- \} \amalg \{ x_{\bar{\imath}} \mid \mu^-(i) > 1 \} \amalg \{ d_{\bar{\jmath}} \mid j \in \mu^+ \} \amalg \{ y_{\bar{\jmath}} \mid \mu^+(j) > 1 \} \amalg \{ r_{\bar{\jmath}} \mid j \in \mu^+ \}.$$

Suppose $\psi \colon X_\mu \to \mathbb{C}^\times$ *satisfies*

(1) $\psi(c_i)$ *is a* $2\mu_i^-$*-th root of unity for all* $c_i \in X_\mu$,
(2) $\psi(x_i)^2 = 1$ *for all* $x_i \in X_\mu$,
(3) $\psi(d_j)$ *is a* μ_j^+*-th root of unity for all* $d_i \in X_\mu$,
(4) $\psi(y_j)^2 = 1$ *for all* $y_i \in X_\mu$, *and*
(5) $\psi(r_i)^2 = 1$ *for all* $r_i \in X_\mu$.

Then ψ *has a unique extension to a linear character of* $C_{W_n}(w_\mu)$*. Moreover, every linear character of* $C_{W_n}(w_\mu)$ *arises in this way.*

Proof. In this proof we set $C = C_{W_n}(w_\mu)$ and let Z_m denote the cyclic group of order m. If λ is a composition, we write $j \in \lambda$ when some part of λ is equal j.

To prove the lemma, it is enough to show that the abelianization $C/[C, C]$ is isomorphic to

$$\Big(\prod_{i\in\mu^-} Z_{2i}\Big)\times\Big(\prod_{\mu^-(i)>1} Z_2\Big)\times\Big(\prod_{j\in\mu^+} Z_j\Big)\times\Big(\prod_{\mu^+(j)>1} Z_2\Big)\times\Big(\prod_{j\in\mu^+} Z_2\Big),$$

and is generated by the image of X_μ.

It is straightforward to check that when S_m acts on $(Z_u)^m$ by permuting the factors, the abelianization of the semidirect product $(Z_u)^m \rtimes S_m$ is isomorphic to $Z_u \times Z_2$, and is generated by the image of a generator of the first direct factor in the product $(Z_u)^m$ and the image of the transposition s_1. Taking $u = 2$ we have in particular that the abelianization of W_m is isomorphic to $Z_2 \times Z_2$, and is generated by the images of t and s_1.

Similarly, when W_m acts on $(Z_u)^m$ by permuting the factors (so the generator t and its conjugates act trivially), it is straightforward to check that the abelianization of the semidirect product $(Z_u)^m \rtimes W_m$ is isomorphic to $Z_u \times Z_2 \times Z_2$, and is generated by the image of a generator of the first direct factor in the product $(Z_u)^m$, the image of the transposition s_1, and the image of t.

It follows from the description of the centralizer C in [13, §4.2] that

$$C \cong \prod_{i\in\mu^-}\Big((Z_{2i})^{\mu^-(i)} \rtimes S_{\mu^-(i)}\Big) \times \prod_{j\in\mu^+}\Big((Z_j)^{\mu^+(j)} \rtimes W_{\mu^+(j)}\Big), \tag{3.1}$$

so to complete the proof it suffices to consider the abelianizations of the direct factors of (3.1) individually. It follows from the remarks above that for $i \in \mu^-$, the abelianization of $(Z_{2i})^{\mu^-(i)} \rtimes S_{\mu^-(i)}$ is isomorphic to $Z_{2i} \times Z_2$, and is generated by the images of $c_{\bar{\imath}}$ and $x_{\bar{\imath}}$. Similarly, for $j \in \mu^+$ the abelianization of $(Z_j)^{\mu^+(j)} \rtimes W_{\mu^+(j)}$ is isomorphic to $Z_j \times Z_2 \times Z_2$, and is generated by the images of $d_{\bar{\jmath}}$, $y_{\bar{\jmath}}$, and $r_{\bar{\imath}}$. □

For a signed partition $\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+)$ of n let φ_μ^B be the character of $C_{W_n}(w_\mu)$ defined (as in the preceding lemma) by

