Parabolic and Equal-Rank Subroot Systems with Applications to Symmetric Spaces and Flag Manifolds

PhD thesis

by

Mohammad Adib Makrooni

Supervisor: Dr. John Burns

School of Mathematics, Statistics and Applied Mathematics

National University of Ireland, Galway

June 2016

Contents

List of Figures

List of Tables

List of symbols

Ad	The adjoint representation of a Lie group
ad	The adjoint representation of a Lie algebra
κ	The Killing form
$\Re$	Root system
r	The rank
α_i	A simple (fundamental) root
ω_i	A simple (fundamental) weight
c_{ij}	Cartan integers
S_α	Reflection with respect to root α
$\tilde{\alpha}$	The highest root
m_i^α	The coefficient of the simple weight ω_i in α when expressed w.r.t the simple weights
m_k	Exponent
n_i^α	The coefficient of the simple root α_i in α when expressed w.r.t the simple roots
s_j	$\sum_{\{n_i^\alpha=j\}} \alpha = s_j\omega_i$
n_j	$\|\{\alpha \in \Re : n_i^\alpha = j\}\|$
d	$Max\{n_i^{\tilde{\alpha}} :\ 1 \leq i \leq r\}$
$d^\vee$	The highest coefficient of the highest short root of the dual root system
h	Coxeter number
g	Dual Coxeter number
ρ	The sum of the simple weights
k_i	$\frac{\langle\tilde{\alpha},\tilde{\alpha}\rangle}{\langle\alpha_i,\alpha_i\rangle}$
$N(\alpha_i)$	$k \in N(\alpha_i)$ if and only if α_k is adjacent to α_i in the Dynkin diagram
π_v	The irreducible representation with highest weight v
χ	Euler characteristic

Declaration

I hereby declare that this Ph.D. thesis is my original work and that I have not obtained a degree in this University or elsewhere on the basis of any of this work. Where use has been made of the work of other people, it has been fully acknowledged and referenced.

Galway, June, 2016

Mohammad Adib Makrooni

Acknowledgement

I would like to express my special appreciation and thanks to my Ph.D. supervisor Dr. John Burns for the continuous support of my Ph.D. study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis.

I would also like to express my heartfelt gratitude to my family, without whose love, support and encouragement, I would not have finished this thesis.

Abstract

Using the algebraic structure of subroot systems in the root system of a complex simple Lie algebra $\mathfrak{g}$, we prove a generalization for compact homogeneous spaces with positive Euler characteristic of the 'strange formula' of Freudenthal and de-Vries. We also derive formulae for the Chern classes of flag manifolds and the defect of the corresponding dual or discriminant varieties.

Chapter 1

Introduction

1.1 Outline of the thesis

The aim of this thesis is to derive relations between subroot systems of a root system of a complex simple Lie algebra and the original root system. We use these relations to study the geometry and topology of related homogeneous spaces, such as symmetric spaces and (generalized) flag manifolds.

In viewing the process of passing to a subroot system as a form of geometric induction, it is natural to relate root theoretic data of the subroot system to that of the parent root system. Whereas there are many kinds of such relations in the literature, the first systematic study along these lines was by R. Carles [12], where he considered the subroot system of roots that are perpendicular to the highest root. He derived formulae for the number of roots and the sum of the positive roots of the subroot system in terms of the corresponding quantities of the original root system. These results were extended to the subroot system consisting of the roots that are perpendicular to the highest short root (in the non-simply laced cases) by J. Burns and M. Clancy in [9].

The root systems considered in [12], [9] are examples of parabolic subroot systems i.e. they correspond naturally to parabolic subalgebras, and we extend these results to all maximal parabolic subroot systems in this thesis. We further extend the results of R. Carles, in that we relate additional root theoretic data of the root system and of the subroot system such as the Coxeter number and the exponents. As an application, we obtain a new description of the exponents of a complex simple Lie algebra, and therefore the Betti numbers of the corresponding Lie group.

In Chapter 2, we consider homogeneous spaces G/K of a semi-simple, compact, connected Lie group G where K is a maximal subgroup of maximal rank. Here the Lie algebra of K is determined by a subroot system obtained from the extended Dynkin diagram of $\mathfrak{g}$, by an algorithm due to Borel and De Siebenthal. For these spaces, we prove a generalization of the 'strange formula' of Freudenthal and de-Vries.

In Chapter 5, we also apply our formulae to computing a topological invariant of G/K introduced by S. Salamon ([24]) to study hyper-Kähler structures. As further applications of the previous chapters, we give geometric interpretations of our results in terms of Chern classes of flag manifolds and the nef value of ample line bundles on such spaces. These nef values and the dimension formulae we obtain in Chapter 4, are precisely the data required to compute the defect of the dual variety corresponding to the flag manifolds in the case of a maximal parabolic subgroup, and we apply these formulae to the study of such manifolds that have positive defect.

1.2 Review of necessary background material

This section recalls some necessary definitions from the theory of Lie algebras, Lie groups and their representations. All the material in this section can be found in [21], [2], [23] and [22].

1.2.1 Lie algebras, Lie groups and their representations

Definition 1.2.1. A vector space $\mathfrak{g}$ over a field $\mathbb{F}$, with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(X, Y) \mapsto [X, Y]$ (called the *bracket* or *commutator* of X and Y), is called a *Lie algebra* over $\mathbb{F}$ if the following axioms are satisfied:

(i) The bracket operation is bilinear.

(ii) $[X, X] = 0$ for all X in $\mathfrak{g}$.

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Definition 1.2.2. A subspace $\mathfrak{k}$ of $\mathfrak{g}$ is called a *subalgebra* if $[X, Y] \in \mathfrak{k}$ whenever $X, Y \in \mathfrak{k}$. Also, $\mathfrak{k}$ is called an *ideal* of $\mathfrak{g}$ if $[X, Y] \in \mathfrak{k}$ whenever $X \in \mathfrak{g}$, $Y \in \mathfrak{k}$.

Definition 1.2.3. A Lie algebra $\mathfrak{g}$ is called *simple* if it has no ideals except itself and 0, and if moreover $[\mathfrak{g}, \mathfrak{g}] \neq 0$.

Definition 1.2.4. A Lie algebra is said to be *semi-simple* if it is isomorphic to a direct sums of simple Lie algebras.

Definition 1.2.5. Let G be a smooth manifold. Then G is called a *Lie group* if:

(i) G is a group.

(ii) The group operations $G \times G \to G, (x, y) \mapsto xy$ and $G \to G, x \mapsto x^{-1}$ are smooth functions.

Definition 1.2.6. A *Lie subgroup* H of a Lie group G is a Lie group that is an abstract subgroup and an immersed submanifold of G.

Theorem 1.2.1. *If G is a Lie group, then the left-invariant vector fields on G form a Lie algebra $\mathfrak{g}$ with the product $[X, Y] = XY - YX$ and $\dim \mathfrak{g} = \dim G$. Also the function $X \mapsto X_e$ defines a linear isomorphism between $\mathfrak{g}$ and $T_e(G)$ as vector spaces where e is the identity element of G, X_e is the tangent vector given by X at the identity and $T_e(G)$ is the tangent space to G at e*

Proposition 1.2.2. *If $f : G_1 \to G_2$ is a homomorphism of Lie groups, then $(df)_e : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of Lie algebras where $(df)_e$ is the differential of f.*

Theorem 1.2.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Conversely, for each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a unique connected Lie subgroup H of G which has $\mathfrak{h}$ as its Lie algebra. Furthermore, normal subgroups of G correspond to ideals in $\mathfrak{g}$.*

Definition 1.2.7. A *representation* of a Lie group G is a Lie group homomorphism $\rho : G \to Gl(V)$, where $Gl(V)$ is the *general linear group* consisting of all invertible endomorphisms of a vector space V, the dimension of the representation is the dimension of the vector space.

By Proposition 1.2.2, similarly we can define a representation of a Lie algebra.

Definition 1.2.8. A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the *general linear Lie algebra* consisting of all endomorphisms of a vector space V.

Definition 1.2.9. A vector space V, endowed with an operation $\mathfrak{g} \times V \to V$ (denoted by $(x, v) \mapsto x.v$) is called a $\mathfrak{g}-$*module* if the following conditions are satisfied:

(i) $(ax + by).v = a(x.v) + b(y.v)$.

(ii) $x.(av + bw) = a(x.v) + b(x.w)$.

(iii) $[x, y].v = x.y.v - y.x.v$ for all $x, y \in \mathfrak{g}; v, w \in V; a, b \in \mathbb{F}$.

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then V may be viewed as an $\mathfrak{g}$-module via the action $x.v = \rho(x)(v)$. Conversely, given a $\mathfrak{g}$-module V, this equation defines a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Example 1.2.1. The *adjoint representation* of G is the Lie group homomorphism $\mathsf{Ad} : G \to Gl(\mathfrak{g})$, given by $\mathsf{Ad}(g) = d\mathsf{I}_g$, where I_g is the conjugation by g. Also, the *adjoint representation* of $\mathfrak{g}$ is the Lie algebra homomorphism $\mathsf{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $\mathsf{ad}(X) = (d\mathsf{Ad})_e(X)$.

Theorem 1.2.4. *The adjoint representation of* $\mathfrak{g}$ *satisfies* $\mathsf{ad}(X)Y = [X, Y]$ *for all* $X, Y \in \mathfrak{g}$.

Definition 1.2.10. A $\mathfrak{g}$-module (G-module) V is said to be *irreducible* if it has precisely two $\mathfrak{g}$-submodules (G-submodules) (itself and 0). Also a representation ρ is called irreducible if its corresponding $\mathfrak{g}$-module (G-module) is irreducible.

Definition 1.2.11. A $\mathfrak{g}$-module (G-module) V is called *semi-simple* if each non-trivial $\mathfrak{g}$-submodule (G-submodule) W has a complementary $\mathfrak{g}$-submodule (G-submodule) W', such that $V = W \oplus W'$.

Definition 1.2.12. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. If $X, Y \in \mathfrak{g}$, define $\kappa(X, Y) := Tr(\mathsf{ad}X\mathsf{ad}Y)$. Then κ is a symmetric bilinear form on $\mathfrak{g}$, called the *Killing form.*

Proposition 1.2.5. *The Killing form of a Lie algebra* $\mathfrak{g}$ *is* Ad*-invariant, that is* $\kappa(X, Y) = \kappa(\mathsf{Ad}(g)X, \mathsf{Ad}(g)Y)$ *for all* $g \in G$ *and* $X, Y \in \mathfrak{g}$.

Definition 1.2.13. A *Cartan subalgebra* $\mathfrak{h}$ of a complex semi-simple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$, such that for all $H \in \mathfrak{h}$, the endomorphism $\mathsf{ad}(H)$ is diagonalizable.

Definition 1.2.14. A *torus* in a Lie group G is a Lie subgroup that is isomorphic to a product $S^1 \times \cdots \times S^1$. A torus T is a *maximal* torus in G, if for any torus S in G with $T \subset S \subset G$ for a torus S, then $T = S$.

Definition 1.2.15. A Lie group G is called *semi-simple* if its Killing form is non-degenerate.

Theorem 1.2.6. *If* G *is a compact semi-simple Lie group, then its Killing form is negative definite.*

Theorem 1.2.7. *Suppose that* ρ *is a finite dimensional irreducible representation of a complex semi-simple Lie algebra* $\mathfrak{g}$ *and let* V *be its corresponding* $\mathfrak{g}$*-module, then* V *decomposes under the action of Cartan subalgebra* $\mathfrak{h}$ *as:*

$$V = \bigoplus_{\lambda \in \Lambda_\rho} V_\lambda,$$

where $\Lambda_\rho \subset \mathfrak{h}^*$, $V_\lambda = \{v \in V : \rho(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$. *Here* Λ_ρ *is the set of linear functionals on* $\mathfrak{h}$ *for which* $V_\lambda \neq \{0\}$, *and an element* λ *is called a weight of the representation.*

Theorem 1.2.8. *A complex semi-simple Lie algebra* $\mathfrak{g}$ *can be decomposed under the adjoint representation, as a direct sum as follows:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{R}} \mathfrak{g}_\alpha,$$

The decomposition is (uniquely determined by $\mathfrak{h}$) is called the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Any element $\alpha \in \mathfrak{h}^*$ such that $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$ is called a root of $\mathfrak{g}$ and the set of all roots which is denoted by $\Re$ is called the root system.

Since the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is non-degenerate, it is possible to identify $\mathfrak{h}$ with $\mathfrak{h}^*$, each $\alpha \in \mathfrak{h}^*$ corresponds to a unique element $H_\alpha \in \mathfrak{h}$ such that $\kappa(H_\alpha, H) = \alpha(H)$, for all $H \in \mathfrak{h}$. Therefore, we can transfer the Killing form of $\mathfrak{h}$ to $\mathfrak{h}^*$ by defining $(\alpha, \beta) = \kappa(H_\alpha, H_\beta)$, for all $\alpha, \beta \in \mathfrak{h}^*$. Also since $\Re$ spans $\mathfrak{h}^*$, there is a basis Π of $\mathfrak{h}^*$ consisting of r roots that generates $\mathfrak{h}^*$ such that any root $\alpha \in \Re$ can be expressed as $\alpha = \sum\limits_{\alpha_i \in \Pi} n_i^\alpha \alpha_i$, where all the n_i^α are either all non-negative, in which case we call α a positive root, or all non-positive in which case we call α a negative root. These roots are called *simple* or *fundamental* roots, r is called the rank of $\mathfrak{g}$ ($rank(\mathfrak{g})$) and Π is called a *fundamental system.* The subsets of negative and positive roots are denoted by $\Re^-$ and $\Re^+$ respectively. We define the height of a positive root with respect to Π, by $ht_{\mathfrak{g}}(\alpha) = \sum\limits_{\alpha_i \in \Pi} n_i^\alpha$ (we write $ht(\alpha)$ when it is clear which Lie algebra we are taking the height in), a root $\tilde{\alpha}$ with the property that $ht(\alpha) \leq ht(\tilde{\alpha})$ for any $\alpha \in \Re$ is called the *highest root.*

Definition 1.2.16. Let $\Re$ be a root system and $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ be a fundamental system. For each $I \subset \{1, 2, \cdots, r\}$, let $\Pi_I = \{\alpha_i : i \in I\}$, then the root system generated by Π_I is denoted by $\Re_I$ and is called the *parabolic subroot system generated by* Π_I.

