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Publication Date	2014-11-06
Publication information	Bishop, Marcus, Matthew Douglass, J., Pfeiffer, Götz, & Röhrle, Gerhard. (2015). Computations for Coxeter arrangements and Solomon's descent algebra III: Groups of rank seven and eight. <i>Journal of Algebra</i> , 423(Supplement C), 1213-1232. doi: <a href="https://doi.org/10.1016/j.jalgebra.2014.10.025">https://doi.org/10.1016/j.jalgebra.2014.10.025</a>
Publisher	Elsevier
Link to publisher's version	<a href="https://doi.org/10.1016/j.jalgebra.2014.10.025">https://doi.org/10.1016/j.jalgebra.2014.10.025</a>
Item record	<a href="http://hdl.handle.net/10379/7006">http://hdl.handle.net/10379/7006</a>

# COMPUTATIONS FOR COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA III: GROUPS OF RANK SEVEN AND EIGHT

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ABSTRACT. In this paper we extend the computations in parts I and II of this series of papers and complete the proof of a conjecture of Lehrer and Solomon expressing the character of a finite Coxeter group  $W$  acting on the  $p^{\text{th}}$  graded component of its Orlik-Solomon algebra as a sum of characters induced from linear characters of centralizers of elements of  $W$  for groups of rank seven and eight. For classical Coxeter groups, these characters are given using a formula that is expected to hold in all ranks.

## 1. INTRODUCTION

Suppose that  $W$  is a finite Coxeter group and that  $V$  is the complexified reflection representation of  $W$ . Let  $\mathcal{A}$  be the set of reflecting hyperplanes of  $W$  in  $V$  and let

$$M = V \setminus \bigcup_{H \in \mathcal{A}} H$$

denote the complement of these hyperplanes in  $V$ . The reflection length of an element  $w \in W$  is the least integer  $p$  such that  $w$  may be written as a product of  $p$  reflections. Clearly, conjugate elements have the same reflection length. Lehrer and Solomon [16, 1.6] conjectured that there is a  $\mathbb{C}W$ -module isomorphism

$$(1.1) \quad H^p(M, \mathbb{C}) \cong \bigoplus_{\mathfrak{c}} \text{Ind}_{C_W(\mathfrak{c})}^W \chi_{\mathfrak{c}} \quad p = 0, \dots, \text{rank}(W)$$

where  $\mathfrak{c}$  runs over a set of representatives of the conjugacy classes of  $W$  with reflection length equal  $p$  and  $\chi_{\mathfrak{c}}$  is a suitable linear character of the centralizer  $C_W(\mathfrak{c})$  of  $\mathfrak{c}$  in  $W$ . Lehrer and Solomon proved (1.1) for symmetric groups. In [8] we proposed an inductive approach to the Lehrer-Solomon conjecture that establishes a direct connection between the character of the Orlik-Solomon algebra of  $W$  and the regular character of  $W$ . This inductive approach has been used to prove (1.1) for dihedral groups [8], symmetric groups [7], and Coxeter groups with rank at most six [2, 3].

In this paper we extend the computations in [2, 3] to finite Coxeter groups of rank seven and eight and complete the proof of the conjectures in [8] that relate the Orlik-Solomon and regular characters of these groups. As a consequence, the conjectures in [8], as well as the Lehrer-Solomon conjecture, are shown to hold for all finite Coxeter groups of rank at most eight. In particular, these conjectures hold for all exceptional finite Coxeter groups.

As described in more detail below, the proof for the Coxeter groups of type  $E_7$  and  $E_8$  uses the techniques developed in [2, 3], except that we use Fleischmann and Janiszczak's computation [10] of the Möbius functions of fixed point sets in the intersection lattice of

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2010 *Mathematics Subject Classification.* 20F55, 20C15, 20C40, 52C35.

*Key words and phrases.* Coxeter group, Orlik-Solomon algebra, descent algebra.

$\mathcal{A}$  to compute the character of the top component of the Orlik-Solomon algebra. We take this opportunity to correct several minor misprints in the table for  $E_8$  in [10]. For Coxeter groups of classical types in this paper we give explicit formulas for the characters  $\chi_c$  in all ranks and then use the methods developed in [2, 3] to verify that (1.1) holds for rank less than or equal eight. The formulas for the characters  $\chi_c$  given below are similar to the formulas in [16].

The rest of this paper is organized as follows. In §2 we establish notation, give a concise review of the constructions and conjectures in [2, 3, 8], and state the main theorem to be proved in this paper (Theorem 1); in §3 we give explicit constructions of the linear characters that we expect to satisfy the conclusion of Theorem 1 for classical groups, and we verify that these characters do indeed satisfy the conclusion of Theorem 1 for groups of type  $B_n$  and  $D_n$  for  $n \leq 8$ ; finally, in §4 we give specific linear characters that satisfy the conclusion of Theorem 1 for the exceptional groups of type  $E_7$  and  $E_8$ , thus completing the proof of the theorem.

## 2. STATEMENT OF THE MAIN THEOREM

We begin by summarizing the constructions in [2, 3, 8]. The reader is referred to these sources for more details and proofs. Let  $(W, S)$  be a finite Coxeter system and let  $A(W)$  be the Orlik-Solomon algebra of  $W$ . The inductive strategy for proving (1.1) proposed in [8] is to decompose the left regular  $\mathbb{C}W$ -module and the Orlik-Solomon algebra of  $W$  (considered as a left  $\mathbb{C}W$ -module) into direct sums, and then relate the characters of the individual summands. The decomposition of  $\mathbb{C}W$  is given by a set of orthogonal idempotents  $\{e_\lambda \mid \lambda \in \Lambda\}$  constructed by Bergeron, Bergeron, Howlett and Taylor [1]. Here  $\Lambda$  denotes the set of *shapes* of  $W$ , that is, the set of subsets of  $S$  modulo the equivalence relation given by conjugacy in  $W$ . Alternately,  $\Lambda$  indexes the conjugacy classes of parabolic subgroups of  $W$ . For each subset  $L$  of  $S$ , Bergeron et al. construct a quasi-idempotent  $e_L$  in the descent algebra of  $W$  and define  $e_\lambda = \sum_{L \in \lambda} e_L$ . Denoting the regular character of  $W$  by  $\rho$  and the character of  $\mathbb{C}W e_\lambda$  by  $\rho_\lambda$ , it follows that

$$(2.1) \quad \rho = \sum_{\lambda \in \Lambda} \rho_\lambda.$$

The set of shapes  $\Lambda$  also indexes the orbits of  $W$  on the lattice of  $\mathcal{A}$  and a construction of Lehrer and Solomon [7, §2] yields a decomposition  $A(W) = \bigoplus_{\lambda \in \Lambda} A_\lambda$ . Denoting the Orlik-Solomon character of  $W$  by  $\omega$  and the character of  $A_\lambda$  by  $\omega_\lambda$ , we have

$$(2.2) \quad \omega = \sum_{\lambda \in \Lambda} \omega_\lambda.$$

Suppose that  $L \subseteq S$ . Then  $(W_L, L)$  is a Coxeter system and we may consider the Orlik-Solomon algebra  $A(W_L)$  of  $W_L$ . For  $J \subseteq L$  we denote by  $e_J^L$  the quasi-idempotents in  $\mathbb{C}W_L$  constructed in [1]. The homogeneous component of  $A(W_L)$  of highest degree is called the *top component* of  $A(W_L)$ . By analogy, the submodule  $\mathbb{C}W_L e_L^L$  of  $\mathbb{C}W_L$  is called the *top component* of  $\mathbb{C}W_L$ . The top components of  $A(W_L)$  and  $\mathbb{C}W_L$  are  $N_W(W_L)$ -stable subspaces of  $A(W)$  and  $\mathbb{C}W$ , respectively. We denote their characters by  $\widetilde{\omega}_L$  and  $\widetilde{\rho}_L$ . If  $\lambda$  is in  $\Lambda$  and  $L$  is in  $\lambda$ , then

$$(2.3) \quad \omega_\lambda = \text{Ind}_{N_W(W_L)}^W \widetilde{\omega}_L \quad \text{and} \quad \rho_\lambda = \text{Ind}_{N_W(W_L)}^W \widetilde{\rho}_L$$

(see [7, Proposition 4.8]). In order to state our main result we need to recall two more definitions. First, an element in  $W_L$  is *cuspidal* if it does not lie in any proper parabolic subgroup of  $W_L$ . Second, for  $\mathfrak{n}$  in  $N_W(W_L)$ , let  $\alpha_L(\mathfrak{n})$  be the determinant of the restriction of  $\mathfrak{n}$  to the space of fixed points of  $W_L$  in  $V$ . Note that  $\alpha_L$  is a linear character of  $N_W(W_L)$ .

**Theorem 1.** *Suppose that  $(W, S)$  is a finite Coxeter system of rank at most eight. Let  $L \subseteq S$  and suppose that  $\mathcal{C}_L$  is a set of representatives of the cuspidal conjugacy classes of  $W_L$ . Then for each  $w \in \mathcal{C}_L$  there exists a linear character  $\varphi_w$  of  $C_W(w)$  such that*

$$\widetilde{\rho}_L = \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w = \alpha_L \epsilon \widetilde{\omega}_L,$$

where  $\epsilon$  is the sign character of  $W$ .