- $\varphi_\mu^B(c_i) = \zeta_{2k}$ where $\mu_i^- = 2^l k$ with k odd, for $1 \leqslant i \leqslant a$,
- $\varphi_\mu^B(d_j) = \zeta_{|d_j|}$ for $1 \leqslant j \leqslant b$,
- $\varphi_\mu^B(x_i) = -1$ for all i such that $\mu_i^- = \mu_{i+1}^-$,
- $\varphi_\mu^B(y_j) = 1$ for all j such that $\mu_j^+ = \mu_{j+1}^+$, and
- $\varphi_\mu^B(r_j) = (-1)^{\mu_j^+ - 1}$ for $1 \leqslant j \leqslant b$.

Theorem 7. *Suppose λ is a partition of m with $0 \leqslant m \leqslant n$ and let $\widetilde{\rho_\lambda}$ be the character of $N_{W_n}(W_\lambda)$ afforded by the top component of $\mathbb{C}W_\lambda$. Then*

$$\widetilde{\rho_\lambda} = \sum_{\overline{\mu}=\mu_\lambda} \operatorname{Ind}_{C_{W_n}(w_\mu)}^{N_{W_n}(W_\lambda)} \varphi_\mu^B$$

for $n \leqslant 8$.

We have verified Theorem 7 using the GAP computer algebra system [19] with the CHEVIE [12] and ZigZag [18] packages to compute both sides of the equality in the theorem. The computation of the character $\widetilde{\rho_\lambda}$ of $N_{W_n}(W_\lambda)$ was described in §2. The sum was computed with the help of some GAP functions. First, we defined a function

```
Lambda2Character( mu, cval, dval, xval, yval, rval )
```

that takes a signed partition `mu` and the character values as in Lemma 6, given as functions from the set of parts of μ to the cyclotomic field, and returns the character ψ of $C_{W_n}(w_\mu)$. Second, we defined a function `BCharacter( mu )` that takes a signed partition, evaluates `Lambda2Character` at appropriate values, and returns the linear character φ_μ^B of $C_{W_n}(w_\mu)$. For any given partition λ of m with $0 \leqslant m \leqslant n$, one can then use the characters φ_μ^B with $\overline{\mu} = \mu_\lambda$ to compute the sum of induced characters $\sum_{\overline{\mu}=\mu_\lambda} \mathrm{Ind}_{C_{W_n}(w_\mu)}^{N_{W_n}(W_\lambda)} \varphi_\mu^B$.

It is tempting to speculate that the characters $\{\varphi_\mu^B \mid \mu \Vdash n,\ \overline{\mu} = \mu_\lambda\}$ satisfy the conclusion of Theorem 1 for the parabolic subgroup W_λ of W_n for all $n \geqslant 2$.

A similar decomposition of the regular character of the Coxeter group of type B_n into characters that are induced from linear characters of element centralizers has been suggested by Bonnafé [4, §10, Ques. (6)]. However, a straightforward calculation shows that his decomposition is different from ours, even for $n = 2$.

3.3. **The characters φ_μ^D.** We regard the Coxeter group of type D_n for $n \geqslant 4$ as the reflection subgroup W_n' of the Coxeter group W_n consisting of signed permutations with an even number of sign changes, so

$$W_n' = \{w \in W_n \mid |\{i \in [n] \mid w(i) < 0\}| \in 2\mathbb{N}\}.$$

Set $t' = s_1 t s_1$. Then W_n' is a reflection subgroup of W_n with Coxeter generating set $S' = \{t', s_1, \dots, s_{n-1}\}$.

The shapes of W' were determined in [11, Proposition 2.3.13] as follows. First, if λ is a partition of m with $m \leqslant n-2$, or if λ is a partition of n containing at most one odd part, set $W_\lambda' = W_n' \cap W_\lambda$. Second, if λ is a partition of n with all even parts, then λ indexes two conjugacy classes. One is represented by $W_{\lambda^+}' = S_\lambda$ and the other is represented by $W_{\lambda^-}' = tS_\lambda t$.