A subroot system $\Re'$ of $\Re$ is *closed* if for any $\alpha, \beta \in \Re'$, $\alpha+\beta \in \Re$ implies $\alpha+\beta \in \Re'$. (In other words, $\alpha + \beta \in \Re'$ whenever $\alpha + \beta$ is a root.)

Theorem 1.2.9. *Let $\mathfrak{g}$, $\mathfrak{h}$ and $\Re$ be as above. Then:*

(i) The set of all roots $\Re$ span a real subspace E in $\mathfrak{h}^*$ of real dimension $r = rank(\mathfrak{g})$.

(ii) If $\alpha \in \Re$, then $-\alpha \in \Re$, but no other scalar multiple of α is a root.

(iii) If $\alpha, \beta \in \Re$, then $\beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Re$.

(iv) If $\alpha, \beta \in \Re$, then $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

A root system with the last property is called a crystallographic root system.

Proposition 1.2.10. *Given $\alpha, \beta \in \Pi$, where $\alpha \neq \beta$, then $(\alpha, \beta) \leqslant 0$.*

Proposition 1.2.11. *Given two non proportional roots $\alpha, \beta \in \Re$, then*

(i) If $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root.

(ii) If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.

(iii) If $(\alpha, \beta) = 0$ and $\alpha + \beta \in \Re$, then $\alpha - \beta \in \Re$.

1.2.2 The Weyl group and classification of root systems

Definition 1.2.17. The *Weyl group W* of a Lie algebra $\mathfrak{g}$ is the reflection group generated by $\{S_\alpha : \alpha \in \Re\}$, where for all $\alpha, \beta \in \Re$,

$$S_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha.$$

The rank of the Weyl group W is defined to be the rank of $\mathfrak{g}$.

In fact, any element of W can be written as a product of simple reflections S_α ($\alpha \in \Pi$). In particular, the product of all simple reflections is called a *Coxeter element*, and its order which is independent of the choice of Coxeter element is denoted by h and is called the *Coxeter number*. It is a well-known fact that $ht(\tilde{\alpha}) = h - 1$.

Definition 1.2.18. A root system $\Re$ is called *irreducible* if there is no orthogonal decomposition $\Re = \Re_1 \cup \Re_2$, such that $\Re_1 \neq 0$ and $\Re_2 \neq 0$.

Lemma 1.2.12. *If $\Re$ is irreducible, then at most two root lengths occur in Φ.*

In the case where two root lengths occur, we call the roots *short* and *long*. We denote the set of long and short roots by $\Re_l$ and $\Re_s$ respectively. If all roots in $\Re$ have equal length, we say $\Re$ is simply laced. Otherwise, we say $\Re$ is non-simply laced. All the root vectors of the same length lie in the same W-orbit.

Lemma 1.2.13. *Given two non-proportional roots α, β such that $(\alpha, \alpha) \leqslant (\beta, \beta)$ then*

$$2\frac{(\alpha, \beta)}{(\beta, \beta)} \in \{-1, 0, 1\}.$$

For a given crystallographic root system $\Re$ and each $\alpha \in \Re$, we denote by $\alpha^\vee$ the *coroot* $\alpha^\vee = \dfrac{2\alpha}{(\alpha, \alpha)}$. They form a crystallographic root system in E, whose Weyl group is canonically isomorphic to W. For all irreducible crystallographic root systems listed at the end of this chapter, $\Re^\vee = \Re$, except when $\Re = B_r, C_r$. In this cases $B_r^\vee = C_r$ and $C_r^\vee = B_r$.

Definition 1.2.19. For a fixed ordering $(\alpha_1, \cdots, \alpha_r)$ of the simple roots of $\Re$, the matrix $[c_{ij}]_{r\times r}$ is called the *Cartan* matrix of $\Re$ where $c_{ij} = (\alpha_i^\vee, \alpha_j)$. Its entries are called *Cartan* integers.

The set Λ of all $\lambda \in E$ such that for all $\alpha \in \Re$, $\dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ is called the set of *weights*. In addition, if all the integers $(\lambda, \alpha^\vee)$ are non-negative, then λ is called a *dominant* weight. Also for a given fundamental system $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ of $\Re$, the *fundamental weights* $\{\omega_1, \cdots, \omega_r\}$ with respect to Π are defined by:

$$(\omega_i, 2\alpha_j) := (\alpha_i, \alpha_j)\delta_{ij}.$$

We denote by ρ the sum of fundamental weights, $\rho = \sum\limits_{i=1}^{r} \omega_i = \frac{1}{2}\sum_{\alpha>0} \alpha$. Since the Weyl group W of $\Re$ preserves the inner product on E, it leaves Λ invariant.

The Cartan matrix tells us how to expand $\{\alpha_1, \cdots, \alpha_r\}$ in terms of $\{\omega_1, \cdots, \omega_r\}$. We can see this directly, write $\alpha_i = \sum\limits_{j=1}^{r} m_{ij}\omega_j, (m_{ij} \in \mathbb{Z})$. Then

$$c_{ki} = (\alpha_i, \alpha_k^\vee) = \sum_{j=1}^{r} m_{ij}(\omega_j, \alpha_k^\vee) = m_{ik}.$$

Therefore $\alpha_i = \sum\limits_{j=1}^{r} c_{ji}\omega_j$. To write ω_j in terms of α_i's, we have only to invert the Cartan matrix.

Proposition 1.2.14. $\sum_{\alpha\in\Re^+} \langle\alpha,\, \gamma\rangle\alpha = g\,\gamma$ *for all* $\gamma \in \mathfrak{h}^*$, g *is a positive integer. If* $\mathfrak{g}$ *is simply laced, then* g *is the Coxeter number of* $\mathfrak{g}$*. In general, if we take an* *invariant inner product* $\langle , \rangle$ *normalised so that* $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2$*, then the Killing form satisfies* $(,) = \frac{1}{2g}\langle , \rangle$*. The positive integer* g *is called the dual Coxeter number.*

For each pair $(i, x) \in \{1, \ldots, r\} \times \mathbb{R}$, let $\Re_{i,x} = \{\alpha \in \Re : \langle \omega_i, \alpha \rangle = x\}$. Now observe that for $j \neq i$ the simple reflection s_j in the hyperplane $\alpha_j = 0$ permutes the elements of $\Re_{i,x}$, so that

$$\langle \sum_{\alpha \in \Re_{i,x}} \alpha,\, \alpha_j \rangle = \langle S_j(\sum_{\alpha \in \Re_{i,x}} \alpha),\, S_j \alpha_j \rangle$$

$$= \langle \sum_{\alpha \in \Re_{i,x}} \alpha,\, S_j \alpha_j \rangle$$

$$= -\langle \sum_{\alpha \in \Re_{i,x}} \alpha,\, \alpha_j \rangle$$

and therefore $\langle \sum_{\alpha \in \Re_{i,x}} \alpha, \alpha_j \rangle = 0$. Thus $\sum_{\alpha \in \Re_{i,x}} \alpha = c_x \omega_i$, for some $c_x \in \mathbb{R}$.

Lemma 1.2.15. *Given a finite dimensional irreducible representation of* V*, with set of weights* Λ*. For each pair* $(i, x) \in \{1, \ldots, r\} \times \mathbb{R}$*, let* $\Lambda_{i,x} := \{\alpha \in \Re^+ | \langle \alpha, \omega_i \rangle = x\}$*, then*

$$\sum_{\lambda \in \Lambda_{i,x}} \lambda = c_{i,x} \omega_i,$$

where $c_{i,x}$ *is zero if and only if* $x = 0$*.*

Definition 1.2.20. The *Dynkin* diagram $\triangle_{\Re}$ is the (directed, multi) graph with r vertices (labelled by the positive simple roots), and $c_{ij}c_{ji}$ edges joining α_i to α_j.

The *extended Dynkin* diagram $\tilde{\triangle}_{\Re}$ is the (directed, multi) graph constructed from $\triangle_{\Re}$ by adding a new vertex $\alpha_0 = -\tilde{\alpha}$ and joining it to any vertex α_i by $n(\alpha_i, \tilde{\alpha}).n(\tilde{\alpha}, \alpha_i)$ edges, where for $\alpha, \beta \in \Re$, $n(\alpha, \beta) = \dfrac{2(\alpha, \beta)}{(\alpha, \alpha)}$, we then write the coefficient of α_i in $\tilde{\alpha}$ over the vertex α_i and 1 over α_0. We now provide the extended Dynkin diagrams of all simple Lie algebras in the following tables.

Extended Dynkin diagram of simply laced Lie algebras

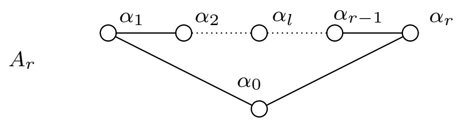

Figure 1.1: Extended Dynkin diagram of A_r

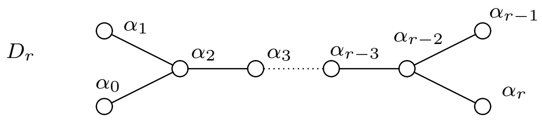

Figure 1.2: Extended Dynkin diagram of D_r

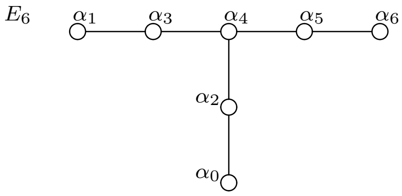

Figure 1.3: Extended Dynkin diagram of E_6

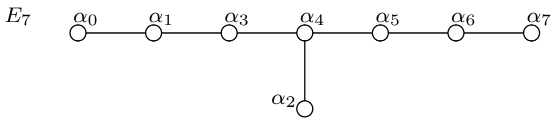

Figure 1.4: Extended Dynkin diagram of E_7

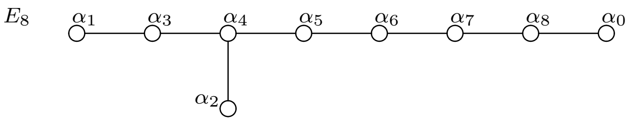

Figure 1.5: Extended Dynkin diagram of E_8

Extended Dynkin diagram of non-simply laced Lie algebras

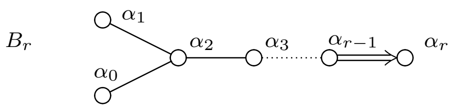

Figure 1.6: Extended Dynkin diagram of B_r

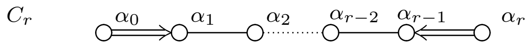

Figure 1.7: Extended Dynkin diagram of C_r

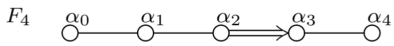

Figure 1.8: Extended Dynkin diagram of F_4

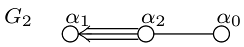

Figure 1.9: Extended Dynkin diagram of G_2

Chapter 2

Homogeneous quotients by equal rank subgroups and their strange formulae

In this chapter, we prove a generalisation of the 'strange formula' of Freudenthal and de-Vries for compact homogeneous spaces with positive Euler characteristic. As a result we obtain a new proof of the original formula ([10]). The 'strange formula' of Freudenthal and de-Vries is the following theorem:

Theorem 2.0.16. [18] $24(\rho, \rho) = \dim G$, *where* G *is a semi-simple, compact, connected Lie group.*

The theorem was first proved in [18] by considering a Taylor expansion of the Weyl character formula and they asked if the formula could be proved by more algebraic or elementary means. Such a proof is given in [8] and it is proved by the comparison of two formulae for the heat kernel in [14]. An inductive proof based on the study of root system embeddings is outlined in [12], and in [15] it is proved by studying the spin representation $Spin(\mathfrak{g}) \to SO(\mathbf{S})$, where $\mathbf{S}$ is the space of spinors on the vector space $\mathfrak{g}$.

Definition 2.0.21. A *homogeneous space* is a manifold M with a transitive action of a Lie group G. Equivalently, it is a manifold of the form G/K, where G is a Lie group and K is the isotropy subgroup of a point $\circ \in M$.

Let G/K be a homogeneous space, $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Suppose that π is the usual projection map $\pi : G \to G/K$, $\pi(g) = gK$, and denote by $(d\pi)_e$ the differential $(d\pi)_e : \mathfrak{g} \to T_\circ(G/K)$, where $\circ = \pi(e) = K$. Then

$$(d\pi)_e(X) = (d\pi)_e(\phi'_X(0)) = \frac{d}{dt}(\pi \circ \exp tX)\Big|_{t=0} = \frac{d}{dt}((\exp tX)K)\Big|_{t=0},$$

in which $X \in \mathfrak{g}$ and $\phi_X(t) = \exp tX$ is the corresponding one-parameter subgroup. As $\exp tX \in K$ for all $X \in \mathfrak{k}$, therefore we obtain that $(d\pi)_e(\mathfrak{k}) = 0$, that is, $\ker d\pi_e = \mathfrak{k}$, hence since $d\pi$ is onto, we get the canonical isomorphism

$$\mathfrak{g}/\mathfrak{k} \cong T_\circ(G/K).$$

Definition 2.0.22. A homogeneous space is called *reductive* if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathsf{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$ for all $k \in K$, that is, $\mathfrak{m}$ is $\mathsf{Ad}(K)$-invariant.