Set  $\alpha_w = \alpha_L|_{C_W(w)}$  when  $w$  is cuspidal in  $W_L$ . Then it follows from (2.3) and the theorem that for  $\lambda$  in  $\Lambda$  we have

$$\rho_\lambda = \text{Ind}_{N_W(W_L)}^W \widetilde{\rho}_L = \text{Ind}_{N_W(W_L)}^W \left( \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w \right) = \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^W \varphi_w$$

and

$$(2.4) \quad \omega_\lambda = \text{Ind}_{N_W(W_L)}^W \widetilde{\omega}_L = \text{Ind}_{N_W(W_L)}^W \left( \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^{N_W(W_L)} (\alpha_w \epsilon \varphi_w) \right) \\ = \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^W (\alpha_w \epsilon \varphi_w).$$

Then the Lehrer-Solomon conjecture for finite Coxeter groups with rank at most eight follows immediately from (2.4) and (2.3) because

$$H^p(M, \mathbb{C}) \cong \bigoplus_{\text{rank}(\lambda)=p} A_\lambda,$$

where  $\text{rank}(\lambda) = |\lambda|$  for any  $\lambda$  in  $\Lambda$  (see [7, §2]).

Using the notation above, set  $\chi_w = \alpha_w \epsilon \varphi_w$ .

**Corollary 2.** *Suppose that  $W$  is a finite Coxeter group of rank at most eight. Then there is a  $CW$ -module isomorphism*

$$H^p(M, \mathbb{C}) \cong \bigoplus_w \text{Ind}_{C_W(w)}^W (\chi_w) \quad p = 0, \dots, \text{rank}(W)$$

where  $w$  runs over a set of representatives of the conjugacy classes of  $W$  with reflection length equal  $p$  and  $\chi_w$  is a linear character of  $C_W(w)$ .

Using (2.1), (2.2), and (2.3), the theorem yields the following corollary, which relates the Orlik-Solomon and the regular characters of  $W$ .

**Corollary 3.** *Suppose that  $W$  is a finite Coxeter group of rank at most eight and that  $\mathcal{R}$  is a set of conjugacy class representatives of  $W$ . Then for each  $w \in \mathcal{R}$  there exists a linear character  $\varphi_w$  of  $C_W(w)$  such that*

$$\rho = \sum_{w \in \mathcal{R}} \text{Ind}_{C_W(w)}^W \varphi_w \quad \text{and} \quad \omega = \epsilon \sum_{w \in \mathcal{R}} \text{Ind}_{C_W(w)}^W (\alpha_w \varphi_w),$$

where  $\epsilon$  is the sign character of  $W$ .

As noted above, [Theorem 1](#) has been proved for symmetric groups, dihedral groups, and Coxeter groups of rank at most six in earlier work. It is also shown in [\[7\]](#) that if the conclusion of [Theorem 1](#) holds for Coxeter groups  $W$  and  $W'$  then it holds for  $W \times W'$ . Therefore, it suffices to consider only irreducible Coxeter groups. In this paper we complete the proof of the theorem by showing that the conclusion of [Theorem 1](#) holds for Coxeter groups of type  $B_7$ ,  $B_8$ ,  $D_7$ ,  $D_8$ ,  $E_7$ , and  $E_8$ .

To prove [Theorem 1](#) for the groups just listed, we follow the approach described in [\[8, §4\]](#) using the GAP computer algebra system [\[19\]](#) with the CHEVIE [\[12\]](#) and ZigZag [\[18\]](#) packages.

STEP 1. Compute  $\widetilde{\rho}_L$ ,  $\widetilde{\omega}_L$ , and verify that  $\widetilde{\rho}_L$  and  $\alpha_L \epsilon \widetilde{\omega}_L$  are equal.

STEP 2. Find linear characters  $\varphi_w$  such that

$$\widetilde{\rho}_L = \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w.$$

The character of  $W_L$  afforded by  $\mathbb{C}W_L e_L^1$  is denoted by  $\rho_L$ . Then  $\widetilde{\rho}_L$  is an extension of  $\rho_L$  to  $N_W(W_L)$ . Similarly, the character of  $W_L$  on  $A(W_L)$  is denoted by  $\omega_L$ , and  $\widetilde{\omega}_L$  is the extension of  $\omega_L$  to  $N_W(W_L)$ . The computation of the top component characters  $\rho_S = \widetilde{\rho}_S$  is described in [\[2, §3.1\]](#). This computation has been implemented in the `ECharacters` function in the `ZigZag` package. The `ECharacters` function takes a finite Coxeter group as its argument and returns the list of the characters  $\rho_\lambda$ . Then  $\rho_{\{S\}} = \widetilde{\rho}_S$ . For  $L$  a proper subset of  $S$ , the computation of  $\widetilde{\rho}_L$ , given  $\rho_L$ , is described in [\[3, §2\]](#).

The computation of the characters  $\widetilde{\omega}_L$ , for  $L$  a proper subset of  $S$ , is described in [\[2, §3\]](#) and [\[3, §2\]](#). The method used in these references is computationally too expensive to compute the character  $\omega_S$  for the groups of rank eight. In this paper we take an alternate approach.

Orlik and Solomon [\[17\]](#) proved that the graded character of an element  $w$  in  $W$  on  $A(W)$  may be computed using the Möbius function of the poset of fixed points of  $w$  in the lattice of  $\mathcal{A}$ . Precisely, let  $\mathcal{L}$  be the intersection lattice of  $\mathcal{A}$  and for  $w$  in  $W$ , let  $\mathcal{L}^w$  denote the subposet of  $w$ -stable subspaces in  $\mathcal{L}$ . Then

$$(2.5) \quad \sum_{i=0}^n \text{Trace}(w, H^i(M, \mathbb{C})) t^i = \sum_{X \in \mathcal{L}^w} \mu_w(X) (-t)^{n - \dim X},$$

where  $\mu_w$  is the Möbius function of  $\mathcal{L}^w$ ,  $n$  is the rank of  $W$ , and  $t$  is an indeterminate. Denote the polynomial in [\(2.5\)](#) by  $P_w(t)$ . Then  $\omega_S(w)$  is the coefficient of  $t^n$  in  $P_w(t)$ .

The polynomials  $P_w(t)$  have been computed by Lehrer for Coxeter groups of types  $A$  and  $B$  [\[14, 15\]](#), and Fleischmann and Janiszczak in all cases [\[9, 10\]](#). When  $W$  has type  $E_7$  or  $E_8$  we used the polynomials calculated by Fleischmann and Janiszczak to find  $\omega_S$ . For the groups of type  $B_7$ ,  $D_7$ ,  $B_8$ , and  $D_8$  we calculated the polynomials  $P_w(t)$  as described below.

The first step is to calculate  $\mathcal{L}$ . The subspaces in  $\mathcal{L}$  are parameterized by the set of pairs  $(\tau, L)$  where  $L \subseteq S$  runs through a fixed set of representatives of the shapes of  $W$  and  $\tau \in W$  is a coset representative of  $N_W(W_L)$  in  $W$ . The pair  $(\tau, L)$  corresponds with the subspace  $\tau X_L$  of  $V$ , where  $X_L = \bigcap_{s \in L} \text{Fix}(s)$ .

The subspace corresponding to  $(\tau, L)$  is contained in the subspace corresponding to  $(\sigma, K)$  if and only if  $\tau X_L \subseteq \sigma X_K$ . This in turn holds if and only if  $\sigma^{-1} \tau X_L \subseteq X_K$  which holds if and only if  $W_K \subseteq W_L^{\tau^{-1} \sigma}$ . This last condition can be checked by calculating a minimal

length representative  $z$  of the  $(W_L, W_K)$ -double coset of  $\tau^{-1}\sigma$ . Then  $W_K \subseteq W_L^{\tau^{-1}\sigma}$  if and only if  $K \subseteq L^z$ . We also remark that it suffices to assume that  $\sigma = 1$ , for the spaces contained in  $(\sigma, K)$  are precisely the spaces  $(\sigma\tau, L)$  for which  $(\tau, L)$  is contained in  $(1, K)$ .

Finally, to determine which subspaces  $\tau X_L$  are in the subposet  $\mathcal{L}^w$ , we observe that  $w\tau X_L = \tau X_L$  if and only if  $\tau^{-1}w\tau X_L = X_L$ . This in turn holds if and only if  $\tau^{-1}w\tau \in N_W(W_L)$ . It remains to calculate  $P_w(t)$  using the formula above, after calculating all the values of  $\mu$  by recursion. Note that if  $w_0$  is central in  $W$  then  $\mathcal{L}^w = \mathcal{L}^{w_0w}$ . In this situation, only one of  $P_w(t)$  or  $P_{w_0w}(t)$  needs to be calculated whenever  $w$  and  $w_0w$  lie in different conjugacy classes.