Suppose that μ is a signed partition of n. If c is a negative cycle, then $\{i \in [n] \mid c(i) < 0\}$ has an odd number of elements and so $w_\mu = c_1 \cdots c_a d_1 \cdots d_b$ lies in W_n' if and only if a is even, that is, if and only if μ^- has an even number of parts. Notice that the individual negative cycles c_i do not lie in W_n', but the products $c_i c_k$ do lie in W_n', and hence in $C_{W_n'}(w_\mu)$.

Now suppose that λ is a partition of m with $0 < m \leqslant n-2$, or that λ is a partition of n with at least one odd part. Then $\{w_\mu \mid \overline{\mu} = \mu_\lambda,\ l(\mu^-) \in 2\mathbb{N}\}$ is a complete set of representatives for the cuspidal conjugacy classes in W_λ'. If λ is a partition of n with all parts even, then the element w_λ in S_λ represents the unique cuspidal class in W_{λ^+}' and the element $tw_\lambda t$ represents the unique cuspidal class in W_{λ^-}' (see [11, Proposition 3.4.12]). Furthermore, because $C_{W_n'}(w) = W_n' \cap C_{W_n}(w)$ for w in W_n', we can define linear characters of $C_{W_n'}(w)$ simply by restricting characters of $C_{W_n}(w)$.

The conclusion of Theorem 1 has been shown to hold whenever W_L is a product of symmetric groups in [7]. Thus, to simplify the exposition, in the following we consider only the parabolic subgroups W_λ where λ is a partition of m with $0 \leqslant m \leqslant n-2$.

Suppose μ is a signed partition of n such that $l(\mu^-)$ is even. It follows from Lemma 6 that there is a linear character ψ_μ of $C_{W_n}(w_\mu)$ such that

- $\psi_\mu(c_i) = \zeta_{|c_i|}$ for $1 \leqslant i \leqslant a$,
- $\psi_\mu(d_j) = \zeta_{|d_j|}$ for $1 \leqslant j \leqslant b$,
- $\psi_\mu(x_i) = -1$ for all i such that $\mu_i^- = \mu_{i+1}^-$,
- $\psi_\mu(y_j) = 1$ for all j such that $\mu_j^+ = \mu_{j+1}^+$, and

- $\psi_\mu(r_j) = -1$ for $1 \leqslant j \leqslant a$.

Define φ_μ^D to be the restriction of ψ_μ to $C_{W_n'}(w_\mu)$. As we have already observed, the individual negative cycles c_i do not lie in W_n'. Similarly, if μ_j^+ is odd, then r_j is not in W_n'. We have

- $\varphi_\mu^D(c_i c_k) = \zeta_{|c_i|}\zeta_{|c_k|}$ for $1 \leqslant i, k \leqslant a$,
- $\varphi_\mu^D(d_j) = \zeta_{|d_j|}$ for $1 \leqslant j \leqslant b$,
- $\varphi_\mu^D(x_i) = -1$ for all i such that $\mu_i^- = \mu_{i+1}^-$,
- $\varphi_\mu^D(y_j) = 1$ for all j such that $\mu_j^+ = \mu_{j+1}^+$, and
- $\varphi_\mu^D(r_j) = -1$ for $1 \leqslant j \leqslant b$ such that μ_j^+ is even, and
- $\varphi_\mu^D(r_i r_k) = 1$ for $1 \leqslant i, k \leqslant b$ such that μ_i^+ and μ_k^+ are odd.

Theorem 8. *Suppose λ is a partition of m with $0 \leqslant m \leqslant n-2$ and let $\widetilde{\rho_\lambda}$ be the character of $N_{W_n'}(W_\lambda')$ afforded by the top component of $\mathbb{C}W_\lambda'$. Then*

$$\widetilde{\rho_\lambda} = \sum_{\substack{\overline{\mu}=\mu_\lambda \\ l(\mu^-)\in 2\mathbb{N}}} \mathrm{Ind}_{C_{W_n'}(w_\mu)}^{N_{W_n'}(W_\lambda')} \varphi_\mu^D$$

for $n \leqslant 8$.