Proposition 2.0.17. [2] *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. There is a one-to-one correspondence between bi-invariant metrics on G and Ad-invariant scalar products on $\mathfrak{g}$, that is $\langle \mathsf{Ad}(g)X, \mathsf{Ad}(g)Y\rangle = \langle X, Y\rangle$ for all $g \in G$ and all $X, Y \in \mathfrak{g}$.*

Theorem 2.0.18. [3] *If ρ is a representation of a compact connected Lie group G on a finite dimensional vector space V, then it is completely reducible. Moreover $V = W_1 \oplus \cdots \oplus W_r$, where for $i \neq j$ the subspaces are mutually orthogonal (with respect to a $\rho-$invariant inner product on V) and each is a non-trivial irreducible subspace*

Let G/K be a homogeneous space with G a semi-simple, compact, connected Lie group. Then by Proposition 2.0.17, every bi-invariant metric of G determines an Ad-invariant scalar product on $\mathfrak{g}$, hence by Theorem 2.0.18, there exists a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the scalar product on $\mathfrak{g}$. In particular from Theorem 1.2.6, we know that if G is a compact semi-simple Lie group, then its Killing form is negative definite, therefore we endow $\mathfrak{g}$ with the inner product $(,)$ which is the negative of the Killing form (which we will also call the Killing form). By Proposition 1.2.5, the Killing form is Ad-invariant, so it gives the $\mathsf{Ad}(K)$-invariant orthogonal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, therefore, as an immediate consequence of the above isomorphism we can identify $\mathfrak{m}$ with the tangent space to $M = G/K$ at the identity coset.

Theorem 2.0.19. [29] *If K is a connected and closed subgroup of a connected compact Lie group G, and rank(G)=rank(K), then:*

$$\chi(G/K) = \frac{|W(G)|}{|W(K)|},$$

where $\chi(G/K)$ is the Euler characteristic of G/K and $W(G), W(K)$ denote the Weyl group of G and K.

Therefore, if G and K have the same rank, then the Euler characteristic of G/K is positive. we will refer throughout this chapter to compact homogeneous spaces with positive Euler characteristic as equal rank homogeneous spaces.

Definition 2.0.23. The *isotropy representation* of the homogeneous space G/K (or simply of K) is the homomorphism

$$\mathsf{Ad}^{G/K} : K \to Gl(T_0(G/K)),$$

defined by $k \mapsto (d\tau_k)_0$, where $k \in K$ and τ_k is the diffeomorphism $\tau_k : G/K \to G/K$. More explicitly, for all $X \in T_0(G/K)$

$$\mathsf{Ad}^{G/K}(k)(X) = (d\tau_k)_0(X).$$

We recall the classification of the maximal subgroups of G of maximal rank in terms of the coefficients of the highest root when expressed w.r.t. the simple roots $\alpha_1, \ldots, \alpha_r$ as $\tilde{\alpha} = \sum_{i=1}^{r} n_i^{\tilde{\alpha}} \alpha_i$, labelled as in [7]. In general, we will express a root α in the form $\alpha = \sum_{i=1}^{r} n_i^{\alpha} \alpha_i$.

Theorem 2.0.20. *([5], [**32**, **p. 278**]) Let G be a compact centreless simple Lie group and let $1 \leq i \leq r$.*

(i) Suppose that $n_i^{\tilde{\alpha}} = 1$, then the centralizer of the circle group $\{\exp(2\pi i t v_i) : t \in \mathbb{R}\}$ (where $v_1, \ldots, v_r$ satisfy $\alpha_i(v_j) = \frac{1}{n_i^{\tilde{\alpha}}}\delta_{ij}$) is a maximal connected subgroup of maximal rank in G with $\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\}$ as a system of simple roots.

(ii) Suppose that $n_i^{\tilde{\alpha}}$ is a prime $p > 1$, then the centralizer of the element $\exp(2\pi i v_i)$ (of order p) is a maximal connected subgroup of maximal rank in G with $\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r, -\tilde{\alpha}\}$ as a system of simple roots.

(iii) Every maximal connected subgroup of maximal rank in G is conjugate to one of the above groups.

We shall make use of a sharpened version of the above Theorem (see [32]). Let G/K be a compact homogeneous space, in which G is a semi-simple, compact, connected Lie group with Lie algebra $\mathfrak{g}$, and K is a maximal subgroup of G of maximal rank, with maximal torus T contained in K. We assume that $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^{\mathbb{C}}$ its complexification. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Re} \mathfrak{g}^{\alpha}$ be the root space decomposition of the complexification of $\mathfrak{g}$. We will denote the root system of K (which contains our maximal torus T) by $\Re_K$ and call $\Re \setminus \Re_K$ the set of complementary roots. The isotropy representation of K complexifies to a representation of K on $\sum_{\alpha \in \Re \setminus \Re_K} \mathfrak{g}^{\alpha}$ and it comes from the adjoint representation of G. Following [32] we will denote this representation by $\mathsf{ad}_{G/K}$ and of course the complementary roots $\Re \setminus \Re_K$ are its weights. We let Ψ denote the set of positive complementary roots and we define $\rho_{\Psi} := \frac{1}{2}\sum_{\alpha \in \Psi} \alpha$.

Theorem 2.0.21. [32] *Denoting the irreducible representation of K with highest weight v by π_v, we have:*

(i) If $n_i^{\tilde{\alpha}} = 1$, then $\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^*$.

(ii) If $n_i^{\tilde{\alpha}} = 2$, then $\mathsf{ad}_{G/K} = \pi_{-\alpha_i}$.

(iii) If $n_i^{\tilde{\alpha}} = 3$, then $\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^*$.

(iv) If $n_i^{\tilde{\alpha}} = 5$*, then* $\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^* + \pi_{-\beta_i} + (\pi_{-\beta_i})^*$*, where* β_i *is the lowest height positive root with* $n_i^{\beta_i} = 2$*.*

2.1 G/K Symmetric

In this section, we consider the cases where $n_i^{\tilde{\alpha}} = 1$ and $n_i^{\tilde{\alpha}} = 2$. In both of these cases G/K is symmetric. Where $n_i^{\tilde{\alpha}} = 1$, the corresponding symmetric space is Hermitian and when $n_i^{\tilde{\alpha}} = 2$, G/K is non-Hermitian symmetric (see [11])

2.1.1 G/K Hermitian symmetric

Theorem 2.1.1. *Let* $M = G/K$ *be an Hermitian symmetric space, then* $24(\rho_\Psi, \rho_\Psi)_M = \dim M$*, where* $(\,,\,)_M = \frac{1}{3}(\,,\,)$*.*

Proof. We are in the situation of case (i) of Theorem 2.0.21, where $n_i^{\tilde{\alpha}} = 1$ and $\Psi = \{\alpha \in \Re^+ \,|\, n_i^\alpha = 1\}$. We will first prove that $\langle 2\rho_\Psi,\, \alpha_i\rangle = g$. Although Ψ may contain both long and short roots, the simple root α_i is necessarily long (because $\tilde{\alpha}$ is), so that by Proposition 1.2.14, we have

$$\sum_{\alpha\in\Re^+} \langle\alpha,\, \omega_i\rangle\alpha = g\,\omega_i = \sum_{\alpha\in\Psi}\langle\alpha,\, \omega_i\rangle\alpha = \sum_{\alpha\in\Psi}\langle\alpha_i,\, \omega_i\rangle\alpha = \sum_{\alpha\in\Psi}\alpha = 2\rho_\Psi,$$

and therefore $\langle 2\rho_\Psi, \alpha_i\rangle = g\langle\alpha_i, \omega_i\rangle = g$. When $\Re$ is simply laced, all the weights of $\pi_{-\alpha_i}$ lie in the same orbit of

$$W' = \langle S_{\alpha_1}, \dots,\, S_{\alpha_{i-1}}, S_{\alpha_{i+1}}, \dots,\, S_{\alpha_r}\rangle,$$

Where W' is the Weyl group of K. Observing that $\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^*$, and Ψ is the set of positive complementary roots which are the weights of $\mathsf{ad}_{G/K}$, we see that W' preserves Ψ so for $w \in W'$, we have $w(2\rho_\Psi) = 2\rho_\Psi$, therefore

$$g = \langle 2\rho_\Psi,\, \alpha_i\rangle = \langle w(2\rho_\Psi),\, w(\alpha_i)\rangle = \langle 2\rho_\Psi,\, w(\alpha_i)\rangle,$$

and by applying all the elements of W' and summing, we have $\langle 2\rho_\Psi, 2\rho_\Psi \rangle = gn$, where $n = |\Psi|$ is the complex dimension of $M = G/K$. We now use the fact that the Killing form satisfies $(,) = \frac{1}{2g}\langle , \rangle$, and we have that the theorem holds in all simply laced cases. In the non-simply laced cases, i.e. B_r $(i = 1)$ and C_r $(i = r)$, while we still have that $\langle 2\rho_\Psi, \alpha_i \rangle = g$, we must consider the W' orbits in Ψ of a long (α_i) and a short root. In the case of B_r ,$\omega_i \in \Psi$ is the highest short root of $\Re$, and again

$$\langle 2\rho_\Psi, \omega_i \rangle = \langle g\omega_i, \omega_i \rangle = g.$$

We have therefore that $\langle 2\rho_\Psi, 2\rho_\Psi \rangle = \langle 2\rho_\Psi, 2\rho_\Psi^L + 2\rho_\Psi^S \rangle$ where $2\rho_\Psi^L$ (respectively $2\rho_\Psi^S$) is the sum of the weights in the W' orbit of long (respectively short) weights. From our observations above $\langle 2\rho_\Psi, 2\rho_\Psi^L + 2\rho_\Psi^S \rangle = gn^L + gn^S = gn$, where n^L (respectively n^S) denotes the number of long (respectively short) positive roots in Ψ. In the case of C_r however, $\omega_2 \in \Psi$ is the highest short root of $\Re$, and since $n_i^{\omega_2} = 1$, (where $\omega_2 = \sum_{i=1}^r n_i^{\omega_2}\alpha_i$) we have that $\langle 2\rho_\Psi, \omega_2 \rangle$ is also equal to g, and again $\langle 2\rho_\Psi, 2\rho_\Psi \rangle = gn$. □

Remark 2.1.1. The appearance of the scaling factor of $1/3$ is quite natural in this context (see Corollary 2.1.5). It is related to the fact that the scalar curvatures S on G, K and M satisfy $S_G = S_K + \frac{3}{4}S_M$. The same factor also appears in an expression for the topological invariant of section 5.1.

2.1.2 G/K Non-Hermitian symmetric

We next consider the non-Hermitian symmetric spaces, among which are the quaternionic analogues of the Hermitian ones (these will be used in Chapter 3). These are symmetric spaces where K is locally a product $H \times Sp(1)$, and as observed in [31] and [17], there is one for each compact, simple Lie group G (unlike the Hermitian symmetric spaces). We will use this fact in our proof of the 'strange formula' for G. For these spaces $\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{g}^{\tilde{\alpha}} \oplus \sum_{n_i^\alpha=0} \mathfrak{g}^\alpha$ and the isotropy representation $\sum_{n_i^\alpha=1} \mathfrak{g}^\alpha$ inherits a quaternionic structure, in the sense that it has real dimension $4m = 2n$ and H acts as a subgroup of $Sp(m)$ (see [31], [17]). G/K has therefore holonomy contained in $Sp(m)Sp(1)$ and is a quaternionic-Kähler manifold. These spaces are called *Wolf spaces*, and our results simplify for these. Before stating our 'strange formula' for the remaining symmetric spaces, we define some quantities of interest. It follows from the observation after Proposition 1.2.14 that $\sum_{\{n_i^\alpha=2\}} \alpha$ is a multiple of ω_i, which we will denote by s_2 (it can be calculated by using the tables in Appendix B), i.e. $\sum_{\{n_i^\alpha=2\}} \alpha = s_2\omega_i$. In [4] the quantity $m(G/K)$ is defined to be the number of roots $\alpha \in \Psi,\ \alpha \neq \alpha_i$ for which $\alpha - \alpha_i$ is a root, which in the Hermitian symmetric cases is the same as the number of roots $\alpha \in \Psi,\ \alpha \neq \alpha_i$ for which m_i^α is positive, where $\alpha = \sum_{j=1}^r m_j^\alpha \omega_j$. We will take the latter as the definition of $m(G/K)$ in what follows and we will prove that $s_2 = g - m(G/K) - 2$, when α_i is long and $s_2 = h - m(G/K) - 2$, when α_i is short, so that the next theorem may also be viewed as a generalisation of Proposition 2.4 in [4].

Theorem 2.1.2. *Let $M = G/K$ be a non-Hermitian symmetric space (corresponding to the deletion of a node α_i from the extended Dynkin diagram), then*

$$24(\rho_\Psi, \rho_\Psi)_M = (1 - \frac{2s_2}{k_i g}) \dim M,$$

where $(,)_M = \frac{1}{3}(,)$, and $k_i = \frac{\langle\tilde{\alpha},\tilde{\alpha}\rangle}{\langle\alpha_i,\alpha_i\rangle}$.