We remark that the polynomials calculated using the method above in types  $B_7$  and  $B_8$  agree with those given by Lehrer [14]. We also note that in the table for  $E_8$  in [10] the values for the classes called  $2A_2$  and  $2D_4$  are missing a minus sign. Also, the second appearance of  $A_1 + E_6(\alpha_2)$  should read  $A_2 + E_6(\alpha_2)$  and the correct polynomial for the class  $E_8(\alpha_3)$  is  $(t-1)(t+1)(t^2+1)(13t^4-1)$ .

**STEP 1** is completed by comparing  $\widetilde{\rho}_L$  and  $\alpha_L \in \widetilde{\omega}_L$  once both have been computed. It remains to complete **STEP 2**: find linear characters  $\varphi_w$  such that

$$\widetilde{\rho}_L = \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w.$$

This is accomplished in the next section for the groups of type  $B_n$  and  $D_n$ , and in the final section for the groups of type  $E_7$  and  $E_8$ .

### 3. CLASSICAL GROUPS

In this section we take  $W$  to be of classical type. For each subset  $L$  of  $S$  and each cuspidal element  $w$  in  $W_L$  we construct a linear character  $\varphi_w$  of  $C_W(w)$ . We then verify that when the rank of  $W$  is at most eight, these characters satisfy the conclusion of **Theorem 1**. For symmetric groups, the characters  $\varphi_w$  are the ones constructed in [7], where it is shown that they satisfy the conclusion of **Theorem 1**. We expect that the characters  $\varphi_w$  constructed in this section satisfy the conclusion of **Theorem 1** for all ranks.

Note that because  $\widetilde{\rho}_L$  only depends on the conjugacy class of  $W_L$ , or equivalently, the shape of  $L$ , we need only consider a suitably chosen representative in each conjugacy class of parabolic subgroups of  $W$ , and because  $\text{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w$  depends only in the conjugacy class of  $w$ , we need only consider one suitably chosen representative in each cuspidal conjugacy class of  $W_L$ .

Our construction follows the same general pattern as the construction in [7], which goes back at least to [16]. It is most naturally phrased in terms of permutations and signed permutations. We begin by reviewing the construction for symmetric groups.

**3.1. The characters  $\varphi_\lambda^A$ .** A *composition* is a non-empty tuple  $\lambda = (\lambda_1, \dots, \lambda_\alpha)$  of positive integers and a *partition* is a composition with the property that  $\lambda_i \geq \lambda_{i+1}$  for  $1 \leq i \leq \alpha - 1$ . The numbers  $\lambda_i$  are called the *parts* of  $\lambda$ , the sum of the parts of  $\lambda$  is denoted by  $|\lambda|$ , and the number of parts of  $\lambda$  is denoted by  $l(\lambda)$ . If  $|\lambda| = n$ , then  $\lambda$  is called a composition, or a partition, of  $n$ . If  $\lambda$  is a partition of  $n$ , we write  $\lambda \vdash n$ . By convention, the empty tuple is a composition, and a partition, of zero.

For a positive integer  $n$ , set  $[n] = \{1, 2, \dots, n\}$  and let  $S_n$  denote the group of permutations of  $[n]$ . If  $s_i$  denotes the transposition that switches  $i$  and  $i + 1$ , then  $S = \{s_1, s_2, \dots, s_{n-1}\}$  is a Coxeter generating set for  $S_n$  such that  $(S_n, S)$  is a Coxeter system of type  $A_{n-1}$ . The product of symmetric groups  $S_{\lambda_1} \times \dots \times S_{\lambda_a}$  is considered as a subgroup,  $S_\lambda$ , of the symmetric group  $S_n$  via the obvious embedding, where  $S_{\lambda_1}$  acts on  $\{1, \dots, \lambda_1\}$ ,  $S_{\lambda_2}$  acts on  $\{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}$ , and so on. Subgroups of  $S_n$  of the form  $S_\lambda$  are called *Young subgroups*. If  $L$  is a subset of  $S$ , then  $\langle L \rangle = S_\lambda$  for a unique composition  $\lambda$  of  $n$ . Moreover, the rule  $L \mapsto \langle L \rangle$  defines a bijection between the set of subsets of  $S$  and the set of Young subgroups of  $S_n$ . Two Young subgroups  $S_\lambda$  and  $S_{\lambda'}$  are conjugate if and only if  $\lambda$  and  $\lambda'$  determine the same partition of  $n$ . In this way, the set of shapes of  $S_n$  is parametrized by partitions of  $n$ .

The set of conjugacy classes in  $S_n$  is also parametrized by partitions of  $n$ . Suppose  $\lambda = (\lambda_1, \dots, \lambda_a)$  is a partition of  $n$ . For  $1 \leq i \leq a$  define  $c_i$  in  $S_n$  by

$$c_i(v) = \begin{cases} v + 1 & \text{if } v = u + 1, \dots, u + \lambda_i - 1, \\ u + 1 & \text{if } v = u + \lambda_i, \\ v & \text{otherwise,} \end{cases} \quad \text{where} \quad u = \sum_{k=1}^{i-1} \lambda_k.$$

(Here and in the formulas below, we use the convention that an empty sum is 0.) Then  $c_i$  is a  $\lambda_i$ -cycle in the direct factor  $S_{\lambda_i}$  of  $S_\lambda$ . Define  $w_\lambda = c_1 c_2 \dots c_a$ . Then

- $w_\lambda$  is a representative of the unique cuspidal conjugacy class in  $S_\lambda$  and
- $\{w_\lambda \mid \lambda \vdash n\}$  is a complete set of conjugacy class representatives in  $S_n$ .

For each  $i$  such that  $\lambda_i = \lambda_{i+1}$  define  $x_i$  in  $S_n$  by

$$x_i(v) = \begin{cases} v + \lambda_i & \text{if } v = u + 1, \dots, u + \lambda_i, \\ v - \lambda_i & \text{if } v = u + \lambda_i + 1, \dots, u + 2\lambda_i, \\ v & \text{otherwise,} \end{cases} \quad \text{where} \quad u = \sum_{k=1}^{i-1} \lambda_k.$$

Then conjugation by  $x_i$  permutes  $\{c_1, \dots, c_a\}$  by exchanging the cycles  $c_i$  and  $c_{i+1}$  and hence  $x_i$  centralizes  $w_\lambda$ . It is well-known that  $C_{S_\lambda}(w_\lambda)$  is the abelian group generated by the cycles  $c_1, \dots, c_a$  and that  $C_{S_n}(w_\lambda)$  is generated by  $C_{S_\lambda}(w_\lambda)$  together with the involutions  $x_i$ , for each  $i$  such that  $\lambda_i = \lambda_{i+1}$ . The abelianization of  $C_{S_n}(w_\lambda)$  is generated by the images of the first  $c_i$  and  $x_i$  for every cycle length. Whenever  $i$  is a part of  $\lambda$  define  $\lambda(i) = \{k \mid \lambda_k = i\}$  and  $\bar{i} = \min\{k \mid \lambda_k = i\}$ .

**Lemma 4.** *Let  $X_\lambda = \{c_{\bar{i}} \mid i \in \lambda\} \amalg \{x_{\bar{i}} \mid \lambda(i) > 1\}$ . Suppose  $\psi: X_\lambda \rightarrow \mathbb{C}^\times$  satisfies*

- (1)  $\psi(c_i)$  is a  $\lambda_i^{\text{th}}$  root of unity for all  $c_i \in X_\lambda$ , and
- (2)  $\psi(x_i)^2 = 1$  for all  $x_i \in X_\lambda$ .

*Then  $\psi$  has a unique extension to a linear character of  $C_{S_n}(w_\lambda)$ . Moreover, every linear character of  $C_{S_n}(w_\lambda)$  arises in this way.*

*Proof.* See the proof of [Lemma 6](#) below. □

For  $k \geq 1$  denote the  $k^{\text{th}}$  root of unity  $e^{2\pi i/k}$  by  $\zeta_k$ . For a partition  $\lambda = (\lambda_1, \dots, \lambda_a)$  of  $n$ , let  $\varphi_\lambda^\Lambda$  be the character of  $C_{S_n}(w_\lambda)$  defined (as in the preceding lemma) by

- $\varphi_\lambda^\Lambda(c_i) = \zeta_{|\lambda_i|}$  for all  $c_i \in X_\lambda$  and
- $\varphi_\lambda^\Lambda(x_i) = 1$  for all  $x_i \in X_\lambda$ .

The next theorem is proved in [\[7\]](#).

**Theorem 5.** *Suppose  $\lambda$  is a partition of  $n$  and let  $\widetilde{\rho}_\lambda$  be the top component character of the parabolic subgroup  $S_\lambda$  of  $S_n$ . Then*

$$\widetilde{\rho}_\lambda = \text{Ind}_{C_{S_n}(w_\lambda)}^{N_{S_n}(S_\lambda)} \varphi_\lambda^A.$$

**3.2. The characters  $\varphi_\mu^B$ .** A *signed partition* is a composition

$$\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+),$$

where  $\mu_1^- \leq \dots \leq \mu_a^-$  and  $\mu_1^+ \geq \dots \geq \mu_b^+$ . Then

$$\mu^- = (\mu_a^-, \dots, \mu_1^-) \quad \text{and} \quad \mu^+ = (\mu_1^+, \dots, \mu_b^+)$$

are partitions. We use this labeling convention for compatibility with [11] and the GAP functions described below, where the conjugacy class representatives are chosen to have minimal length. If  $|\mu^-| + |\mu^+| = n$ , then  $\mu$  is a signed partition of  $n$  and we write  $\mu \vdash n$ .