The verification of Theorem 8 parallels the verification of Theorem 7. The computation of the character $\widetilde{\rho_\lambda}$ of $N_{W_n}(W_\lambda)$ was described in §2. The sum was computed with the help of further GAP functions. We defined a function `DCharacter( mu )` that takes a signed partition `mu`, evaluates the above function `Lambda2Character` at appropriate values, and returns the linear character ψ_μ of $C_{W_n}(w_\mu)$. Restriction to $C_{W_n'}(w_\mu)$ yields the linear character φ_μ^D. For any given partition λ of m with $0 \leqslant m \leqslant n-2$, one can then use the characters φ_μ^D with $\overline{\mu} = \mu_\lambda$ to compute the sum of induced characters $\sum_{\substack{\overline{\mu}=\mu_\lambda \\ l(\mu^-)\in 2\mathbb{N}}} \mathrm{Ind}_{C_{W_n'}(w_\mu)}^{N_{W_n'}(W_\lambda')} \varphi_\mu^D$.

It is tempting to speculate that the characters φ_μ^D satisfy the conclusion of Theorem 1 for W_n' for all n and all partitions λ of m with $0 \leqslant m \leqslant n-2$. Note that, in general, the characters φ_μ^D are different from the restrictions of the characters φ_μ^B to the centralizers in W_n'.

4. Exceptional groups

As in [3, §4], we only need to consider subsets L of S (up to conjugacy) such that W_L is not bulky in W, W_L has rank at least three, and W_L is not a direct product of Coxeter groups of type A. The pairs (W, W_L) are given by type in the following table.

W	W_L
E_7	D_4, A_1D_4, D_5, A_1D_5, E_6
E_8	D_4, A_1D_4, D_5, A_2D_4, A_1D_5, D_6, E_6, A_2D_5, A_1D_6, D_7

Because of space considerations, we do not list the values of the characters $\widetilde{\rho_L} = \alpha_L\epsilon\widetilde{\omega_L}$. Instead we list the characters φ_w that satisfy the conclusion of Theorem 1 for each pair (W, L) when L is a proper subset of S in Table 1 and Table 2. However, see [3, §3.1] for an example with all the data included.

In the tables we exhibit a set of generators of the centralizer of each cuspidal class representative of W_L. We use the symbol w_i to denote a representative of the i^{th}

conjugacy class of a group in the list of conjugacy classes returned by the command `ConjugacyClasses` in GAP [19]. At each generator of $C_W(w_i)$ we display the value of the character φ_{w_i}, denoted simply by φ_i. The symbol w_0 represents the longest element of W, while w_L represents the longest element of W_L when L is a proper subset of S. We use the symbols $1, 2, \ldots, n$ to denote the elements of S. For $p \geqslant 1$ we denote the p^{th} root of unity $e^{2\pi i/p}$ by ζ_p. Finally, r represents the reflection with respect to the highest long root in the root system of W. We sometimes express generators in terms of longest elements of certain parabolic subgroups of W. For this purpose, we fix the following subsets of S.

$$\begin{aligned} E &= \{2,3,4,5\}, & F &= \{1,2,3,4,5\}, \\ G &= \{2,3,4,5,6\}, & H &= \{1,2,3,4,5,6\}, \\ I &= \{2,3,4,5,6,7\}, & J &= \{2,3,4,5,6,7,8\}. \end{aligned}$$

For the subgroup of type E_6 of $W(E_8)$ we modified the cuspidal conjugacy class representatives to match those used in [3]. Namely, we took $w_{15} = 123456$, $w_{14} = 24w_{15}$, $w_{12} = 13456r_{E_6}$, $w_{10} = w_{15}^2$, and $w_4 = 12356r_{E_6}$ where r_{E_6} is the reflection corresponding with the highest root in the E_6 subsystem. For the subgroup of type A_1E_6 of $W(E_8)$ the representatives $(w_8, w_{20}, w_{24}, w_{28}, w_{30})$ were obtained from the representatives $(w_4, w_{10}, w_{12}, w_{14}, w_{15})$ for the subgroup of type E_6 by multiplying by the generator 8.