Proof. Here we are dealing with case (ii) of Theorem 2.0.21, where $n_i^{\tilde{\alpha}} = 2$ and $\Psi = \{\alpha \in \Re^+ \,|\, n_i^\alpha = 1\}$, as the roots α with $n_i^\alpha = 2$ are roots of K. We will give a slightly roundabout proof (rather than using Proposition 1.2.14) in the simply laced case in order to relate s_2 to $m(G/K)$ and other quantities. For any root system we let $R^+(i)$ denote the sum of the positive roots of G whose i^{th}-coordinate relative to the fundamental weights is positive so that by [9, **Proposition 2.3**], $R^+(i)$ has i^{th} coordinate equal to g if $\alpha_i \in \Re_L^+$, and equal to h if $\alpha_i \in \Re_S^+$. We first consider the simply laced cases, and prove that $\langle 2\rho_\Psi, \alpha_i\rangle = g - 2s_2$. To do so, Suppose that $P_1 = \{\alpha \in \Psi : m_i^\alpha = -1\}$ and $P_2 = \{\alpha \in \Re^+ : n_i^\alpha = 2, m_i^\alpha = 1\}$. Let $\alpha \in P_1$, for such α, $\langle\alpha_i, \alpha\rangle$ is negative and therefore $\alpha + \alpha_i$ is a root with $n_i^\alpha = 2$, and $m_i^\alpha = 1$, so $\alpha + \alpha_i \in P_2$. Conversely, suppose that $\alpha \in P_2$, as $m_i^\alpha = 1 > 0$, so we can reflect α with s_{α_i} and get a root $\alpha - \alpha_i$ in which $n_i^{\alpha-\alpha_i} = 1$, so $\alpha - \alpha_i \in \Psi$. Also since $m_i^{\alpha_i} = 2$, then $m_i^{\alpha-\alpha_i} = -1$ which means $\alpha - \alpha_i \in P_1$. Therefore, P_1 and P_2 have the same number of elements. We note that $|P_2| = s_2$, because any $\alpha \in \{\beta \in \Re^+ : n_i^\beta = 2\}$ must have $m_i^\alpha \in \{0, 1\}$ (in the simply laced cases), since otherwise $\alpha + \alpha_i$ would be a root with $n_i^{\alpha+\alpha_i} > 2$. Similarly, for any positive root with $n_i^\alpha = 0$, m_i^α is necessarily non-positive, as otherwise $\alpha - \alpha_i$ would be a root with coefficients of both signs when expressed with respect to the simple roots, so that $g = s_2 + m(G/K) + 2$. Summing the roots in Ψ and using the observation following Proposition 1.2.14, we have $\sum_{\alpha \in \Psi} \alpha = c\,\omega_i$ and since the ω_i coefficient of $R^+(i) = g$, we have that $c = g - 2s_2$, and $\langle 2\rho_\Psi,\, \alpha_i \rangle = g - 2s_2$. As all complementary weights lie in the same W' orbit in the simply laced cases we have

$$\langle 2\rho_\Psi, 2\rho_\Psi \rangle = (g - 2s_2)\, dim\, M.$$

We now consider the non-simply laced cases where $\alpha_i \in \Re_L^+$. By Proposition 1.2.14,

$$\sum_{\alpha \in \Re^+} \langle \alpha,\, \omega_i \rangle \alpha = 2\rho_\Psi + 2s_2\omega_i = g\,\omega_i,$$

so that $\langle 2\rho_\Psi,\, \alpha_i \rangle = g - 2s_2$. Taking the W'- orbit of α_i we have that $\langle 2\rho_\Psi, 2\rho_\Psi^L \rangle = (g - 2s_2) n^L$ where n^L and $2\rho_\Psi^L$ denote the number of long positive roots in Ψ and their sum respectively. Let β denote the highest short root. If $\tilde{\alpha} = \omega_{p_0}$ (the case of C_r is similar) and α_j is the closest short node to α_{p_0} in the Dynkin diagram, then $\alpha := \tilde{\alpha} - \alpha_{p_0} - \cdots - \alpha_{j-1}$ is a positive root where $\{\alpha_{p_0}, \cdots, \alpha_{j-1}, \alpha_j\}$ is the set of nodes on the shortest path in the Dynkin diagram from α_{p_0} to α_j, and $\beta = \alpha - \alpha_j$. In addition, since $\alpha_{p_0}, \cdots, \alpha_{j-1}$ are all long, so too is α and $\alpha \in \Psi$. Since α is in the same W'- orbit as α_i we have $\langle 2\rho_\Psi,\, \alpha \rangle = \langle 2\rho_\Psi,\, \alpha_i \rangle = g - 2s_2$, and therefore, as $2\rho_\Psi$ is a multiple of ω_i,

$$\langle 2\rho_\Psi,\, \beta \rangle = \langle 2\rho_\Psi,\, \alpha - \alpha_j \rangle = \langle 2\rho_\Psi,\, \alpha \rangle = g - 2s_2.$$

Now taking the W'-orbit of β, as above we obtain $\langle 2\rho_\Psi, 2\rho_\Psi^S \rangle = (g - 2s_2) n^S$ where n^S and $2\rho_\Psi^S$ denote the number of short positive roots in Ψ and their sum respectively, so that $\langle 2\rho_\Psi, 2\rho_\Psi \rangle = \langle 2\rho_\Psi, 2\rho_\Psi^L + 2\rho_\Psi^S \rangle = (g - 2s_2)(n^L + n^S)$. Finally when $\alpha_i \in \Re_S^+$, we have

$$\sum_{\alpha \in \Re^+} \langle \alpha,\, \omega_i \rangle \alpha = \frac{1}{k_i}(2\rho_\Psi + 2s_2\omega_i) = g\,\omega_i,$$

so that $\langle 2\rho_\Psi, \alpha_i\rangle = g - \frac{2s_2}{k_i}$. If α_i is the closest short node to α_{p_0} in the Dynkin diagram, then Ψ contains only short roots, and we are done. Otherwise, as above we have that the long root $\alpha = \beta + \alpha_j \in \Psi$ and $\langle 2\rho_\Psi, \alpha\rangle = \langle 2\rho_\Psi, \beta\rangle = g - \frac{2s_2}{k_i}$. □

Wolf spaces

Corollary 2.1.3. *Let $M = G/K$ be a Wolf space (corresponding to the deletion of the node α_i, from the extended Dynkin diagram, where $\tilde{\alpha}$ is a multiple of ω_i) , then*

$$24(\rho_\Psi, \rho_\Psi)_M = (\frac{g-2}{g}) \dim M,$$

where $(,)_M = \frac{1}{3}(,)$.

Proof. Recall that $n_i^{\tilde{\alpha}} = 2$, so $\tilde{\alpha}$ is either ω_i or $2\omega_i$. When $\tilde{\alpha} = \omega_i$, then $\langle \tilde{\alpha}, \omega_i\rangle = \langle \tilde{\alpha}, \tilde{\alpha}\rangle = 2$. On the other side

$$\langle \tilde{\alpha}, \omega_i\rangle = \langle 2\alpha_i, \omega_i\rangle = 2\langle \alpha_i, \omega_i\rangle,$$

so $\langle \alpha_i, \omega_i\rangle = 1$, α_i is therefore long and $k_i = 1$. The next highest long root $\alpha = S_{\alpha_i}\tilde{\alpha}$ has $n_i^\alpha = 1$, as does the highest short root, so that $s_2 = 1$. As Wolf spaces are special types of non-Hermitian symmetric spaces so by Theorem 2.1.2, it follows that:

$$24(\rho_\Psi, \rho_\Psi)_M = (1 - \frac{2s_2}{k_i g}) \dim M = (1 - \frac{2}{g}) \dim M.$$

When $\tilde{\alpha} = 2\omega_i$, (for C_r) the argument is similar,

$$2 = \langle \tilde{\alpha}, \tilde{\alpha}\rangle = \langle \tilde{\alpha}, 2\omega_i\rangle = \langle 2\alpha_i, 2\omega_i\rangle,$$

so $\langle \alpha_i, \omega_i\rangle = 1/2$, therefore α_i is short, so $k_i = 2$. Also as $\tilde{\alpha}$ is the only root with $n_i^\alpha = 2$, therefore $\sum_{\{n_i^\alpha = 2\}} \alpha = \tilde{\alpha} = 2\omega_i$, so $s_2 = 2$. □

Remark 2.1.2. We should note that in those cases where two nodes (to be deleted) are the image of each other under an automorphism of the extended Dynkin diagram but not under an automorphism of the Dynkin diagram itself, the quantity $\langle 2\rho_\Psi, 2\rho_\Psi\rangle$ is not an invariant of their common G/K. For example in E_6, choosing $n_2^{\tilde{\alpha}} = 2 = n_3^{\tilde{\alpha}}$ in Theorem 2.1.2 (with simple roots labelled as in [7]), we obtain isomorphic maximal connected subgroups of maximal rank (with root systems $A_1 \oplus A_5$). However the corresponding values of s_2 are 1 and 3 respectively, but for $n_3^{\tilde{\alpha}} = 2 = n_5^{\tilde{\alpha}}$, we have $s_2 = 3$ in both cases. Nor indeed is the quantity $\langle 2\rho_\Psi, 2\rho\rangle = \langle 2\rho_\Psi, 2\rho_\Psi\rangle + \langle 2\rho_\Psi, 2\rho_K\rangle$. Again, this can be seen in E_6, by choosing $n_2^{\tilde{\alpha}} = 2 = n_3^{\tilde{\alpha}}$ in Theorem 2.1.2, where the corresponding values of $\langle 2\rho_\Psi, 2\rho\rangle$ are 220 and 180 respectively. However the quantity

$$\langle 2\rho_\Psi, 2\rho_\Psi\rangle + 2\langle 2\rho_\Psi, 2\rho_K\rangle = \langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle$$

is an invariant of the corresponding symmetric space (depending only on its dimension and the group G), as we will now prove. In the example of E_6, choosing $n_2^{\tilde{\alpha}} = 2$ it has the value $220 + 20 = 240$, and choosing $n_3^{\tilde{\alpha}} = 2$ it also has the value $240 = 180 + 60$. For this reason it may be more appropriate to consider the following corollary a 'strange formula' for symmetric spaces.

Corollary 2.1.4. *Let $M = G/K$ be a symmetric space (with K maximal of maximal rank), then*

$$\langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle = gn.$$

Proof. We recall from Theorem 2.1.1 that when M is Hermitian symmetric, $\Psi = \{\alpha \in \Re^+ \,|\, n_i^\alpha = 1\}$ and $2\rho_\Psi = g\omega_i$, also all the roots α with $n_i^\alpha = 0$ are roots of K. Now since $\langle 2\rho_\Psi, 2\rho_K\rangle = \langle g\omega_i, 2\rho_K\rangle = 0$, then

$$\langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle = \langle 2\rho_\Psi, 2\rho_\Psi\rangle + 2\langle 2\rho_\Psi, 2\rho_K\rangle = \langle 2\rho_\Psi, 2\rho_\Psi\rangle = gn.$$

In the non-Hermitian symmetric cases we first give the proof when M corresponds to the deletion of a node α_i from the extended Dynkin diagram with $\alpha_i \in \Re_L^+$. By Theorem 2.1.2 and the fact that $2\rho_\Psi = \sum_{\alpha\in\Psi} \alpha = (g - 2s_2)\,\omega_i$, we have

$$\langle 2\rho_\Psi, 2\rho_\Psi\rangle + 2\langle 2\rho_\Psi, 2\rho_K\rangle = (g - 2s_2)n + 4n_2(g - 2s_2),$$

where $n_2 = |\{\alpha \in \Re^+ : n_i^\alpha = 2\}|$. By Proposition 1.2.14, we have

$$g\,\langle \omega_i,\, \omega_i\rangle = \sum_{\alpha\in\Re^+} \langle \alpha,\, \omega_i\rangle^2 = n + 4n_2,$$

so that

$$\langle 2\rho_\Psi, 2\rho_\Psi\rangle + 2\langle 2\rho_\Psi, 2\rho_K\rangle = g(g-2s_2)\langle \omega_i,\, \omega_i\rangle.$$

As $n_i^\alpha = 1$ for $\alpha \in \Psi$, $\alpha_i \in \Re_L^+$ and $\sum_{\alpha\in\Psi} \alpha = (g-2s_2)\,\omega_i$, we have

$$\langle (g-2s_2)\,\omega_i, \omega_i\rangle = n.$$

When $\alpha_i \in \Re_S^+$, $2\rho_\Psi = (gk_i - 2s_2)\omega_i$, and $\langle 2\rho_\Psi, \omega_i\rangle = (gk_i - 2s_2)\langle \omega_i,\, \omega_i\rangle$. On the other hand $\langle 2\rho_\Psi, \omega_i\rangle = \dfrac{n}{k_i}$, so that $\langle \omega_i,\, \omega_i\rangle = \dfrac{n}{k_i(gk_i - 2s_2)}$.

From Theorem 2.1.2, we have $\langle 2\rho_\Psi, 2\rho_\Psi\rangle + 2\langle 2\rho_\Psi, 2\rho_K\rangle = \frac{1}{k_i}(gk_i - 2s_2)(n+4n_2)$, and by Proposition 1.2.14, $g\,\langle \omega_i,\, \omega_i\rangle = \sum_{\alpha\in\Re^+} \langle \alpha,\, \omega_i\rangle^2 = \frac{1}{k_i^2}(n+4n_2)$. We now have $\langle 2\rho_\Psi, 2\rho_\Psi\rangle + 2\langle 2\rho_\Psi, 2\rho_K\rangle = k_i g\langle \omega_i,\, \omega_i\rangle(gk_i - 2s_2) = gn$. □

2.1.3 The strange formula for G

Definition 2.1.1. Suppose that G is a semi-simple, compact Lie group and let $\mathfrak{g}$ be its Lie algebra. Also suppose that there is an $\mathsf{Ad}(G)$-invariant scalar product on $\mathfrak{g}$ which yields the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, then the restriction of the scalar product to $\mathfrak{m}$ induces a G-invariant Riemannian metric on G/K which is referred to as a *normal homogeneous Riemannian metric*

The normal homogeneous Riemannian metric on G/K induced by the negative of the Killing form of $\mathfrak{g}$ is called the *standard homogeneous Riemannian metric.*

Corollary 2.1.5. *(The 'strange formula' of Freudenthal and de-Vries)*

$$24(\rho, \rho) = dim\, G.$$

Proof. We apply Corollary 2.1.4 to a symmetric space quotient of G (the corresponding Wolf space for example), so that

$$24(\rho,\rho) = \frac{3}{g}\langle 2\rho, 2\rho\rangle = 3n + \frac{3}{g}\langle 2\rho_K, 2\rho_K\rangle = 3n + 24(\rho_K,\rho_K).$$

Since G/K is a symmetric space, we know from [30] that the normal homogeneous metric on M, induced by the negative of the Killing form, is Einstein (with Einstein constant $C = \frac{1}{2}$), i.e. has constant Ricci curvature $Ric = \frac{1}{4}(,) + \frac{1}{2}A$, where for tangent vectors X, Y to M, $A(X,Y) = -\mathrm{Tr}_{\mathfrak{k}} P_{\mathfrak{k}}(\mathsf{ad}X \circ \mathsf{ad}Y)$, and $P_{\mathfrak{k}}$ is the projection of $\mathfrak{g}$ onto $\mathfrak{k}$ (see [30]). For an orthonormal basis $\{Z_i\}$ of $\mathfrak{k}$ (with respect to the Killing form of $\mathfrak{g}$) we have the expression

$$A(X,Y) = -\sum_i([X,[Y,Z_i]],Z_i) = -\sum_i([Z_i,[Z_i,X]],Y).$$

However (still following [30]), taking the trace of the Ricci curvature and using the above expression for $A(X,Y)$ yields the alternative form for the Einstein constant

$$C = \frac{1}{4} + \frac{1}{2}\sum_j(\dim K_j)\,(1-q_j)/(\dim(G/K) = \frac{1}{2}$$

so that $n = \sum_j(\dim K_j)\,(1-q_j)$, where K_j are the simple factors of K and the Killing form on K_j is $(,)_{K_j} = q_j(,)_{|K_j}$. Therefore

$$\begin{aligned}
3n + 24(\rho_K,\rho_K) &= 2n + n + 24(\rho_K,\rho_K) \\
&= 2n + \sum_j \dim K_j - \sum_j q_j(\dim K_j) + \sum_j 24(\rho_{K_j},\rho_{K_j}) \\
&= 2n + \dim K - q_j\frac{1}{q_j}\sum_j 24(\rho_{K_j},\rho_{K_j}) + \sum_j 24(\rho_{K_j},\rho_{K_j})
\end{aligned}$$

(by induction), so that

$$24(\rho,\rho) = \dim G/K + \dim K = \dim G.$$

□

2.2 G/K Non-symmetric

We now consider the non-symmetric cases i.e. where $n_i^{\tilde{\alpha}} \in \{3,5\}$. Similar to the cases where $n_i^{\tilde{\alpha}} = 2$ we define s_j and n_j as follows: $\sum_{\{n_i^\alpha = j\}} \alpha = s_j \omega_i$, and $n_j = |\{\alpha \in \Re : n_i^\alpha = j\}|$.