A *signed permutation* of  $n$  is a permutation  $w: \pm[n] \rightarrow \pm[n]$  such that  $w(-i) = -w(i)$  for  $1 \leq i \leq n$ . Signed permutations will be identified with their restrictions to functions  $[n] \rightarrow \pm[n]$  without comment. Let  $W_n$  denote the group of all signed permutations of  $n$ . Let  $S$  denote the set of transpositions  $s_i = (i \ i+1)$  for  $1 \leq i \leq n-1$ , together with the signed permutation  $t$  defined by  $t(1) = -1$  and  $t(i) = i$  for  $2 \leq i \leq n$ . Then  $(W_n, S)$  is a Coxeter system of type  $B_n$ .

For a signed partition  $\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+)$  of  $n$  define  $W_\mu$  to be the subgroup

$$W_{\mu_1^-} \times \dots \times W_{\mu_a^-} \times S_{\mu_1^+} \times \dots \times S_{\mu_b^+}$$

of  $W_n$ . Here, a similar identification to that for the embedding of  $S_\lambda = S_{\lambda_1^+} \times \dots \times S_{\lambda_a}$  in  $S_n$  is used. Thus,  $W_{\mu_1^-}$  acts on  $\{1, \dots, \mu_1^-\}$ ,  $W_{\mu_2^-}$  acts on  $\{\mu_1^- + 1, \dots, \mu_1^- + \mu_2^-\}$ ,  $\dots$ ,  $S_{\mu_1^+}$  acts on  $\{|\mu^-| + 1, \dots, |\mu^-| + \mu_1^+\}$ , and so on.

The conjugacy classes of parabolic subgroups of  $W_n$ , and hence the set of shapes of  $W_n$ , are parametrized by the set of partitions of  $m$  for  $0 \leq m \leq n$ . Suppose  $0 \leq m \leq n$  and  $\lambda$  is a partition of  $m$ . Define  $\mu_\lambda$  to be the signed partition  $((n-m), \lambda)$  of  $n$  and define  $W_\lambda = W_{\mu_\lambda}$ . Then the subgroups  $W_\lambda$  for  $\lambda$  a partition of  $m$  with  $0 \leq m \leq n$  form a set of representatives for the conjugacy classes of parabolic subgroups of  $W_n$ .

The set of signed partitions of  $n$  indexes the conjugacy classes in  $W_n$  as follows. Let  $\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+)$  be a signed partition of  $n$ . For  $1 \leq i \leq a$ , define  $c_i$  in  $W_n$  by

$$c_i(v) = \begin{cases} v+1 & \text{if } v = u+1, \dots, u + \mu_i^- - 1, \\ -(u+1) & \text{if } v = u + \mu_i^-, \\ v & \text{if } v \in [n] \setminus \{u+1, \dots, u + \mu_i^-\}, \end{cases} \quad \text{where } u = \sum_{k=1}^{i-1} \mu_k^-.$$

Then  $c_i$  is a negative  $\mu_i^-$ -cycle in the direct factor  $W_{\mu_i^-}$  of  $W_\mu$ . Similarly, for  $1 \leq j \leq b$  define  $d_j$  in  $W_n$  by

$$d_j(v) = \begin{cases} v+1 & \text{if } v = u+1, \dots, u + \mu_j^+ - 1, \\ u+1 & \text{if } v = u + \mu_j^+, \\ v & \text{if } v \in [n] \setminus \{u+1, \dots, u + \mu_j^+\}, \end{cases} \quad \text{where } u = |\mu^-| + \sum_{k=1}^{j-1} \mu_k^+$$

Then  $d_j$  is a positive  $\mu_j^+$ -cycle in the direct factor  $S_{\mu_j^+}$  of  $W_\mu$ . Finally, define

$$w_\mu = c_1 \cdots c_a d_1 \cdots d_b.$$



Then  $\{w_\mu \mid \mu \Vdash n\}$  is a set of conjugacy class representatives in  $W_n$ .

For a signed partition  $\mu$  of  $n$ , define  $\bar{\mu}$  to be the signed partition  $((|\mu^-|), \mu^+)$  of  $n$ . Then for a partition  $\lambda$  of  $m$  with  $0 \leq m \leq n$ ,  $\{w_\mu \mid \bar{\mu} = \mu_\lambda\}$  is a set of representatives of the cuspidal conjugacy classes in the parabolic subgroup  $W_\lambda$ .

Suppose  $\lambda$  is a partition of  $m$  with  $0 \leq m \leq n$  and  $\mu$  is a signed partition such that  $\bar{\mu} = \mu_\lambda$ . For each  $i$  such that  $\mu_i^- = \mu_{i+1}^-$  define  $x_i$  in  $W_n$  by

$$x_i(v) = \begin{cases} v + \mu_i^- & \text{if } v = u + 1, \dots, u + \mu_i^-, \\ v - \mu_i^- & \text{if } v = u + \mu_i^- + 1, \dots, u + 2\mu_i^-, \\ v & \text{if } v \in [n] \setminus \{u + 1, \dots, u + 2\mu_i^-\}, \end{cases} \quad \text{where } u = \sum_{k=1}^{i-1} \mu_k^-.$$

Then conjugation by  $x_i$  permutes  $\{c_1, \dots, c_a, d_1, \dots, d_b\}$  by exchanging the negative cycles  $c_i$  and  $c_{i+1}$  and hence  $x_i$  centralizes  $w_\mu$ . Next, for each  $j$  such that  $\mu_j^+ = \mu_{j+1}^+$  define  $y_j$  in  $W_n$  by

$$y_j(v) = \begin{cases} v + \mu_j^+ & \text{if } v = u + 1, \dots, u + \mu_j^+, \\ v - \mu_j^+ & \text{if } v = u + \mu_j^+ + 1, \dots, u + 2\mu_j^+, \\ v & \text{if } v \in [n] \setminus \{u + 1, \dots, u + 2\mu_j^+\}, \end{cases} \quad \text{where } u = m + \sum_{k=1}^{j-1} \mu_k^+.$$

Then conjugation by  $y_j$  permutes  $\{c_1, \dots, c_a, d_1, \dots, d_b\}$  by exchanging the positive cycles  $d_j$  and  $d_{j+1}$  and hence  $y_j$  centralizes  $w_\mu$ . Finally, for  $1 \leq j \leq b$  define  $r_j$  in  $W_n$  by

$$r_j(v) = \begin{cases} -v & \text{if } v = u + 1, \dots, u + \mu_j^+, \\ v & \text{if } v \in [n] \setminus \{u + 1, \dots, u + \mu_j^+\}, \end{cases} \quad \text{where } u = m + \sum_{k=1}^{j-1} \mu_k^+.$$

Then  $r_j$  centralizes  $c_i$  for  $1 \leq i \leq a$  and  $d_k$  for  $1 \leq k \leq b$  and hence centralizes  $w_\mu$ .

It is not hard to see that  $C_{W_\lambda}(w_\mu)$  is generated by the elements  $c_i$ , for  $1 \leq i \leq a$ ;  $d_j$ , for  $1 \leq j \leq b$ ; and  $x_i$ , for  $1 \leq i \leq a$  such that  $\mu_i^- = \mu_{i+1}^-$ . By [13, Proposition 4.4], the elements  $r_j$ , for  $1 \leq j \leq b$ , and  $y_j$ , for  $1 \leq j \leq b$  with  $\mu_j^+ = \mu_{j+1}^+$ , generate a complement to  $C_{W_\lambda}(w_\mu)$  in  $C_{W_n}(w_\mu)$ .

With the preceding notation, define the following. Whenever  $i$  is a part of  $\mu^-$  define  $\mu^-(i) = |\{k \mid \mu_k^- = i\}|$  and  $\bar{i} = \min\{k \mid \mu_k^- = i\}$ . Similarly, whenever  $j$  is a part of  $\mu^+$  define  $\mu^+(j) = |\{k \mid \mu_k^+ = j\}|$  and  $\bar{j} = \min\{k \mid \mu_k^+ = j\}$ .

**Lemma 6.** *Let*

$$X_\mu = \{c_{\bar{i}} \mid i \in \mu^-\} \amalg \{x_{\bar{i}} \mid \mu^-(i) > 1\} \amalg \{d_{\bar{j}} \mid j \in \mu^+\} \amalg \{y_{\bar{j}} \mid \mu^+(j) > 1\} \amalg \{r_{\bar{j}} \mid j \in \mu^+\}.$$

Suppose  $\psi: X_\mu \rightarrow \mathbb{C}^\times$  satisfies

- (1)  $\psi(c_i)$  is a  $2\mu_i^-$ -th root of unity for all  $c_i \in X_\mu$ ,
- (2)  $\psi(x_i)^2 = 1$  for all  $x_i \in X_\mu$ ,
- (3)  $\psi(d_j)$  is a  $\mu_j^+$ -th root of unity for all  $d_j \in X_\mu$ ,
- (4)  $\psi(y_j)^2 = 1$  for all  $y_j \in X_\mu$ , and
- (5)  $\psi(r_i)^2 = 1$  for all  $r_i \in X_\mu$ .