4.1. **Proof of Theorem 1 when W has type E and $L = S$.** To prove Theorem 1 when W has type E and $L = S$ we proceed as in the case when L is a proper subset of S except that we use the methods described in §2 to compute the Orlik-Solomon character $\widetilde{\omega_S} = \omega_S$. Again, we do not list the values of the characters $\widetilde{\rho_L} = \alpha_L \epsilon \widetilde{\omega_L}$. In Table 3 and Table 4 we list the characters φ_d that satisfy the conclusion of Theorem 1 when $L = S$ for the groups of types E_7 and E_8 respectively. Here the conjugacy classes are labeled by Carter diagrams [6] and we denote the character φ_{w_d} by φ_d where d is a Carter diagram.

As in [2, 3], we give additional information about regular conjugacy classes. If w is in W and ζ is an eigenvalue of w on V, then we denote the determinant of the representation of $C_W(w)$ on the ζ-eigenspace of w by $\det|_\zeta$. By Springer's theory of regular elements [20] the centralizer $C_W(w)$ is a complex reflection group acting on an eigenspace of w whenever w is a regular element. In each table we indicate which classes are regular in the column labeled Reg, which is to be interpreted as follows. If w is regular and φ_w is a power of $\det|_\zeta$ for some ζ then we indicate this power in the Reg column. However, if w is regular but φ_w is *not* a power of $\det|_\zeta$ for any ζ, then we indicate this by writing ♠ in the Reg column. Whenever practical we describe the structure of $C_W(w)$ in terms of Z_m and S_m, which denote the cyclic group of size m and the symmetric group on m letters.

When $C_W(w_d)$ acts as a complex reflection group on an eigenspace of w_d we often specify the character φ_d by listing its values on generators of its "Dynkin diagram" presentation [5]. We also exhibit its Dynkin diagram in these cases. Note that vertices of the Dynkin diagram connected by single edges are conjugate in W and thereby take the same character values, which we list only once.

For the group $W(E_8)$ we have slightly modified the representatives of the cuspidal conjugacy classes supplied by GAP. Namely, we observe that $w_{E_8(a_8)}$ can be taken to be

$w_0 w_{A_2^4}$ and then $C_W\left(w_{E_8(a_8)}\right) = C_W\left(w_{A_2^4}\right)$. Similar observations hold for the classes A_4^2 and $E_8(a_6)$.

Appendix A. Tables

Table 1: $W = W(E_7)$, $L \subsetneq S$

L	Type	Characters
$\{2,3,4,5\}$	D_4	$\varphi_3 : (2,4,7,w_F,w_G) \mapsto (-1,-1,1,1,1)$
		$\varphi_9 : (7,2w_G,245w_H) \mapsto (1,\zeta_4,\zeta_4)$
		$\varphi_{11} : (w_{11},7,2w_F2,w_G) \mapsto (\zeta_3,1,1,1)$
$\{2,3,4,5,7\}$	A_1D_4	$\varphi_6 : (3,4,5,7,r,w_F) \mapsto (-1,-1,-1,-1,1,1)$
		$\varphi_{18} : (w_{18},23,r,w_F234) \mapsto (1,-1,1,\zeta_4)$
		$\varphi_{22} : (2345,7,r,2w_F2) \mapsto (\zeta_3,-1,1,1)$
$\{2,3,4,5,6\}$	D_5	$\varphi_7 : (w_7,2,3,4,r,w_0) \mapsto (\zeta_4,-1,-1,-1,1,-1)$
		$\varphi_{15} : (w_{15},r,w_0) \mapsto (\zeta_{12},1,-1)$
		$\varphi_{17} : (w_{17},r,w_0) \mapsto (\zeta_8,1,-1)$
$\{1,2,3,4,5,7\}$	A_1D_5	$\varphi_{14} : (w_{14},2,4,5,7,w_0) \mapsto (\zeta_4,-1,-1,-1,-1,1)$
		$\varphi_{30} : (w_{30},7,w_0) \mapsto (\zeta_{12},-1,1)$
		$\varphi_{34} : (w_{34},7,w_0) \mapsto (\zeta_8,-1,1)$
$\{1,2,3,4,5,6\}$	E_6	$\varphi_4 : (2343,134256,w_0) \mapsto (\zeta_3,1,1)$
		$\varphi_{10} : \left(w_{15}^2,123654,w_0\right) \mapsto (\zeta_3,1,1)$
		$\varphi_{12} : (w_{12},2,4,w_0) \mapsto (\zeta_3,-1,-1,1)$
		$\varphi_{14} : (w_{14},w_0) \mapsto (\zeta_9,1)$
		$\varphi_{15} : (w_{15},w_0) \mapsto (-1,1)$