Theorem 2.2.1. *Let $M = G/K$, where K is the connected centraliser of an element of prime order $p > 2$ in $\mathsf{Ad}(G)$, then:*

(i) For $p = 3$, we have $24(\rho_\Psi, \rho_\Psi)_M = (\frac{1}{3} - \frac{s_3}{k_i g})\frac{8}{3} dim\, M$.

(ii) For $p = 5$, we have $24(\rho_\Psi, \rho_\Psi)_M = (\frac{1}{5} - \frac{s_5}{k_i g})4 \dim\, M$.

Proof. We give the proof in both simply and non-simply laced cases, starting with the case $p = 3$. We first note that $\Psi = \{\alpha \in \Re^+ \,|\, n_i^\alpha \in \{1,2\}\}$, as the roots α with $n_i^\alpha = 3$ are roots of K. We recall that $\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^*$, where the weights Λ of $\pi_{-\alpha_i}$ consist of the negative roots $-\alpha$ with $n_i^\alpha = 1$ together with the positive roots α with $n_i^\alpha = 2$. These two subsets of roots are interchanged by the reflection $S_{-\tilde{\alpha}}$. The sum of the weights $\sum_{\lambda \in \Lambda} \lambda$ is therefore fixed by all simple reflections of K, so that

$$\langle \sum_{\lambda \in \Lambda} \lambda, \alpha_j \rangle = \langle S_{\alpha_j}(\sum_{\lambda \in \Lambda} \lambda), S_{\alpha_j} \alpha_j \rangle = \langle \sum_{\lambda \in \Lambda} \lambda, -\alpha_j \rangle = 0.$$

Similarly $\sum_{\lambda \in \Lambda} \lambda$ is orthogonal to $\tilde{\alpha}$ and is therefore the zero vector, so

$$0 = \sum_{\lambda \in \Lambda} \lambda = \sum_{\{n_i^\alpha = 2\}} \alpha - \sum_{\{n_i^\alpha = 1\}} \alpha = s_2 \omega_i - s_1 \omega_i,$$

therefore $s_1 = s_2$, or $2n_2 = n_1$. By Proposition 1.2.14

$$g\langle \omega_i, \omega_i \rangle = \sum_{\alpha \in \Re^+} \langle \alpha, \omega_i \rangle^2 = \sum_{n_i^\alpha = 1} \langle \alpha, \omega_i \rangle^2 + \sum_{n_i^\alpha = 2} \langle \alpha, \omega_i \rangle^2 + \sum_{n_i^\alpha = 3} \langle \alpha, \omega_i \rangle^2.$$

Since in simply laced cases all roots are long so $\langle \alpha_i, \omega_i \rangle = 1$, then $n_1 + 4n_2 + 9n_3 = g\langle \omega_i, \omega_i \rangle$. From $s_j \langle \omega_i, \omega_i \rangle = j n_j$, we have

$$s_1 \langle \omega_i, \omega_i \rangle + 2 s_2 \langle \omega_i, \omega_i \rangle + 3 s_3 \langle \omega_i, \omega_i \rangle = g \langle \omega_i, \omega_i \rangle,$$

and $s_1 = s_2 = \frac{g}{3} - s_3$. As $-\alpha_i$ is the highest weight of $\pi_{-\alpha_i}$ and

$$\langle 2\rho_\Psi, \alpha_i\rangle = \langle \sum_{\alpha\in\Psi} \alpha, \alpha_i\rangle = \langle \sum_{n_i^\alpha=1} \alpha + \sum_{n_i^\alpha=2} \alpha, \alpha_i\rangle = s_1 + s_2.$$

By applying elements of W' we have that $\langle 2\rho_\Psi, \sum_{\{n_i^\alpha=1\}} \alpha\rangle = n_1(s_1+s_2)$. Similarly $\langle 2\rho_\Psi, \sum_{\{n_i^\alpha=2\}} \alpha\rangle = 2n_2(s_1+s_2)$, so that

$$\langle 2\rho_\Psi, 2\rho_\Psi\rangle = (n_1 + 2n_2)(s_1+s_2) = (\frac{g}{3} - s_3)\frac{8}{3}\dim M.$$

We remark that the above conditions $s_1 = s_2$ and $n_1 + 4n_2 + 9n_3 = g$ are equivalent to the condition that the normal homogeneous metric on M, induced by the negative of the Killing form, is Einstein, namely that $\langle -\alpha_i, -\alpha_i + 2\rho_K\rangle = \langle -\lambda_i, -\lambda_i + 2\rho_K\rangle$, where the representation $(\pi_{-\alpha_i})^*$ has highest weight $-\lambda_i$, the negative of the lowest height positive root with $n_i^{\lambda_i} = 2$.

Similarly, in non-simply laced cases, the sum of the weights $\sum_{\lambda\in\Lambda}\lambda$ is fixed by all simple reflections of K, so again we have $s_1 = s_2$ or $n_1 = 2n_2$. By Proposition 1.2.14 and the fact that in non-simply laced cases $\langle \alpha_i, \omega_i\rangle = \dfrac{1}{k_i}$, we have

$$g\langle \omega_i, \omega_i\rangle = \sum_{\alpha\in\Re^+} \langle \alpha,\, \omega_i\rangle^2 = \frac{1}{k_i^2}n_1 + \frac{4}{k_i^2}n_2 + \frac{9}{k_i^2}n_3,$$

so that $s_1 + 2s_2 + 3s_3 = gk_i$ and $s_1 = s_2 = gk_i/3 - s_3$. By applying the elements of W', we again have

$$\langle 2\rho_\Psi, 2\rho_\Psi\rangle = \frac{1}{k_i}(n_1 + 2n_2)(s_1+s_2) = (\frac{g}{3} - \frac{s_3}{k_i})\frac{8}{3}\dim M.$$

For the case where $n_i^{\tilde{\alpha}} = 5$ we have $\Psi = \{\alpha \in \Re^+ \,|\, n_i^\alpha \in \{1,2,3,4\}\}$. By a similar argument to above we obtain that $s_1 = s_4$ and $s_2 = s_3$ and $n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 = g\langle \omega_i, \omega_i\rangle$, that yields the additional equation $s_1 + s_2 + s_3 + s_4 = 2(\frac{g}{5} - s_5)$ so that $\langle 2\rho_\Psi, 2\rho_\Psi\rangle = (g - 5s_5)\frac{4}{5}\dim M$. The proof for non-simply laced cases can be easily obtained by using the above argument. □

Corollary 2.2.2. *Let $M = G/K$ where K is the connected centraliser of an element of prime order $p > 2$ in $Ad(G)$, then:*

(i) For $p = 3$, we have $\langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle = \frac{8}{9}gn$.

(ii) For $p = 5$, we have $\langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle = \frac{4}{5}gn$.

Proof. As the proof is similar to that of Corollary 2.1.4, we only give the proof for the cases $p = 3$ and $\alpha_i \in \Re_L^+$. In the proof of Theorem 2.2.1, we saw that $2\rho_\Psi = (s_1 + s_2)\omega_i = 2(\frac{g}{3} - s_3)\omega_i$, so that $\langle 2\rho_\Psi, \omega_i\rangle = 2(\frac{g}{3} - s_3)\langle\omega_i,\, \omega_i\rangle$. On the other hand, as $\Psi = \{\alpha \in \Re^+ \,|\, n_i^\alpha \in \{1, 2\}\}$, we also have $\langle 2\rho_\Psi, \omega_i\rangle = n_1 + 2n_2 = \frac{4}{3}n$, so that $\langle\omega_i,\, \omega_i\rangle = \frac{\frac{4}{3}n}{2(\frac{g}{3} - s_3)}$.
By Theorem 2.2.1, $\langle 2\rho_\Psi, 2\rho_\Psi\rangle + 2\langle 2\rho_\Psi, 2\rho_K\rangle = (\frac{g}{3} - s_3)(\frac{8n}{3} + 12n_3) = (\frac{g}{3} - s_3)(8n_2 + 12n_3) = (\frac{g}{3} - s_3)(8n_2 + 12n_3) = (\frac{g}{3} - s_3)\frac{4}{3}(6n_2 + 9n_3) = (\frac{g}{3} - s_3)\frac{4}{3}g\langle\omega_i,\, \omega_i\rangle = \frac{8}{9}gn$. □

We summarise our formulae in the next proposition. To make our formulae more uniform we introduce the the quantity s defined as follows: For $n_i^{\tilde{\alpha}} = t$, let $S := \{\alpha \in \Re_K : n_i^\alpha = t\}$ and let $\sum_{\alpha \in S} \alpha = s\,\omega_i$, observing that $s = 0$, when $n_i^{\tilde{\alpha}} = 1$.

Theorem 2.2.3. *Let G be a compact centreless simple Lie group with K a maximal connected subgroup of G of maximal rank. Let $M = G/K$ and let $2\rho_\Psi$ denote the sum of the positive complementary roots, then:*

(i) For $n_i^{\tilde{\alpha}} = 1$, we have $\langle 2\rho_\Psi, 2\rho_\Psi\rangle = (g - \frac{s}{k_i})\dim M$.

(ii) For $n_i^{\tilde{\alpha}} = 2$, we have $\langle 2\rho_\Psi, 2\rho_\Psi\rangle = (g - \frac{2s}{k_i})\dim M$.

(iii) For $n_i^{\tilde{\alpha}} = 3$, we have $\langle 2\rho_\Psi, 2\rho_\Psi\rangle = (g - \frac{3s}{k_i})\frac{8}{9}\dim M$.

(iv) For $n_i^{\tilde{\alpha}} = 5$, we have $\langle 2\rho_\Psi, 2\rho_\Psi\rangle = (g - \frac{5s}{k_i})\frac{4}{5}\dim M$.

Chapter 3

The Wolf sequence and exponents

Some years ago R. Carles ([12]) considered the subroot system obtained by taking the orthogonal complement of the highest root in an irreducible (reduced crystallographic) root system. These are parabolic subroot systems (their Dynkin diagram is that of the original root system with one or two nodes deleted), and he related root theoretic data such as the cardinality and the sum 2ρ of the positive roots of this subsystem to those of the original root system (see also [20]). In this chapter and the next chapter we extend these results in two directions. Firstly, in this chapter, we relate additional root theoretic data of such a subsystem to that of its parent, such as the Coxeter number and the exponents and secondly, in the next chapter, we consider all maximal parabolic sub-root systems. For an irreducible (reduced, crystallographic) root system $\Re$ (of rank r) with highest root $\tilde{\alpha} = \sum_{i=1}^{r} n_i^{\tilde{\alpha}} \alpha_i$ (when expressed in terms of the simple roots $\alpha_1, \ldots, \alpha_r$), we denote by $\Re'$ the root system $\Re' = \{\alpha \in \Re : \langle \alpha,\ \tilde{\alpha} \rangle = 0\}$ (not necessarily irreducible), we also denote by $\tilde{\alpha}'$ the element of maximal height in $\Re'$.

If ζ is a primitive h-th root of unity, then the eigenvalues of a Coxeter element are of the form ζ^m, where $0 < m < h$. ([22]) The *exponents* are the various m involved, written as $m_1 \leq m_2 \leq \cdots \leq m_r$, and they satisfy the duality conditions $m_i + m_{r+1-i} = h$, and as $m_1 = 1$, we have that $m_r = h - 1$. We therefore focus attention on the second exponent m_2.

3.1 The second exponent

Definition 3.1.1. A path γ in the extended Dynkin diagram $\tilde{\triangle}_{\Re}$ of an irreducible crystallographic root system $\Re$ is a sequence of ordered nodes $\{\alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_s}\}$ where $\alpha_{i_{j-1}}$ is adjacent to α_{i_j}. We define the length $L(\gamma)$ of the path γ to be the number of nodes in the path and a node α_{i_k} is called a branch node when it is adjacent to more than two nodes.

Let $d = Max\{n_i^{\tilde{\alpha}} : \ 1 \leq i \leq r\}$ and let $d^\vee$ be similarly defined for the coroot $\tilde{\alpha}^\vee$, so that $d^\vee$ is the highest coefficient of the highest short root of the dual root system.

Proposition 3.1.1. *Let γ be the path in the extended Dynkin diagram $\tilde{\triangle}_{\Re}$ of an irreducible crystallographic root system $\Re$ starting at the node $\alpha_0 = -\tilde{\alpha}$ and ending at the first branch node or first node with multiple connections. Let m denote the maximum number of edges in a connection to any adjacent node of the first branch node or first node with multiple connections, then:*

(i) Then $L(\gamma) = d^\vee$ and

(ii) $m + d^\vee$ is an exponent of $\Re$. In particular for a simply laced root system, the second smallest exponent $m_2 = 1 + d$.