Then  $\psi$  has a unique extension to a linear character of  $C_{W_n}(w_\mu)$ . Moreover, every linear character of  $C_{W_n}(w_\mu)$  arises in this way.

*Proof.* In this proof we set  $C = C_{W_n}(w_\mu)$  and let  $Z_m$  denote the cyclic group of order  $m$ . If  $\lambda$  is a composition, we write  $j \in \lambda$  when some part of  $\lambda$  is equal  $j$ .

To prove the lemma, it is enough to show that the abelianization  $C/[C, C]$  is isomorphic to

$$\left( \prod_{i \in \mu^-} Z_{2j} \right) \times \left( \prod_{\mu^-(i) > 1} Z_2 \right) \times \left( \prod_{j \in \mu^+} Z_j \right) \times \left( \prod_{\mu^+(j) > 1} Z_2 \right) \times \left( \prod_{j \in \mu^+} Z_2 \right),$$

and is generated by the image of  $X_\mu$ .

It is straightforward to check that when  $S_m$  acts on  $(Z_u)^m$  by permuting the factors, the abelianization of the semidirect product  $(Z_u)^m \rtimes S_m$  is isomorphic to  $Z_u \times Z_2$ , and is generated by the image of a generator of the first direct factor in the product  $(Z_u)^m$  and the image of the transposition  $s_1$ . Taking  $u = 2$  we have in particular that the abelianization of  $W_m$  is isomorphic to  $Z_2 \times Z_2$ , and is generated by the images of  $t$  and  $s_1$ .

Similarly, when  $W_m$  acts on  $(Z_u)^m$  by permuting the factors (so the generator  $t$  and its conjugates act trivially), it is straightforward to check that the abelianization of the semidirect product  $(Z_u)^m \rtimes W_m$  is isomorphic to  $Z_u \times Z_2 \times Z_2$ , and is generated by the image of a generator of the first direct factor in the product  $(Z_u)^m$ , the image of the transposition  $s_1$ , and the image of  $t$ .

It follows from the description of the centralizer  $C$  in [13, §4.2] that

$$(3.1) \quad C \cong \prod_{i \in \mu^-} \left( (Z_{2i})^{\mu^-(i)} \rtimes S_{\mu^-(i)} \right) \times \prod_{j \in \mu^+} \left( (Z_j)^{\mu^+(j)} \rtimes W_{\mu^+(j)} \right),$$

so to complete the proof it suffices to consider the abelianizations of the direct factors of (3.1) individually. It follows from the remarks above that for  $i \in \mu^-$ , the abelianization of  $(Z_{2i})^{\mu^-(i)} \rtimes S_{\mu^-(i)}$  is isomorphic to  $Z_{2i} \times Z_2$ , and is generated by the images of  $c_{\bar{i}}$  and  $x_{\bar{i}}$ . Similarly, for  $j \in \mu^+$  the abelianization of  $(Z_j)^{\mu^+(j)} \rtimes W_{\mu^+(j)}$  is isomorphic to  $Z_j \times Z_2 \times Z_2$ , and is generated by the images of  $d_{\bar{j}}$ ,  $y_{\bar{j}}$ , and  $r_{\bar{i}}$ .  $\square$

For a signed partition  $\mu = (\mu_1^-, \dots, \mu_a^-, \mu_1^+, \dots, \mu_b^+)$  of  $n$  let  $\varphi_\mu^B$  be the character of  $C_{W_n}(w_\mu)$  defined (as in the preceding lemma) by

- $\varphi_\mu^B(c_i) = \zeta_{2k}$  where  $\mu_i^- = 2^l k$  with  $k$  odd, for  $1 \leq i \leq a$ ,
- $\varphi_\mu^B(d_j) = \zeta_{|d_j|}$  for  $1 \leq j \leq b$ ,
- $\varphi_\mu^B(x_i) = -1$  for all  $i$  such that  $\mu_i^- = \mu_{i+1}^-$ ,
- $\varphi_\mu^B(y_j) = 1$  for all  $j$  such that  $\mu_j^+ = \mu_{j+1}^+$ , and
- $\varphi_\mu^B(r_j) = (-1)^{\mu_j^+ - 1}$  for  $1 \leq j \leq b$ .

**Theorem 7.** *Suppose  $\lambda$  is a partition of  $m$  with  $0 \leq m \leq n$  and let  $\widetilde{\rho}_\lambda$  be the character of  $N_{W_n}(W_\lambda)$  afforded by the top component of  $\mathbb{C}W_\lambda$ . Then*

$$\widetilde{\rho}_\lambda = \sum_{\bar{\mu} = \mu_\lambda} \text{Ind}_{C_{W_n}(w_\mu)}^{N_{W_n}(W_\lambda)} \varphi_\mu^B$$

for  $n \leq 8$ .

We have verified Theorem 7 using the GAP computer algebra system [19] with the CHEVIE [12] and ZigZag [18] packages to compute both sides of the equality in the theorem. The computation of the character  $\widetilde{\rho}_\lambda$  of  $N_{W_n}(W_\lambda)$  was described in §2. The sum was computed with the help of some GAP functions. First, we defined a function

```
Lambda2Character( mu, cval, dval, xval, yval, rval )
```

that takes a signed partition  $\mu$  and the character values as in [Lemma 6](#), given as functions from the set of parts of  $\mu$  to the cyclotomic field, and returns the character  $\psi$  of  $C_{W_n}(w_\mu)$ . Second, we defined a function `BCharacter`(  $\mu$  ) that takes a signed partition, evaluates `Lambda2Character` at appropriate values, and returns the linear character  $\varphi_\mu^B$  of  $C_{W_n}(w_\mu)$ . For any given partition  $\lambda$  of  $m$  with  $0 \leq m \leq n$ , one can then use the characters  $\varphi_\mu^B$  with  $\bar{\mu} = \mu_\lambda$  to compute the sum of induced characters  $\sum_{\bar{\mu}=\mu_\lambda} \text{Ind}_{C_{W_n}(w_\mu)}^{N_{W_n}(W_\lambda)} \varphi_\mu^B$ .

It is tempting to speculate that the characters  $\{\varphi_\mu^B \mid \mu \vdash n, \bar{\mu} = \mu_\lambda\}$  satisfy the conclusion of [Theorem 1](#) for the parabolic subgroup  $W_\lambda$  of  $W_n$  for all  $n \geq 2$ .

A similar decomposition of the regular character of the Coxeter group of type  $B_n$  into characters that are induced from linear characters of element centralizers has been suggested by Bonnafé [[4](#), §10, Ques. (6)]. However, a straightforward calculation shows that his decomposition is different from ours, even for  $n = 2$ .

**3.3. The characters  $\varphi_\mu^D$ .** We regard the Coxeter group of type  $D_n$  for  $n \geq 4$  as the reflection subgroup  $W'_n$  of the Coxeter group  $W_n$  consisting of signed permutations with an even number of sign changes, so

$$W'_n = \{w \in W_n \mid |\{i \in [n] \mid w(i) < 0\}| \in 2\mathbb{N}\}.$$

Set  $t' = s_1 t s_1$ . Then  $W'_n$  is a reflection subgroup of  $W_n$  with Coxeter generating set  $S' = \{t', s_1, \dots, s_{n-1}\}$ .

The shapes of  $W'$  were determined in [[11](#), Proposition 2.3.13] as follows. First, if  $\lambda$  is a partition of  $m$  with  $m \leq n-2$ , or if  $\lambda$  is a partition of  $n$  containing at most one odd part, set  $W'_\lambda = W'_n \cap W_\lambda$ . Second, if  $\lambda$  is a partition of  $n$  with all even parts, then  $\lambda$  indexes two conjugacy classes. One is represented by  $W'_{\lambda^+} = S_\lambda$  and the other is represented by  $W'_{\lambda^-} = tS_\lambda t$ .

Suppose that  $\mu$  is a signed partition of  $n$ . If  $c$  is a negative cycle, then  $\{i \in [n] \mid c(i) < 0\}$  has an odd number of elements and so  $w_\mu = c_1 \cdots c_a d_1 \cdots d_b$  lies in  $W'_n$  if and only if  $a$  is even, that is, if and only if  $\mu^-$  has an even number of parts. Notice that the individual negative cycles  $c_i$  do not lie in  $W'_n$ , but the products  $c_i c_k$  do lie in  $W'_n$ , and hence in  $C_{W'_n}(w_\mu)$ .