Table 2: $W = W(E_8)$, $L \subsetneq S$

L	Type	Characters
$\{2,3,4,5\}$	D_4	$\varphi_3 : (2,4,7,8,w_F,w_G) \mapsto (-1,-1,1,1,1,1)$
		$\varphi_9 : (7,8,2w_G,254w_F) \mapsto (1,1,\zeta_4,-\zeta_4)$
		$\varphi_{11} : (2345,7,8,25w_F,w_G) \mapsto (\zeta_3,1,1,1,1)$
$\{2,3,4,5,8\}$	A_1D_4	$\varphi_6 : (3,4,7,w_F,w_J,w_I) \mapsto (-1,-1,-1,1,-1,-1)$
		$\varphi_{18} : (r,254w_F,3w_J) \mapsto (1,\zeta_4,-\zeta_4)$
		$\varphi_{22} : (2345,r,25w_F,w_J) \mapsto (\zeta_3,1,1,-1)$
$\{1,2,3,4,5\}$	D_5	$\varphi_7 : (2,4,1342543,7,8,w_0,r) \mapsto (-1,-1,\zeta_4,1,1,-1,1)$
		$\varphi_{15} : (1234254,7,8,r,w_0) \mapsto (\zeta_{12},1,1,1,-1)$
		$\varphi_{17} : (13425,7,8,r,w_0) \mapsto (\zeta_8,1,1,1,-1)$
$\{2,3,4,5,7,8\}$	A_2D_4	$\varphi_9 : (3,4,78,w_F,w_J) \mapsto (-1,-1,\zeta_3,1,1)$
		$\varphi_{27} : (78,254w_F,3w_J) \mapsto (\zeta_3,\zeta_4,\zeta_4)$
		$\varphi_{33} : (2345,78,25w_F,w_J) \mapsto (\zeta_3,\zeta_3,1,1)$
$\{1,2,3,4,5,7\}$	A_1D_5	$\varphi_{14} : (2,4,1342543,7,w_0,r) \mapsto (-1,-1,\zeta_4,-1,1,1)$
		$\varphi_{30} : (1234254,7,w_0,r) \mapsto (\zeta_{12},-1,1,1)$
		$\varphi_{34} : (13425,7,r,w_0) \mapsto (\zeta_8,-1,1,1)$

Table 2: (continued)