Proof. In the case of a crystallographic Coxeter group, we have the following standard description ([13]) of the exponents. For $\alpha \in \Re^+$, let t_k be the number of positive roots of height k. Then $t_k - t_{k+1}$ is the number of times k occurs as an exponent m_k. We first consider the simply laced cases and observe that if $\alpha = \sum_{k \in P} a_k \omega_k + \sum_{l \in Q} b_l \omega_l$ (with all a_k positive and all b_l negative) is the only root of height $ht(\alpha)$ then the number of positive roots of height $ht(\alpha) - 1 = |P|$, since all roots are contained in the same Weyl group orbit. For $\Re$ of type A_r, $\tilde{\triangle}_{\Re}$ has no branch node and therefore $L(\gamma) = 1 = d$. Also since $\tilde{\alpha} = \omega_1 + \omega_r$, we have that $t_{h-1} = 1$ and $t_{h-2} = |P| = 2$. Therefore, $h-2 = h-(d+1)$ is an exponent and by duality so is $d+1$. We assume therefore that $d \geq 2$ and recall that $\alpha_i = \sum_{k=1}^{r} c_{ki}\omega_k$ where c_{ki} are the Cartan integers. In the simply laced cases $c_{ji} = c_{ij} = -1$ for adjacent vertices i and j, and are zero otherwise. Therefore we have that $\alpha_i = 2\omega_i - \sum_{k \in N(\alpha_i)} \omega_k$, where $k \in N(\alpha_i)$ if and only if α_k is adjacent to α_i in the Dynkin diagram . We now begin a list of positive roots of decreasing height in correspondence with the nodes of the path $\gamma = \{\alpha_0, \alpha_{p_0}, \alpha_{p_1}, \ldots, \alpha_{p_b}\}$ (where α_{p_b} is the branch node) as follows: At height $h-1$ we have the highest root $\tilde{\alpha} = -\alpha_0 = \omega_{p_0}$ and $t_{h-1} = 1$ (see figure 3.1). Applying the simple reflection S_{p_0} we obtain the next highest root:

$$-\alpha_0 - \alpha_{p_0} = \omega_{p_0} - 2\omega_{p_0} + \sum_{k \in N(\alpha_{p_0})} \omega_k = -\omega_{p_0} + \omega_{p_1}$$

and $t_{h-2} = 1$. Continuing our list (reflecting with S_{p_1}) at height $h-3$ we obtain the root $-\alpha_0 - \alpha_{p_0} - \alpha_{p_1} = -\omega_{p_0} + \omega_{p_1} - 2\omega_{p_1} + \sum_{k \in N(\alpha_{p_1})} \omega_k = -\omega_{p_0} - \omega_{p_1} + \omega_{p_0} + \omega_{p_2} = -\omega_{p_1} + \omega_{p_2}$ and $t_{h-3} = 1$. The positive roots corresponding to the last two nodes of γ are:

$$-\alpha_0 - \alpha_{p_0} - \alpha_{p_1} \cdots - \alpha_{p_{b-1}} = -\omega_{p_{b-1}} + \omega_{p_b},$$

and

$$\beta = -\alpha_0 - \alpha_{p_0} - \alpha_{p_1} \cdots - \alpha_{p_b} = -\omega_{p_b} + \omega_{p_{b+1}} + \omega_{p_{b+2}}.$$

In the special case of D_4 we have $\beta = -\omega_2 + \omega_1 + \omega_3 + \omega_4$. This last root β is of height $h-(b+2)$. Therefore at heights $h-(b+2)$ and $h-(b+3)$ we have $t_{h-(b+2)} = 1$ and $t_{h-(b+3)} = 2$ (or 3 for D_4) so that $h-(b+3)$ and by duality of exponents $b+3$ is also an exponent (of multiplicity two for D_4). As $L(\gamma) = b+2$, we now prove that $d = b+2$ to finish the proof in the simply laced cases. By an observation of N. Iwahori, which appears in [9], the node coefficients of any simple path in $\tilde{\Delta}_{\Re}$, starting at a pendant node and ending at the first node with a branch or multiple connection, form an arithmetic progression with common difference equal to the coefficient of the terminal node. Since the coefficient of the α_0 node is one and by the observation of N. Iwahori, that of the branch node is d, we have a bijection between the sets $\{\alpha_0, \alpha_{p_0}, \alpha_{p_1}, \ldots, \alpha_{p_b}\}$ and $\{1, 2, \ldots, d\}$.

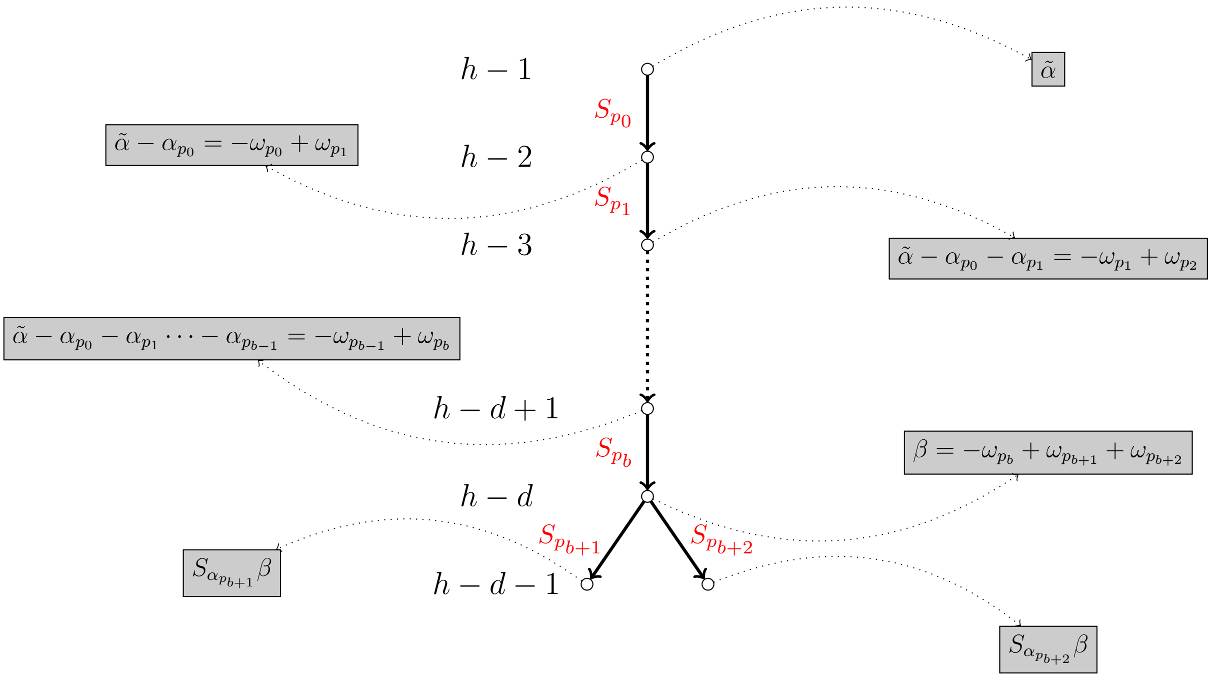

Figure 3.1: The second smallest exponent in simply laced cases

Next, we consider the non-simply laced cases. The case of B_r is in fact covered by the above simply laced argument as the path γ in the extended Dynkin diagram will first encounter a branch node (at α_2) rather than one with a multiple connection, so that $m=1$. In the other cases, the extended Dynkin diagram has no branch node and $\tilde{\alpha}=-\alpha_0=k\omega_{p_0}$, where $k=1$, except for the root system C_r where $k=2$. Again we consider the path $\gamma=\{\alpha_0,\alpha_{p_0},\alpha_{p_1},\ldots,\alpha_{p_{b-1}}\}$, where the node $\alpha_{p_{b-1}}$ has a multiple connection (with m edges) to α_{p_b}, (in the C_r case $\gamma=\{\alpha_0=\alpha_{p_{b-1}}\}$ and $\alpha_{p_0}=\alpha_{p_b}$). The sequence of positive roots corresponding to γ is:

$$\{\tilde{\alpha}=-\alpha_0,-\alpha_0-\alpha_{p_0},\ldots,-\alpha_0-\alpha_{p_0}-\alpha_{p_1}-\cdots-\alpha_{p_{b-1}}\}$$

and the roots have heights $h-1,h-2,\ldots,h-(b+1)$. They are also the only roots at these heights. Applying S_{p_b} (see figure 3.2) to

$$-\alpha_0-\alpha_{p_0}-\alpha_{p_1}-\cdots-\alpha_{p_{b-1}}=-\omega_{p_{b-1}}-c_{p_bp_{b-1}}\omega_{p_b}=-\omega_{p_{b-1}}+m\omega_{p_b},$$

we obtain the root $-\alpha_0-\alpha_{p_0}-\alpha_{p_1}-\cdots-m\alpha_{p_b}$ of height $h-m-(b+1)$. Since the α_{p_b} string through $-\alpha_0-\alpha_{p_0}-\alpha_{p_1}-\cdots-\alpha_{p_{b-1}}$ is unbroken, the highest short root has height $h-(b+2)$ (and is the unique root of this height). In the case where

$m = 2$, we, therefore, have that $h - m - (b+1)$ is the second highest exponent. When $m = 3$ (i.e. the case of G_2), that $h - m - (b+1) = 1$ is the highest height at which there are two roots is a simple count (of the six positive roots, two of which are simple). Using the proof of Iwahori's observation in [9] we have that d is the coefficient of α_{p_b} in the expression of $\tilde{\alpha}$ w.r.t. the simple roots, so that we have a bijection between the sets $\{\alpha_0, \alpha_{p_0}, \alpha_{p_1}, \ldots, \alpha_{p_b}\}$ and $\{1, 2, \ldots, d\}$. Therefore by the above, since $d^\vee$ is the highest coefficient of the highest short root of the dual root system, we have that $d^\vee = d - 1$ for the non-simply laced cases (except for B_r where $d^\vee = d$) and the result follows.

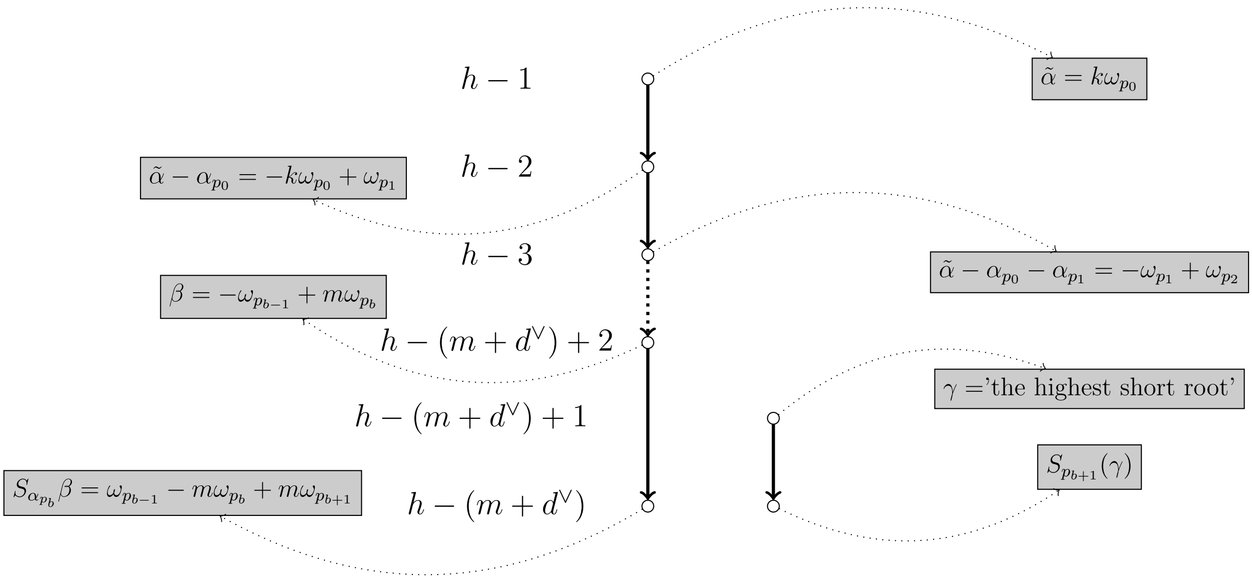

Figure 3.2: The second smallest exponent in non-simply laced cases

□

Theorem 3.1.2. *Let the highest root $\tilde{\alpha}$ of an irreducible (reduced, crystallographic) root system $\Re$ be expressed as $\tilde{\alpha} = \sum_{i=1}^{r} n_i^{\tilde{\alpha}} \alpha_i$, and let $d = Max\{n_i^{\tilde{\alpha}} : 1 \leq i \leq r\}$ then:*

$$ht(\tilde{\alpha}') = h - 2d^\vee - 1.$$

Proof. We first consider the simply laced cases. For $\Re$ of type A_r we have $d = 1$ and $h = r + 1$. Since $\tilde{\alpha} = \omega_1 + \omega_r$ we have that $\Re'$ is of type A_{r-2} and therefore $ht(\tilde{\alpha}') = r - 2 = r + 1 - 2 - 1 = h - 2d - 1$. Similarly we verify the case of D_4. For other type root systems $\tilde{\alpha} = -\alpha_0 = \omega_{p_0}$ and we extend the list of roots obtained in

Proposition 3.1.1, from the path $\gamma = \{\alpha_0, \alpha_{p_0}, \alpha_{p_1}, \ldots, \alpha_{p_b}\}$ to include $\tilde{\alpha}'$ as follows (see figure 3.3); The final root β (of height $h-d$) in the list was:

$$\beta = -\alpha_0 - \alpha_{p_0} - \alpha_{p_1} \cdots - \alpha_{p_b} = -\omega_{p_b} + \omega_{p_{b+1}} + \omega_{p_{b+2}}.$$

We append to the list the roots $S_{\alpha_{p_{b+1}}}\beta$ and $S_{\alpha_{p_{b+2}}}S_{\alpha_{p_{b+1}}}\beta = S_{\alpha_{p_{b+2}}}S_{\alpha_{p_{b+1}}}\beta$ and observe that:

$$S_{\alpha_{p_{b+2}}}S_{\alpha_{p_{b+1}}}\beta = \omega_{p_b} - \omega_{p_{b+1}} - \omega_{p_{b+2}} + \sum_{k \in N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})} \omega_k = -\beta + \sum_{k \in N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})} \omega_k$$

where $N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})$ is the set of simple roots adjacent to $\alpha_{p_{b+1}}$ or $\alpha_{p_{b+2}}$ but not in the path $\gamma = \{\alpha_0, \alpha_{p_0}, \alpha_{p_1}, \ldots, \alpha_{p_b}\}$, i.e.

$$N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}}) = N(\alpha_{p_{b+1}}, \alpha_{p_{b+2}}) \setminus \{\alpha_{p_b}\}.$$

We remark for future reference that $N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})$ has cardinality at most two. Our list of roots is now in correspondence with the nodes of the path $\gamma \cup \{\alpha_{p_{b+1}}, \alpha_{p_{b+2}}\}$. Since $S_{\alpha_{p_{b+2}}}S_{\alpha_{p_{b+1}}}\beta$ has ω_{p_b} coefficient equal to $+1$ when expressed w.r.t. the fundamental weights, we can extend the list of decreasing height roots by reflecting with the simple reflections corresponding to the nodes of the path $-(\gamma \setminus \{\alpha_0\})$ (the minus sign indicating that the path is traversed in the opposite direction), terminating with the root $-\omega_{p_0} + \sum_{k \in N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})} \omega_k$.