Now suppose that  $\lambda$  is a partition of  $m$  with  $0 < m \leq n-2$ , or that  $\lambda$  is a partition of  $n$  with at least one odd part. Then  $\{w_\mu \mid \bar{\mu} = \mu_\lambda, l(\mu^-) \in 2\mathbb{N}\}$  is a complete set of representatives for the cuspidal conjugacy classes in  $W'_\lambda$ . If  $\lambda$  is a partition of  $n$  with all parts even, then the element  $w_\lambda$  in  $S_\lambda$  represents the unique cuspidal class in  $W'_{\lambda^+}$  and the element  $tw_\lambda t$  represents the unique cuspidal class in  $W'_{\lambda^-}$  (see [[11](#), Proposition 3.4.12]). Furthermore, because  $C_{W'_n}(w) = W'_n \cap C_{W_n}(w)$  for  $w$  in  $W'_n$ , we can define linear characters of  $C_{W'_n}(w)$  simply by restricting characters of  $C_{W_n}(w)$ .

The conclusion of [Theorem 1](#) has been shown to hold whenever  $W_\lambda$  is a product of symmetric groups in [[7](#)]. Thus, to simplify the exposition, in the following we consider only the parabolic subgroups  $W_\lambda$  where  $\lambda$  is a partition of  $m$  with  $0 \leq m \leq n-2$ .

Suppose  $\mu$  is a signed partition of  $n$  such that  $l(\mu^-)$  is even. It follows from [Lemma 6](#) that there is a linear character  $\psi_\mu$  of  $C_{W_n}(w_\mu)$  such that

- $\psi_\mu(c_i) = \zeta_{|c_i|}$  for  $1 \leq i \leq a$ ,
- $\psi_\mu(d_j) = \zeta_{|d_j|}$  for  $1 \leq j \leq b$ ,
- $\psi_\mu(x_i) = -1$  for all  $i$  such that  $\mu_i^- = \mu_{i+1}^-$ ,
- $\psi_\mu(y_j) = 1$  for all  $j$  such that  $\mu_j^+ = \mu_{j+1}^+$ , and

- $\psi_\mu(r_j) = -1$  for  $1 \leq j \leq a$ .

Define  $\varphi_\mu^D$  to be the restriction of  $\psi_\mu$  to  $C_{W'_n}(w_\mu)$ . As we have already observed, the individual negative cycles  $c_i$  do not lie in  $W'_n$ . Similarly, if  $\mu_j^+$  is odd, then  $r_j$  is not in  $W'_n$ . We have

- $\varphi_\mu^D(c_i c_k) = \zeta_{|c_i|} \zeta_{|c_k|}$  for  $1 \leq i, k \leq a$ ,
- $\varphi_\mu^D(d_j) = \zeta_{|d_j|}$  for  $1 \leq j \leq b$ ,
- $\varphi_\mu^D(x_i) = -1$  for all  $i$  such that  $\mu_i^- = \mu_{i+1}^-$ ,
- $\varphi_\mu^D(y_j) = 1$  for all  $j$  such that  $\mu_j^+ = \mu_{j+1}^+$ , and
- $\varphi_\mu^D(r_j) = -1$  for  $1 \leq j \leq b$  such that  $\mu_j^+$  is even, and
- $\varphi_\mu^D(r_i r_k) = 1$  for  $1 \leq i, k \leq b$  such that  $\mu_i^+$  and  $\mu_k^+$  are odd.

**Theorem 8.** *Suppose  $\lambda$  is a partition of  $m$  with  $0 \leq m \leq n-2$  and let  $\widetilde{\rho}_\lambda$  be the character of  $N_{W'_n}(W'_\lambda)$  afforded by the top component of  $\mathbb{C}W'_\lambda$ . Then*

$$\widetilde{\rho}_\lambda = \sum_{\substack{\bar{\mu} = \mu_\lambda \\ \iota(\mu^-) \in 2\mathbb{N}}} \text{Ind}_{C_{W'_n}(w_\mu)}^{N_{W'_n}(W'_\lambda)} \varphi_\mu^D$$

for  $n \leq 8$ .

The verification of [Theorem 8](#) parallels the verification of [Theorem 7](#). The computation of the character  $\widetilde{\rho}_\lambda$  of  $N_{W'_n}(W'_\lambda)$  was described in [§2](#). The sum was computed with the help of further GAP functions. We defined a function `DCharacter( mu )` that takes a signed partition  $\mu$ , evaluates the above function `Lambda2Character` at appropriate values, and returns the linear character  $\psi_\mu$  of  $C_{W'_n}(w_\mu)$ . Restriction to  $C_{W'_n}(w_\mu)$  yields the linear character  $\varphi_\mu^D$ . For any given partition  $\lambda$  of  $m$  with  $0 \leq m \leq n-2$ , one can then use the characters  $\varphi_\mu^D$  with  $\bar{\mu} = \mu_\lambda$  to compute the sum of induced characters  $\sum_{\substack{\bar{\mu} = \mu_\lambda \\ \iota(\mu^-) \in 2\mathbb{N}}} \text{Ind}_{C_{W'_n}(w_\mu)}^{N_{W'_n}(W'_\lambda)} \varphi_\mu^D$ .

It is tempting to speculate that the characters  $\varphi_\mu^D$  satisfy the conclusion of [Theorem 1](#) for  $W'_n$  for all  $n$  and all partitions  $\lambda$  of  $m$  with  $0 \leq m \leq n-2$ . Note that, in general, the characters  $\varphi_\mu^D$  are different from the restrictions of the characters  $\varphi_\mu^B$  to the centralizers in  $W'_n$ .

#### 4. EXCEPTIONAL GROUPS

As in [[3](#), §4], we only need to consider subsets  $L$  of  $S$  (up to conjugacy) such that  $W_L$  is not bulky in  $W$ ,  $W_L$  has rank at least three, and  $W_L$  is not a direct product of Coxeter groups of type A. The pairs  $(W, W_L)$  are given by type in the following table.

$W$	$W_L$
$E_7$	$D_4, A_1 D_4, D_5, A_1 D_5, E_6$
$E_8$	$D_4, A_1 D_4, D_5, A_2 D_4, A_1 D_5, D_6, E_6, A_2 D_5, A_1 D_6, D_7$

Because of space considerations, we do not list the values of the characters  $\widetilde{\rho}_L = \alpha_L \epsilon \widetilde{\omega}_L$ . Instead we list the characters  $\varphi_w$  that satisfy the conclusion of [Theorem 1](#) for each pair  $(W, L)$  when  $L$  is a proper subset of  $S$  in [Table 1](#) and [Table 2](#). However, see [[3](#), §3.1] for an example with all the data included.

In the tables we exhibit a set of generators of the centralizer of each cuspidal class representative of  $W_L$ . We use the symbol  $w_i$  to denote a representative of the  $i^{\text{th}}$

conjugacy class of a group in the list of conjugacy classes returned by the command `ConjugacyClasses` in GAP [19]. At each generator of  $C_W(w_i)$  we display the value of the character  $\varphi_{w_i}$ , denoted simply by  $\varphi_i$ . The symbol  $w_0$  represents the longest element of  $W$ , while  $w_L$  represents the longest element of  $W_L$  when  $L$  is a proper subset of  $S$ . We use the symbols  $1, 2, \dots, n$  to denote the elements of  $S$ . For  $p \geq 1$  we denote the  $p^{\text{th}}$  root of unity  $e^{2\pi i/p}$  by  $\zeta_p$ . Finally,  $r$  represents the reflection with respect to the highest long root in the root system of  $W$ . We sometimes express generators in terms of longest elements of certain parabolic subgroups of  $W$ . For this purpose, we fix the following subsets of  $S$ .

$$\begin{aligned} E &= \{2, 3, 4, 5\}, & F &= \{1, 2, 3, 4, 5\}, \\ G &= \{2, 3, 4, 5, 6\}, & H &= \{1, 2, 3, 4, 5, 6\}, \\ I &= \{2, 3, 4, 5, 6, 7\}, & J &= \{2, 3, 4, 5, 6, 7, 8\}. \end{aligned}$$

For the subgroup of type  $E_6$  of  $W(E_8)$  we modified the cuspidal conjugacy class representatives to match those used in [3]. Namely, we took  $w_{15} = 123456$ ,  $w_{14} = 24w_{15}$ ,  $w_{12} = 13456r_{E_6}$ ,  $w_{10} = w_{15}^2$ , and  $w_4 = 12356r_{E_6}$  where  $r_{E_6}$  is the reflection corresponding with the highest root in the  $E_6$  subsystem. For the subgroup of type  $A_1E_6$  of  $W(E_8)$  the representatives  $(w_8, w_{20}, w_{24}, w_{28}, w_{30})$  were obtained from the representatives  $(w_4, w_{10}, w_{12}, w_{14}, w_{15})$  for the subgroup of type  $E_6$  by multiplying by the generator  $\delta$ .

**4.1. Proof of Theorem 1 when  $W$  has type  $E$  and  $L = S$ .** To prove Theorem 1 when  $W$  has type  $E$  and  $L = S$  we proceed as in the case when  $L$  is a proper subset of  $S$  except that we use the methods described in §2 to compute the Orlik-Solomon character  $\widetilde{\omega}_S = \omega_S$ . Again, we do not list the values of the characters  $\widetilde{\rho}_L = \alpha_L \in \widetilde{\omega}_L$ . In Table 3 and Table 4 we list the characters  $\varphi_d$  that satisfy the conclusion of Theorem 1 when  $L = S$  for the groups of types  $E_7$  and  $E_8$  respectively. Here the conjugacy classes are labeled by Carter diagrams [6] and we denote the character  $\varphi_{w_d}$  by  $\varphi_d$  where  $d$  is a Carter diagram.