L	Type	Characters
$\{2,3,4,5,6,7\}$	D_6	$\varphi_4 : (2,4,5,6,7,r,w_J) \mapsto (-1,-1,-1,-1,-1,1,1)$
		$\varphi_{14} : (2,54234,6576,r,w_J) \mapsto (-1,\zeta_4,-1,1,1)$
		$\varphi_{21} : (2,4,76542345,r,w_J) \mapsto (-1,-1,\zeta_3,1,1)$
		$\varphi_{27} : (543654765,43w_J,r) \mapsto (-1,\zeta_3,1)$
		$\varphi_{33} : (24234567,3w_J,r) \mapsto (\zeta_8,\zeta_4,1)$
		$\varphi_{35} : (234567,r,w_J) \mapsto (\zeta_5,1,1)$
$\{1,2,3,4,5,6\}$	E_6	$\varphi_4 : (56,14234542,8,r,w_0) \mapsto \left(\zeta_3,\zeta_3^2,1,1,1\right)$
		$\varphi_{10} : (w_{15},234543,8,r,w_0) \mapsto (1,\zeta_3,1,1,1)$
		$\varphi_{12} : (w_{12},2345432,r_{E_6},8,r,w_0) \mapsto (\zeta_3,-1,-1,1,1,1)$
		$\varphi_{14} : (w_{14},8,r,w_0) \mapsto (\zeta_9,1,1,1)$
		$\varphi_{15} : (w_{15},8,r,w_0) \mapsto (-1,1,1,1)$
$\{1,2,3,4,5,7,8\}$	A_2D_5	$\varphi_{21} : (2,4,1342543,78,w_0) \mapsto (-1,-1,\zeta_4,\zeta_3,-1)$
		$\varphi_{45} : (1234254,78,w_0) \mapsto (\zeta_{12},\zeta_3,-1)$
		$\varphi_{51} : (13425,78,w_0) \mapsto (\zeta_8,\zeta_3,-1)$
$\{1,2,3,4,5,6,8\}$	A_1E_6	$\varphi_8 : (56,14234542,8,w_0) \mapsto \left(\zeta_3,\zeta_3^2,-1,-1\right)$
		$\varphi_{20} : (w_{30},234543,w_0) \mapsto (-1,\zeta_3,-1)$
		$\varphi_{24} : (w_{24},2345432,r_{E_6},8,w_0) \mapsto (\zeta_6,-1,-1,-1,-1)$
		$\varphi_{28} : (24w_{30},w_0) \mapsto (\zeta_{18},-1)$
		$\varphi_{30} : (w_{30},8,w_0) \mapsto (1,-1,-1)$
$\{2,3,4,5,6,7,8\}$	D_7	$\varphi_{10} : (w_{10},2,3,4,5,6,w_0) \mapsto (\zeta_4,-1,-1,-1,-1,-1,-1)$
		$\varphi_{20} : (234,5465,7687,w_0) \mapsto (\zeta_4,-1,-1,-1)$
		$\varphi_{31} : (w_{31},2,42345,w_0) \mapsto (\zeta_{12},-1,-\zeta_4,-1)$
		$\varphi_{41} : (w_{41},2,3,4,w_0) \mapsto (\zeta_8,-1,-1,-1,-1)$
		$\varphi_{47} : (w_{47},w_0) \mapsto (\zeta_{24},-1)$
		$\varphi_{52} : (w_{52},w_0) \mapsto (\zeta_{20},-1)$
		$\varphi_{54} : (w_{54},w_0) \mapsto (\zeta_{12},-1)$

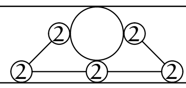
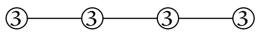
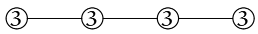
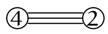
Table 3: $W = W(E_7)$, $L = S$

d	$C_W(w_d)$	Gen	φ_d	Reg
A_1^7	W	S	ϵ	$\det\vert_{-1}$
$A_1^3D_4$		$w_{A_1^3D_4}$	ζ_6	
		$2,4,5,6,7$	-1	
$E_7(a_4)$	G_{26}	②═③—③	$-1,\zeta_3,\zeta_3$	♠
$A_1D_6(a_2)$		$w_{A_1D_6(a_2)}$	$-\zeta_3$	
		2	-1	
		1342654231456	-1	
		$(76)^{542345}$	ζ_3	
$A_1A_3^2$		$w_{E_7(a_2)}$	1	
		42543	ζ_4	

Table 3: (continued)

d	$C_W(w_d)$	Gen	φ_d	Reg
A_1D_6		$w_{A_1D_6}$	ζ_{10}	
		2	-1	
		4	-1	
A_2A_5		$w_{A_2A_5}$	-1	
		34	ζ_3	
		2345	ζ_3^2	
$E_7(a_1)$	Z_{14}	⑭	ζ_{14}	$\det\|_{\zeta_{14}}$
A_7	$Z_8 \times Z_2 \times Z_2$	w_{A_7}	ζ_8	
		w_0	-1	
		2	-1	
E_7	Z_{18}	⑱	ζ_{18}	$\det\|_{\zeta_{18}}$
$E_7(a_2)$	$Z_{12} \times Z_2$	$w_{E_7(a_2)}$	1	
		w_0	-1	
$E_7(a_3)$	Z_{30}	$w_{E_7(a_3)}$	ζ_{30}	