We now show that this terminal root is in fact $\tilde{\alpha}'$. Since $\tilde{\alpha} = -\alpha_0 = \omega_{p_0}$ and $<\tilde{\alpha}, \tilde{\alpha}> = 2 = <\tilde{\alpha}, \omega_{p_0}>$ the coefficient $n_{p_0}^{\tilde{\alpha}}$ of α_{p_0} in the expression of the highest root w.r.t. the simple roots is equal to two. In forming the list of decreasing height roots starting with $\tilde{\alpha}$ we have applied the simple reflection $S_{\alpha_{p_0}}$ twice (once in the path γ and once in the path $-(\gamma \setminus \{\alpha_0\})$) so that the terminal root

$$\delta = \sum_{k \in N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})} \omega_k - \omega_{p_0}$$

has α_{p_0} coefficient equal to zero when expressed w.r.t. the simple roots. It is therefore orthogonal to $\tilde{\alpha} = \omega_{p_0}$ and is by definition a root of

$$\Re' = \{\alpha \in \Re : \langle \alpha,\ \tilde{\alpha} \rangle = 0\}.$$

If this root δ were not the highest root $\tilde{\alpha}'$ of $\Re'$ there would be a sequence of simple reflections $\{S_{\alpha_{k_1}}, \ldots, S_{\alpha_{k_s}}\}$ with $k_i \neq p_0$ and roots $\{S_{\alpha_{k_1}}\tilde{\alpha}', \ldots, S_{\alpha_{k_s}} \ldots S_{\alpha_{k_1}}\tilde{\alpha}'\}$ of decreasing height, from $\tilde{\alpha}'$ down to δ as all roots lie in the same Weyl group orbit. However in the expression for δ w.r.t the fundamental weights the only negative coefficient is that of ω_{p_0}. This contradiction means that the root $\delta = \tilde{\alpha}'$. It is now clear that $N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})$ has cardinality at most two and has cardinality equal to two only when $\Re'$ is of type A. Finally

$$ht(\tilde{\alpha}') = ht(\beta) - 2 - (b+1) = ht(\beta) - 2 - (d-1) = h - 2d - 1.$$

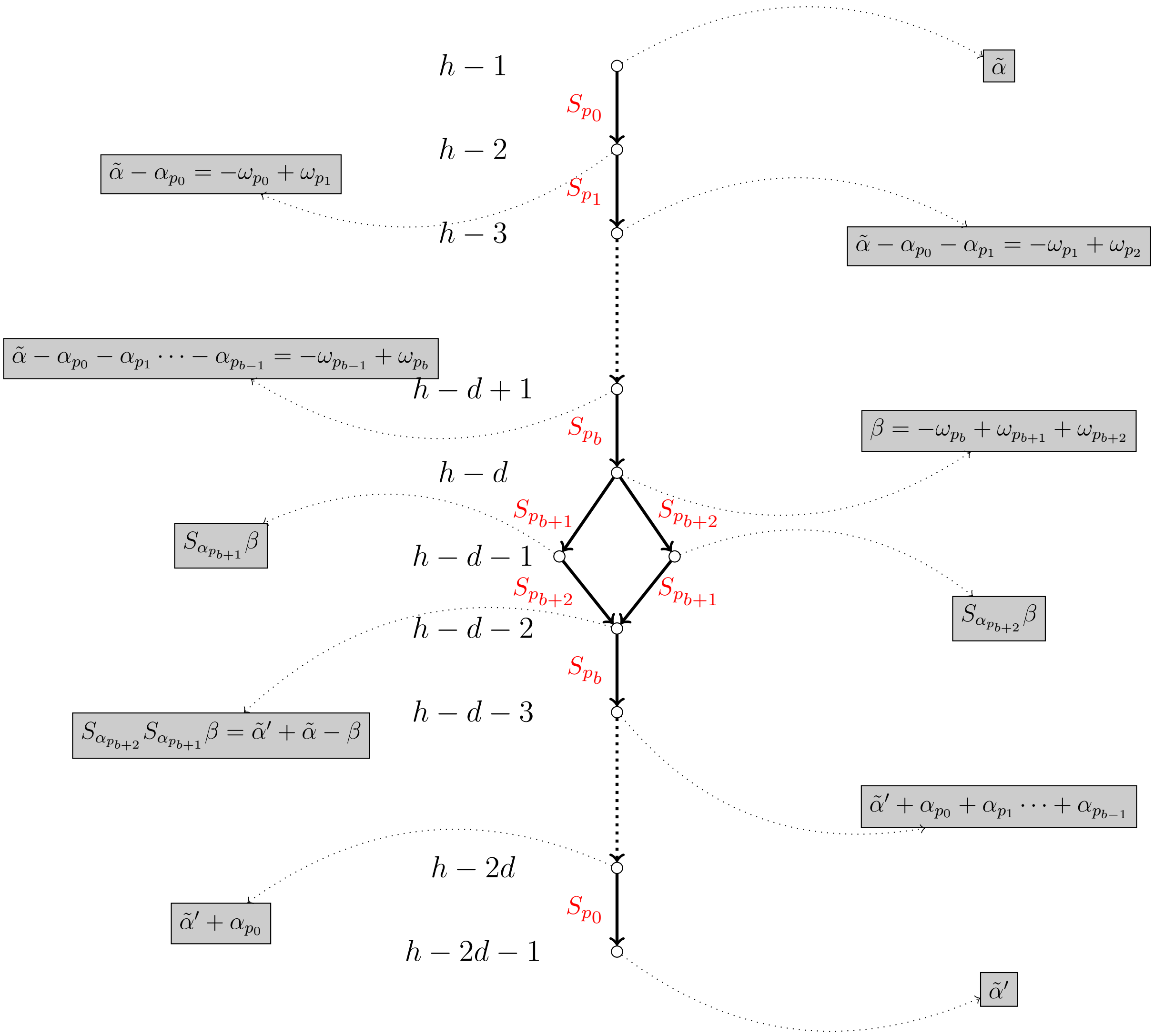

Figure 3.3: $\Re'$ in simply laced cases

In the non-simply laced cases (see figure 3.4), we similarly extend the corresponding list of roots obtained in Proposition 3.1.1 (from the path $\gamma=\{\alpha_0,\alpha_{p_0},\alpha_{p_1},\ldots,\alpha_{p_{b-1}}\}$) to include $\tilde{\alpha}'$. The final root β in the list was $\beta=-\alpha_0-\alpha_{p_0}-\alpha_{p_1}\cdots-\alpha_{p_{b-1}}$ to which we append the root:

$$S_{p_b}\beta=-\alpha_0-\alpha_{p_0}-\alpha_{p_1}-\cdots-m\alpha_{p_b}=\omega_{p_{b-1}}-m\omega_{p_b}+m\omega_{p_{b+1}}$$

of height $h-m-(b+1)$ obtained above. Since this root has $\omega_{p_{b-1}}$ coefficient equal to $+1$ when expressed w.r.t. the fundamental weights, we can extend the list of decreasing height roots by reflecting with the simple reflections corresponding to the nodes of the path $-(\gamma\setminus\{\alpha_0\})$ (the minus sign indicating that the path is traversed in the opposite direction). Since the simple reflection $S_{\alpha_{p_0}}$ was applied twice (once in the path γ and once in the path $-(\gamma \setminus \{\alpha_0\})$, except of course for C_r where $S_{\alpha_{p_0}}\tilde{\alpha} = \tilde{\alpha}'$) has α_{p_0} coefficient equal to zero when expressed w.r.t. the simple roots. It is therefore a root of $\Re'$ and by the same argument used in the simply laced case, is the highest long, and therefore highest root $\tilde{\alpha}'$ of $\Re'$.

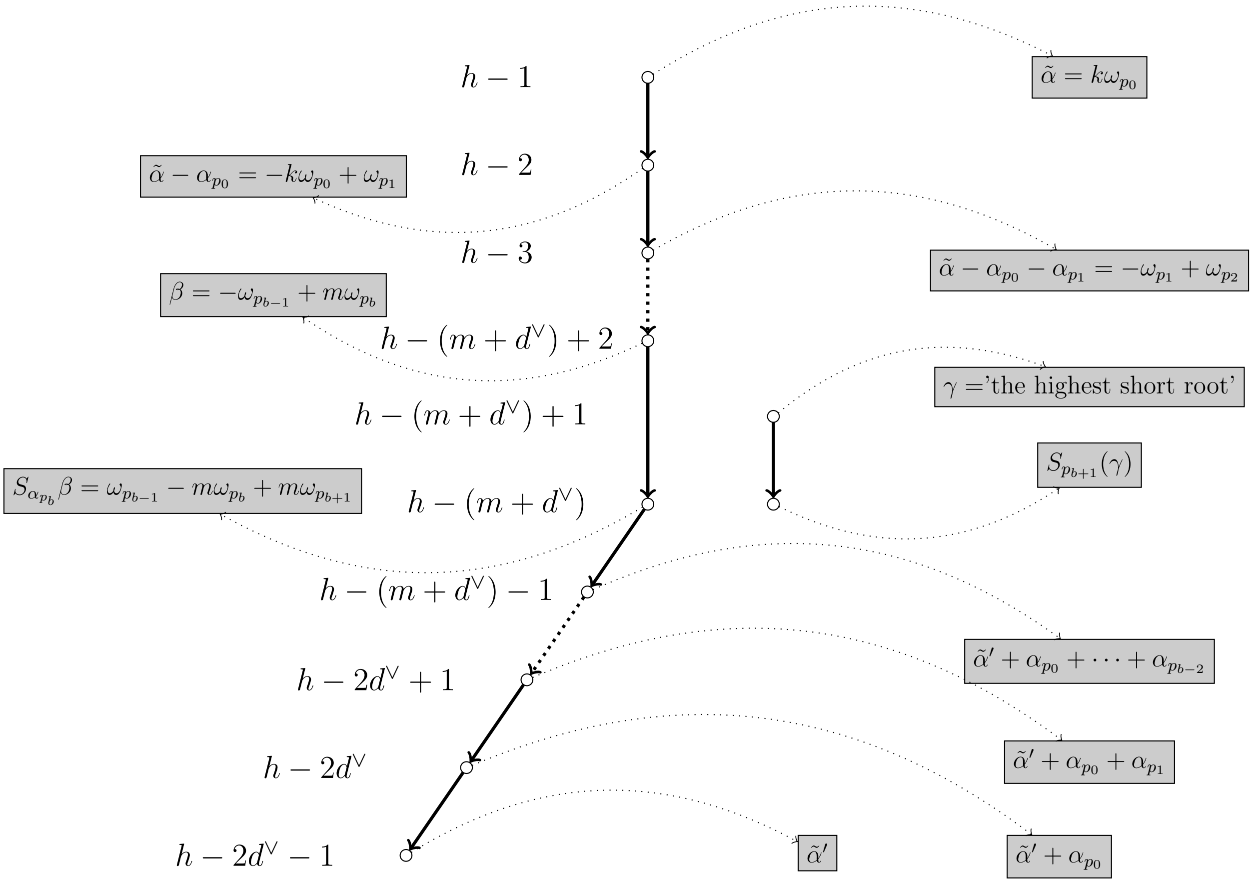

Figure 3.4: $\Re'$ in non-simply laced cases

□

Corollary 3.1.3. *(i) $h' = h - 2d^\vee$, if $\Re'$ is irreducible and $h' = h - 2d^\vee + 2$ otherwise.*

(ii) For $d \geq 3$ (the exceptional systems) $d' = d-2$. and $d-1$ is an exponent of $\Re'$.

(iii) For $r > 5$ and $d \leq 2$ (the classical systems) $d' = d$.