As in [2, 3], we give additional information about regular conjugacy classes. If  $w$  is in  $W$  and  $\zeta$  is an eigenvalue of  $w$  on  $V$ , then we denote the determinant of the representation of  $C_W(w)$  on the  $\zeta$ -eigenspace of  $w$  by  $\det|_\zeta$ . By Springer's theory of regular elements [20] the centralizer  $C_W(w)$  is a complex reflection group acting on an eigenspace of  $w$  whenever  $w$  is a regular element. In each table we indicate which classes are regular in the column labeled `Reg`, which is to be interpreted as follows. If  $w$  is regular and  $\varphi_w$  is a power of  $\det|_\zeta$  for some  $\zeta$  then we indicate this power in the `Reg` column. However, if  $w$  is regular but  $\varphi_w$  is *not* a power of  $\det|_\zeta$  for any  $\zeta$ , then we indicate this by writing  $\spadesuit$  in the `Reg` column. Whenever practical we describe the structure of  $C_W(w)$  in terms of  $Z_m$  and  $S_m$ , which denote the cyclic group of size  $m$  and the symmetric group on  $m$  letters.

When  $C_W(w_d)$  acts as a complex reflection group on an eigenspace of  $w_d$  we often specify the character  $\varphi_d$  by listing its values on generators of its ‘‘Dynkin diagram’’ presentation [5]. We also exhibit its Dynkin diagram in these cases. Note that vertices of the Dynkin diagram connected by single edges are conjugate in  $W$  and thereby take the same character values, which we list only once.

For the group  $W(E_8)$  we have slightly modified the representatives of the cuspidal conjugacy classes supplied by GAP. Namely, we observe that  $w_{E_8(a_8)}$  can be taken to be

$w_0w_{A_2^4}$  and then  $C_W(w_{E_8(a_8)}) = C_W(w_{A_2^4})$ . Similar observations hold for the classes  $A_4^2$  and  $E_8(a_6)$ .

## APPENDIX A. TABLES

 Table 1:  $W = W(E_7)$ ,  $L \subsetneq S$ 

L	Type	Characters
{2, 3, 4, 5}	D <sub>4</sub>	$\varphi_3 : (2, 4, 7, w_F, w_G) \mapsto (-1, -1, 1, 1, 1)$ $\varphi_9 : (7, 2w_G, 245w_H) \mapsto (1, \zeta_4, \zeta_4)$ $\varphi_{11} : (w_{11}, 7, 2w_{F2}, w_G) \mapsto (\zeta_3, 1, 1, 1)$
{2, 3, 4, 5, 7}	A <sub>1</sub> D <sub>4</sub>	$\varphi_6 : (3, 4, 5, 7, r, w_F) \mapsto (-1, -1, -1, -1, 1, 1)$ $\varphi_{18} : (w_{18}, 23, r, w_{F234}) \mapsto (1, -1, 1, \zeta_4)$ $\varphi_{22} : (2345, 7, r, 2w_{F2}) \mapsto (\zeta_3, -1, 1, 1)$
{2, 3, 4, 5, 6}	D <sub>5</sub>	$\varphi_7 : (w_7, 2, 3, 4, r, w_0) \mapsto (\zeta_4, -1, -1, -1, 1, -1)$ $\varphi_{15} : (w_{15}, r, w_0) \mapsto (\zeta_{12}, 1, -1)$ $\varphi_{17} : (w_{17}, r, w_0) \mapsto (\zeta_8, 1, -1)$
{1, 2, 3, 4, 5, 7}	A <sub>1</sub> D <sub>5</sub>	$\varphi_{14} : (w_{14}, 2, 4, 5, 7, w_0) \mapsto (\zeta_4, -1, -1, -1, -1, 1)$ $\varphi_{30} : (w_{30}, 7, w_0) \mapsto (\zeta_{12}, -1, 1)$ $\varphi_{34} : (w_{34}, 7, w_0) \mapsto (\zeta_8, -1, 1)$
{1, 2, 3, 4, 5, 6}	E <sub>6</sub>	$\varphi_4 : (2343, 134256, w_0) \mapsto (\zeta_3, 1, 1)$ $\varphi_{10} : (w_{15}^2, 123654, w_0) \mapsto (\zeta_3, 1, 1)$ $\varphi_{12} : (w_{12}, 2, 4, w_0) \mapsto (\zeta_3, -1, -1, 1)$ $\varphi_{14} : (w_{14}, w_0) \mapsto (\zeta_9, 1)$ $\varphi_{15} : (w_{15}, w_0) \mapsto (-1, 1)$

 Table 2:  $W = W(E_8)$ ,  $L \subsetneq S$ 

L	Type	Characters
{2, 3, 4, 5}	D <sub>4</sub>	$\varphi_3 : (2, 4, 7, 8, w_F, w_G) \mapsto (-1, -1, 1, 1, 1, 1)$ $\varphi_9 : (7, 8, 2w_G, 254w_F) \mapsto (1, 1, \zeta_4, -\zeta_4)$ $\varphi_{11} : (2345, 7, 8, 25w_F, w_G) \mapsto (\zeta_3, 1, 1, 1, 1)$
{2, 3, 4, 5, 8}	A <sub>1</sub> D <sub>4</sub>	$\varphi_6 : (3, 4, 7, w_F, w_J, w_I) \mapsto (-1, -1, -1, 1, -1, -1)$ $\varphi_{18} : (r, 254w_F, 3w_J) \mapsto (1, \zeta_4, -\zeta_4)$ $\varphi_{22} : (2345, r, 25w_F, w_J) \mapsto (\zeta_3, 1, 1, -1)$
{1, 2, 3, 4, 5}	D <sub>5</sub>	$\varphi_7 : (2, 4, 1342543, 7, 8, w_0, r) \mapsto (-1, -1, \zeta_4, 1, 1, -1, 1)$ $\varphi_{15} : (1234254, 7, 8, r, w_0) \mapsto (\zeta_{12}, 1, 1, 1, -1)$ $\varphi_{17} : (13425, 7, 8, r, w_0) \mapsto (\zeta_8, 1, 1, 1, -1)$
{2, 3, 4, 5, 7, 8}	A <sub>2</sub> D <sub>4</sub>	$\varphi_9 : (3, 4, 78, w_F, w_J) \mapsto (-1, -1, \zeta_3, 1, 1)$ $\varphi_{27} : (78, 254w_F, 3w_J) \mapsto (\zeta_3, \zeta_4, \zeta_4)$ $\varphi_{33} : (2345, 78, 25w_F, w_J) \mapsto (\zeta_3, \zeta_3, 1, 1)$
{1, 2, 3, 4, 5, 7}	A <sub>1</sub> D <sub>5</sub>	$\varphi_{14} : (2, 4, 1342543, 7, w_0, r) \mapsto (-1, -1, \zeta_4, -1, 1, 1)$ $\varphi_{30} : (1234254, 7, w_0, r) \mapsto (\zeta_{12}, -1, 1, 1)$ $\varphi_{34} : (13425, 7, r, w_0) \mapsto (\zeta_8, -1, 1, 1)$

Table 2: (continued)

L	Type	Characters
$\{2, 3, 4, 5, 6, 7\}$	$D_6$	$\varphi_4 : (2, 4, 5, 6, 7, r, w_J) \mapsto (-1, -1, -1, -1, -1, 1, 1)$ $\varphi_{14} : (2, 54234, 6576, r, w_J) \mapsto (-1, \zeta_4, -1, 1, 1)$ $\varphi_{21} : (2, 4, 76542345, r, w_J) \mapsto (-1, -1, \zeta_3, 1, 1)$ $\varphi_{27} : (543654765, 43w_J, r) \mapsto (-1, \zeta_3, 1)$ $\varphi_{33} : (24234567, 3w_J, r) \mapsto (\zeta_8, \zeta_4, 1)$ $\varphi_{35} : (234567, r, w_J) \mapsto (\zeta_5, 1, 1)$
$\{1, 2, 3, 4, 5, 6\}$	$E_6$	$\varphi_4 : (56, 14234542, 8, r, w_0) \mapsto (\zeta_3, \zeta_3^2, 1, 1, 1)$ $\varphi_{10} : (w_{15}, 234543, 8, r, w_0) \mapsto (1, \zeta_3, 1, 1, 1)$ $\varphi_{12} : (w_{12}, 2345432, r_{E_6}, 8, r, w_0) \mapsto (\zeta_3, -1, -1, 1, 1, 1)$ $\varphi_{14} : (w_{14}, 8, r, w_0) \mapsto (\zeta_9, 1, 1, 1)$ $\varphi_{15} : (w_{15}, 8, r, w_0) \mapsto (-1, 1, 1, 1)$
$\{1, 2, 3, 4, 5, 7, 8\}$	$A_2D_5$	$\varphi_{21} : (2, 4, 1342543, 78, w_0) \mapsto (-1, -1, \zeta_4, \zeta_3, -1)$ $\varphi_{45} : (1234254, 78, w_0) \mapsto (\zeta_{12}, \zeta_3, -1)$ $\varphi_{51} : (13425, 78, w_0) \mapsto (\zeta_8, \zeta_3, -1)$
$\{1, 2, 3, 4, 5, 6, 8\}$	$A_1E_6$	$\varphi_8 : (56, 14234542, 8, w_0) \mapsto (\zeta_3, \zeta_3^2, -1, -1)$ $\varphi_{20} : (w_{30}, 234543, w_0) \mapsto (-1, \zeta_3, -1)$ $\varphi_{24} : (w_{24}, 2345432, r_{E_6}, 8, w_0) \mapsto (\zeta_6, -1, -1, -1, -1)$ $\varphi_{28} : (24w_{30}, w_0) \mapsto (\zeta_{18}, -1)$ $\varphi_{30} : (w_{30}, 8, w_0) \mapsto (1, -1, -1)$
$\{2, 3, 4, 5, 6, 7, 8\}$	$D_7$	$\varphi_{10} : (w_{10}, 2, 3, 4, 5, 6, w_0) \mapsto (\zeta_4, -1, -1, -1, -1, -1, -1)$ $\varphi_{20} : (234, 5465, 7687, w_0) \mapsto (\zeta_4, -1, -1, -1)$ $\varphi_{31} : (w_{31}, 2, 42345, w_0) \mapsto (\zeta_{12}, -1, -\zeta_4, -1)$ $\varphi_{41} : (w_{41}, 2, 3, 4, w_0) \mapsto (\zeta_8, -1, -1, -1, -1)$ $\varphi_{47} : (w_{47}, w_0) \mapsto (\zeta_{24}, -1)$ $\varphi_{52} : (w_{52}, w_0) \mapsto (\zeta_{20}, -1)$ $\varphi_{54} : (w_{54}, w_0) \mapsto (\zeta_{12}, -1)$