Table 4: $W = W(E_8)$, $L = S$

d	$C_W(w_d)$	Gen	φ_d	Reg
A_1^8	W	S	ϵ	$\det\|_{-1}$
$D_4(a_1)^2$	G_{31}		-1	♠
$A_1^4D_4$		1	-1	
		$w_{A_1^4D_4}$	ζ_3	
		$w_{A_1E_7(a_4)}$	ζ_3^2	
		$w_{A_3D_5(a_1)}$	ζ_3	
A_2^4	G_{32}		ζ_3	♠
$E_8(a_8)$	G_{32}		ζ_3	♠
$A_1E_7(a_4)$		2	-1	
		$w_{A_1^4D_4}$	ζ_3	
		† y	$-\zeta_3$	
D_4^2		2	-1	
		w_E	1	
		‡ z	ζ_6	
$A_1^2A_3^2$		$w_{A_2E_6(a_2)}$	1	
		5	-1	
		$7w_G$	ζ_4	
$D_8(a_3)$	G_9		$\zeta_4, -1$	$\det\|_{\zeta_8}$
$A_1^2D_6$	$Z_{10} \times S_5$	$w_{A_1^2D_6}$	ζ_5	
		$3, 4, 5, 6$	-1	

† $y = 13427654234567876$
‡ $z = 1342543654276548765$

Table 4: (continued)

d	$C_W(w_d)$	Gen	φ_d	Reg
A_4^2	G_{16}	⑤——⑤	ζ_5	$\det\|_{\zeta_5}$
$E_8(a_6)$	G_{16}	⑤——⑤	ζ_5	$\det\|_{\zeta_{10}}$
$A_2E_6(a_2)$		$w_{A_2E_6(a_2)}$	ζ_3	
		$w_{A_1^2A_3^2}$	1	
		24	ζ_3	
		2345	ζ_3^2	
		$87w_I$	ζ_3	
$E_8(a_3)$	G_{10}	④══③	$-1, \zeta_3$	$(\det\|_{\zeta_{12}})^2$
$A_1A_2A_5$		$w_{A_1A_2A_5}$	1	
		7	-1	
		8	-1	
		34	ζ_3	
		2345	ζ_3^2	
$D_8(a_1)$	$Z_{12} \times S_3$	$w_{D_8(a_1)}$	ζ_3	
		48	-1	
		4578	1	
D_8	$Z_{14} \times Z_2$	w_{D_8}	ζ_7	
		2	-1	
A_1A_7		$w_{A_1A_7}$	ζ_8	
		2	-1	
		34254	ζ_4	
A_1E_7		$w_{A_1E_7}$	ζ_9	
		3	-1	
		4	-1	
A_8	$Z_{18} \times Z_3$	w_{A_8}	ζ_9	
		$13w_0$	ζ_3	
$E_8(a_4)$	$Z_{18} \times Z_3$	$w_{E_8(a_4)}$	ζ_9	
		34	ζ_3	
$E_8(a_2)$	Z_{20}	⑳	ζ_5	$(\det\|_{\zeta_{20}})^4$
$A_3D_5(a_1)$		$w_{A_3D_5(a_1)}$	ζ_6	
		354	ζ_4	
		136542	-1	
A_2E_6		$w_{A_2E_6}$	ζ_6	
		34	ζ_3	
		2435	ζ_3	
$E_8(a_7)$		$w_{E_8(a_7)}$	ζ_6	
		2354	ζ_3	
		2454	ζ_3	
$A_1E_7(a_2)$		$w_{A_1E_7(a_2)}$	-1	
		$2, 4, w_0$	-1	
$E_8(a_1)$	Z_{24}	㉔	ζ_{12}	$(\det\|_{\zeta_{24}})^2$

Table 4: (continued)

d	$C_W(w_d)$	Gen	φ_d	Reg
$D_8(a_2)$	$Z_{30} \times Z_2$	$w_{D_8(a_2)}$	ζ_{15}	
		5	-1	
$E_8(a_5)$	Z_{30}	㉚	ζ_{15}	$(\det\|_{\zeta_{15}})^2$
E_8	Z_{30}	㉚	ζ_{15}	$(\det\|_{\zeta_{30}})^2$

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