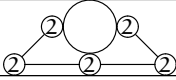
Table 3:  $W = W(E_7)$ ,  $L = S$ 

d	$C_W(w_d)$	Gen	$\varphi_d$	Reg
$A_1^7$	$W$	$S$	$\epsilon$	$\det _{-1}$
$A_1^3D_4$		$w_{A_1^3D_4}$ 2, 4, 5, 6, 7	$\zeta_6$ -1	
$E_7(a_4)$	$G_{26}$	$\textcircled{2} \text{---} \textcircled{3} \text{---} \textcircled{3}$	$-1, \zeta_3, \zeta_3$	$\spadesuit$
$A_1D_6(a_2)$		$w_{A_1D_6(a_2)}$ 2 1342654231456 (76) <sup>542345</sup>	$-\zeta_3$ -1 -1 $\zeta_3$	
$A_1A_3^2$		$w_{E_7(a_2)}$ 42543	1 $\zeta_4$	

Table 3: (continued)

d	$C_W(w_d)$	Gen	$\varphi_d$	Reg
$A_1D_6$		$w_{A_1D_6}$ 2 4	$\zeta_{10}$ -1 -1	
$A_2A_5$		$w_{A_2A_5}$ 34 2345	-1 $\zeta_3$ $\zeta_3^2$	
$E_7(\alpha_1)$	$Z_{14}$	$\textcircled{14}$	$\zeta_{14}$	$\det _{\zeta_{14}}$
$A_7$	$Z_8 \times Z_2 \times Z_2$	$w_{A_7}$ $w_0$ 2	$\zeta_8$ -1 -1	
$E_7$	$Z_{18}$	$\textcircled{18}$	$\zeta_{18}$	$\det _{\zeta_{18}}$
$E_7(\alpha_2)$	$Z_{12} \times Z_2$	$w_{E_7(\alpha_2)}$ $w_0$	1 -1	
$E_7(\alpha_3)$	$Z_{30}$	$w_{E_7(\alpha_3)}$	$\zeta_{30}$	

Table 4:  $W = W(E_8)$ ,  $L = S$

d	$C_W(w_d)$	Gen	$\varphi_d$	Reg
$A_1^8$	$W$	$S$	$\epsilon$	$\det _{-1}$
$D_4(\alpha_1)^2$	$G_{31}$		-1	$\spadesuit$
$A_1^4D_4$		1 $w_{A_1^4D_4}$ $w_{A_1E_7(\alpha_4)}$ $w_{A_3D_5(\alpha_1)}$	-1 $\zeta_3$ $\zeta_3^2$ $\zeta_3$	
$A_2^4$	$G_{32}$	$\textcircled{3} - \textcircled{3} - \textcircled{3} - \textcircled{3}$	$\zeta_3$	$\spadesuit$
$E_8(\alpha_8)$	$G_{32}$	$\textcircled{3} - \textcircled{3} - \textcircled{3} - \textcircled{3}$	$\zeta_3$	$\spadesuit$
$A_1E_7(\alpha_4)$		2 $w_{A_1^4D_4}$ $\dagger y$	-1 $\zeta_3$ $-\zeta_3$	
$D_4^2$		2 $w_E$ $\ddagger z$	-1 1 $\zeta_6$	
$A_1^2A_3^2$		$w_{A_2E_6(\alpha_2)}$ 5 $7w_G$	1 -1 $\zeta_4$	
$D_8(\alpha_3)$	$G_9$	$\textcircled{4} \equiv \textcircled{2}$	$\zeta_4, -1$	$\det _{\zeta_8}$
$A_1^2D_6$	$Z_{10} \times S_5$	$w_{A_1^2D_6}$ 3, 4, 5, 6	$\zeta_5$ -1	

$\dagger y = 13427654234567876$   
 $\ddagger z = 1342543654276548765$



Table 4: (continued)

d	$C_W(w_d)$	Gen	$\varphi_d$	Reg
$A_4^2$	$G_{16}$	$\textcircled{5} \text{---} \textcircled{5}$	$\zeta_5$	$\det  _{\zeta_5}$
$E_8(a_6)$	$G_{16}$	$\textcircled{5} \text{---} \textcircled{5}$	$\zeta_5$	$\det  _{\zeta_{10}}$
$A_2E_6(a_2)$		$w_{A_2E_6(a_2)}$	$\zeta_3$	
		$w_{A_1^2A_3^2}$	1	
		24	$\zeta_3$	
		2345	$\zeta_3^2$	
		$87w_I$	$\zeta_3$	
$E_8(a_3)$	$G_{10}$	$\textcircled{4} \text{---} \textcircled{3}$	$-1, \zeta_3$	$(\det  _{\zeta_{12}})^2$
$A_1A_2A_5$		$w_{A_1A_2A_5}$	1	
		7	-1	
		8	-1	
		34	$\zeta_3$	
		2345	$\zeta_3^2$	
$D_8(a_1)$	$Z_{12} \times S_3$	$w_{D_8(a_1)}$	$\zeta_3$	
		48	-1	
		4578	1	
$D_8$	$Z_{14} \times Z_2$	$w_{D_8}$	$\zeta_7$	
		2	-1	
$A_1A_7$		$w_{A_1A_7}$	$\zeta_8$	
		2	-1	
		34254	$\zeta_4$	
$A_1E_7$		$w_{A_1E_7}$	$\zeta_9$	
		3	-1	
		4	-1	
$A_8$	$Z_{18} \times Z_3$	$w_{A_8}$	$\zeta_9$	
		$13w_0$	$\zeta_3$	
$E_8(a_4)$	$Z_{18} \times Z_3$	$w_{E_8(a_4)}$	$\zeta_9$	
		34	$\zeta_3$	
$E_8(a_2)$	$Z_{20}$	$\textcircled{20}$	$\zeta_5$	$(\det  _{\zeta_{20}})^4$
$A_3D_5(a_1)$		$w_{A_3D_5(a_1)}$	$\zeta_6$	
		354	$\zeta_4$	
		136542	-1	
$A_2E_6$		$w_{A_2E_6}$	$\zeta_6$	
		34	$\zeta_3$	
		2435	$\zeta_3$	
$E_8(a_7)$		$w_{E_8(a_7)}$	$\zeta_6$	
		2354	$\zeta_3$	
		2454	$\zeta_3$	
$A_1E_7(a_2)$		$w_{A_1E_7(a_2)}$	-1	
		2, 4, $w_0$	-1	
$E_8(a_1)$	$Z_{24}$	$\textcircled{24}$	$\zeta_{12}$	$(\det  _{\zeta_{24}})^2$

Table 4: (continued)

$d$	$C_W(w_d)$	Gen	$\varphi_d$	Reg
$D_8(\mathfrak{a}_2)$	$Z_{30} \times Z_2$	$w_{D_8(\mathfrak{a}_2)}$ 5	$\zeta_{15}$ -1	
$E_8(\mathfrak{a}_5)$	$Z_{30}$	$\textcircled{30}$	$\zeta_{15}$	$(\det _{\zeta_{15}})^2$
$E_8$	$Z_{30}$	$\textcircled{30}$	$\zeta_{15}$	$(\det _{\zeta_{30}})^2$

**Acknowledgments:** The authors would like to acknowledge support from the DFG-priority program SPP1489 *Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory*. This work was partially supported by a grant from the Simons Foundation (Grant #245399 to J. Matthew Douglass).

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