Proof. (i) is an immediate consequence of Theorem 3.1.2 when $\Re'$ is irreducible. In all other cases $\Re'$ has two irreducible components, one of which is a root system of type A_1, for which the Coxeter number is 2. We first consider the simply laced case. Let $d \geq 3$ and let t'_k and t''_k denote the number of positive roots of height k in $\Re'$ and $\Re \setminus \Re'$ respectively. By Proposition 3.1.1, we know that $d-1$ is not an exponent of $\Re$ and therefore

$$t_{d-1} = t'_{d-1} + t''_{d-1} = t_d = t'_d + t''_d.$$

In addition, since $h-(1+d)$ is the second largest exponent we have that $t_{h-1} = 1 = \cdots = t_{h-d} = 1$ and $t_{h-1-d} = 2$. However by Theorem 3.1.2,

$$t_{h-1} = t''_{h-1} = \cdots = t_{h-d} = t''_{h-d} = 1$$

and $t_{h-1-d} = t''_{h-1-d} = 2$ since $ht(\tilde{\alpha}') = h-2d-1$. We now observe that for $\alpha \in \Re\setminus\Re'$ we have $-S_{\tilde{\alpha}}(\alpha) = \tilde{\alpha} - \alpha \in \Re \setminus \Re'$ and therefore $t''_1 = t''_{h-2}, \ldots$ and $t''_d = t''_{h-1-d}$. Since $t''_{h-1-d} = 2$ and $t''_{h-d} = 1$ we have that $t''_d = 2$ and $t''_{d-1} = 1$. However, as $t'_{d-1} + t''_{d-1} = t'_d + t''_d$, we have $t'_{d-1} + 1 = t'_d + 2$ so that $d-1$ is an exponent of $\Re'$ and $1 + d' = d - 1$. □

3.2 Diamonds

Definition 3.2.1. For a simply laced irreducible (reduced) root system $\Re$ the sequence of roots

$$\{\tilde{\alpha}, \ldots, \beta, S_{\alpha_{p_{b+1}}}\beta, S_{\alpha_{p_{b+2}}}\beta, S_{\alpha_{p_{b+1}}}S_{\alpha_{p_{b+2}}}\beta, \ldots, \tilde{\alpha}'\}$$

as described in the proof of Theorem 3.1.2 will be called the $\tilde{\alpha}'$ string through $\tilde{\alpha}$ with diamond $\{\beta, S_{\alpha_{p_{b+1}}}\beta, S_{\alpha_{p_{b+2}}}\beta, S_{\alpha_{p_{b+1}}}S_{\alpha_{p_{b+2}}}\beta\}$. For $\Re$ of type A the $\tilde{\alpha}'$ string through $\tilde{\alpha}$ consists only of the diamond.

Following Carles [12] we now define inductively a sequence of embedded root systems which we call *the Wolf sequence* because in 2.1.2, the same sequence of operations performed in the extended Dynkin diagram yields a sequence of Wolf spaces. Let $\Re^{(0)} := \Re$, $\tilde{\alpha}^{(0)} := \tilde{\alpha}$ and let $d^{(0)} = d$. Let $\Re^{(1)} := \Re'$, $\tilde{\alpha}^{(1)} := \tilde{\alpha}'$ and let $d^{(1)} = d'$

as previously defined. In the cases where $\Re^{(1)}$ is irreducible (A_r and all exceptional root systems) let $\Re^{(2)} := \{\alpha \in \Re^{(1)} : \langle \alpha,\ \tilde{\alpha}^{(1)} \rangle = 0\}$ and let $\tilde{\alpha}^{(2)}$ denote its highest root, with highest coefficient denoted by $d^{(2)}$. In those cases where $\Re^{(1)}$ is reducible (B_r, D_r) it is a sum of an A_1 with an irreducible root system (of the same type as $\Re$ but of lower rank) and we define $\Re^{(2)}$ in this irreducible component of $\Re^{(1)}$ as above. Continuing this process inductively until we obtain a root system that is of type A_2 or a sum of root systems of type A_1 we obtain the following sequences.

Embedded Root Systems

A_n **even**

$$\Re^{(0)} = A_n \longrightarrow \Re^{(1)} = A_{n-2} \longrightarrow \Re^{(2)} = A_{n-4} \longrightarrow \cdots \longrightarrow A_2$$

A_n **odd**

$$\Re^{(0)} = A_n \longrightarrow \Re^{(1)} = A_{n-2} \longrightarrow \Re^{(2)} = A_{n-4} \longrightarrow \cdots \longrightarrow A_1$$

B_n **even**

$$\Re^{(0)} = B_n \longrightarrow \Re^{(1)} = A_1 + B_{n-2} \longrightarrow \Re^{(2)} = A_1 + B_{n-4} \longrightarrow \cdots \longrightarrow A_1$$

B_n **odd**

$$\Re^{(0)} = B_n \longrightarrow \Re^{(1)} = A_1 + B_{n-2} \longrightarrow \Re^{(2)} = A_1 + B_{n-4} \longrightarrow \cdots \longrightarrow A_1 + A_1$$

C_n

$$\Re^{(0)} = C_n \longrightarrow \Re^{(1)} = C_{n-1} \longrightarrow \cdots \longrightarrow A_1$$

D_n **even**

$$\Re^{(0)} = D_n \longrightarrow \Re^{(1)} = A_1 + D_{n-2} \longrightarrow \Re^{(2)} = A_1 + D_{n-4} \longrightarrow \cdots \longrightarrow A_1 + A_1 + A_1$$

D_n **odd**

$$\Re^{(0)} = D_n \longrightarrow \Re^{(1)} = A_1 + D_{n-2} \longrightarrow \Re^{(2)} = A_1 + D_{n-4} \longrightarrow \cdots \longrightarrow A_1$$

E_6

$$\Re^{(0)} = E_6 \longrightarrow \Re^{(1)} = A_5 \longrightarrow \Re^{(2)} = A_3 \longrightarrow A_1$$

E_7

$$\Re^{(0)} = E_7 \longrightarrow \Re^{(1)} = D_6 \longrightarrow \Re^{(2)} = A_1 + D_4 \longrightarrow \Re^{(3)} = A_1 + A_1 + A_1$$

E_8

$\Re^{(0)} = E_8 \longrightarrow \Re^{(1)} = E_7 \longrightarrow \Re^{(2)} = D_6 \longrightarrow \Re^{(3)} = A_1 + D_4 \longrightarrow \Re^{(4)} = A_1 + A_1 + A_1$

F_4

$\Re^{(0)} = F_4 \longrightarrow \Re^{(1)} = C_3 \longrightarrow \Re^{(2)} = C_2 \longrightarrow \Re^{(3)} = A_1$

G_2

$\Re^{(0)} = G_2 \longrightarrow \Re^{(1)} = A_1$

Theorem 3.2.1. *The exponents $m_1 \leq m_2 \leq \cdots \leq m_r$ of a simply laced root system $\Re$ are given as follows:*

$$m_1 = 1, m_2 = 1 + d^{(0)}, \ldots, m_{\lfloor (r+1)/2 \rfloor} = 1 + d^{(0)} + d^{(1)} + \cdots + d^{(\lfloor (r-3)/2 \rfloor)}$$

$$m_r = h-1, m_{r-1} = h-1-d^{(0)}, \ldots, m_{r+1-\lfloor (r+1)/2 \rfloor} = h-1-d^{(0)}-d^{(1)}-\cdots-d^{(\lfloor (r-3)/2 \rfloor)}\}$$

Proof. By Proposition 3.1.1, we may assume that the rank is greater than four. We recall from Proposition 3.1.1 that $h-(d+1)$ is an exponent of $\Re$ because $t_{h-d} = 1 = |\{\beta\}|$ and $t_{h-d-1} = 2 = |\{S_{\alpha_{p_{b+1}}}\beta, S_{\alpha_{p_{b+2}}}\beta\}|$, where $\beta = -\omega_{p_b} + \omega_{p_{b+1}} + \omega_{p_{b+2}}$. In other words the diamond (less its base vertex) of the $\tilde{\alpha}'$ string through $\tilde{\alpha}$ accounts for this exponent. Similarly we will prove that the next highest exponent is accounted for by a copy (in a sense explained below) of the $\tilde{\alpha}^{(2)}$ string through $\tilde{\alpha}'$ attached to the diamond of the $\tilde{\alpha}'$ string through $\tilde{\alpha}$. Clearly the number t_{h-d-2} of roots of height $h-d-2$ is determined by the set $N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})$. We first consider the case where $\Re'$ is of type A i.e. $|N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})| = 2$, so that $t_{h-d-2} = 3$ and therefore $h-1-d-1 = h-1-d-d'$ is an exponent. We now assume that $\Re'$ is not of type A, i.e. $N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}}) = \{\alpha_k\}$, which we take as adjacent to $\alpha_{p_{b+2}}$. Then $S_{\alpha_{p_{b+2}}}\beta = \omega_{p_{b+1}} - \omega_{p_{b+2}} + \omega_k$ and $S_{\alpha_{p_{b+1}}}\beta = -\omega_{p_{b+1}} + \omega_{p_{b+2}}$, where we have used the fact that $\alpha_{p_{b+1}}$ and $\alpha_{p_{b+2}}$ are never adjacent to one another. Since $S_{\alpha_{p_{b+1}}}S_{\alpha_{p_{b+2}}}\beta = S_{\alpha_{p_{b+2}}}S_{\alpha_{p_{b+1}}}\beta$ the only other root at height $h-d-2$ is $S_{\alpha_k}S_{\alpha_{p_{b+2}}}\beta$ so that $t_{h-d-2} = 2 = t_{h-d-1}$. We now show that there are two roots at each subsequent height until we reach height $h-1-d-d'$, one in the $\tilde{\alpha}'$ string through $\tilde{\alpha}$ and the other obtained as follows. Recall that by Theorem 3.1.2, the highest root of $\Re'$ is

$$\tilde{\alpha}' = \sum_{k \in N^*(\alpha_{p_{b+1}}, \alpha_{p_{b+2}})} \omega_k - \tilde{\alpha} = \omega_k - \tilde{\alpha},$$

and from above that $S_{\alpha_{p_{b+2}}}\beta = \omega_k + \omega_{p_{b+1}} - \omega_{p_{b+2}}$. There is therefore a copy of the $\tilde{\alpha}^{(2)}$ string through $\tilde{\alpha}'$ (accounting for the second root at the above heights) issuing from $S_{\alpha_{p_{b+2}}}\beta$, (modulo changing any simple reflection by $S_{\alpha_{p_{b+2}}}$ when descending from $\tilde{\alpha}'$ to a simple reflection by $S_{\alpha_{p_{b+1}}}$ when descending from $S_{\alpha_{p_{b+2}}}\beta$). When we reach the apex of the diamond in the $\tilde{\alpha}^{(2)}$ string through $\tilde{\alpha}'$ we have seen that the height below this an exponent of $\Re'$, and by Proposition 3.1.1, this height is d' below that of $\tilde{\alpha}'$. Therefore the copy of the $\tilde{\alpha}^{(2)}$ string through $\tilde{\alpha}'$ attached at $S_{\alpha_{p_{b+2}}}\beta$ (at height $h-1-d$) makes $h-1-d-d'$ an exponent of $\Re$, because it is clear from the signs of the ω_i coefficients of the roots in the $\tilde{\alpha}'$ string through $\tilde{\alpha}$ that they all have a unique ancestor (with the exception of β) so that the string is disjoint from the copy of the $\tilde{\alpha}^{(2)}$ string through $\tilde{\alpha}'$. However $\tilde{\alpha}' - \alpha_k$ (at height $h-2d-2$) does not, so that this height is the first possibility of intersection of the two strings. In fact for root systems of type E and D_r, $r \leq 5$ we can continue the above process, accounting for the exponents down to height

$$m_{r+1-\lfloor (r+1)/2 \rfloor} = h - 1 - d^{(0)} - d^{(1)} - \cdots - d^{(\lfloor (r-3)/2 \rfloor)},$$

because by Corollary 3.1.3:

$$h - (1 + d^{(0)} + d^{(1)} + \cdots + d^{(\lfloor (r-3)/2 \rfloor)}) \geq h - (2d+2).$$

In the case of D_r, $r \geq 5$ the intersections of the various strings and their copies are easily described because all the $\tilde{\alpha}^{(i+1)}$ strings through $\tilde{\alpha}^{(i)}$ are isomorphic as graphs. In the copy of the $\tilde{\alpha}^{(i+1)}$ string through $\tilde{\alpha}^{(i)}$ which is itself in the copy of $\Re'$ attached at $S_{\alpha_{p_{b+2}}}\beta$, the copy of the node $\tilde{\alpha}^{(i+1)}$ is the same node as $(S_{\alpha_{p_{b+2}}}\beta)^{(i)}$ (the node corresponding to $S_{\alpha_{p_{b+2}}}\beta$ in the $\tilde{\alpha}^{(i+1)}$ string through $\tilde{\alpha}^{(i)}$), and likewise in the iterated attached copies. The pairs of central nodes in the diamonds of these iterated attached copies make every second height (descending from height $h-1$) an exponent until we come to an iterated attached copy of D_4 (if r is even) whose triple diamond will make $r-1$ an exponent of multiplicity 2, or an iterated attached copy of D_5 (when r is odd) in which case the multiplicity of $r-1$ is one. □

Corollary 3.2.2. *The exponents $m_1 \leq m_2 \leq \cdots \leq m_r$ of a non-simply laced root system $\Re$ except of type B_r are given as follows:*

$$m_1 = 1, m_2 = d^\vee + m^{(0)}, \ldots, m_{\lfloor (r+1)/2 \rfloor} = d^\vee + m^{(0)} + m^{(1)} + \cdots + m^{(\lfloor (r-3)/2 \rfloor)},$$
$$m_r = h - 1, m_{r-1} = h - d^\vee - m^{(0)}, \ldots,$$
$$m_{r+1-\lfloor (r+1)/2 \rfloor} = h - d^\vee - m^{(0)} - m^{(1)} - \cdots - m^{(\lfloor (r-3)/2 \rfloor)}$$

Proof. Based on Theorem 3.2.1, the second exponent could be reached when $S_{\alpha_{p_b}}(\beta)$ is reflected by $S_{\alpha_{p_{b+1}}}$ and $S_{\alpha_{p_{b+2}}}$, in other word, the second exponent occurs at the hight one less than the top of the diamond $S_{\alpha_{p_b}}(\beta)$ in which α_{p_b} is a branch node(in simply laced cases and B_r), or it occurs when the long root β is reflected by $S_{\alpha_{p_b}}(\beta)$ in which α_{p_b} is a short simple root, then $ht(S_{\alpha_{p_b}}(\beta)) - m$ is an exponent (non-simply laces cases except B_r). □

Remark 3.2.1. The exponents of root systems of type B_r can be obtained by using Theorem 3.2.1 since in this case $m = 1$ and $d^\vee = d$.

Example 3.2.1. As an example we compute the exponents for the case $\Re = \Re^{(0)} = E_8$. From above we have $\Re^{(1)} = E_7$ and $\Re^{(2)} = D_6$. This gives the values $d^{(0)} = 6$, $d^{(1)} = 4$ and $d^{(2)} = 2$, so that:

$$m_1 = 1, m_2 = 1 + d^{(0)} = 7, m_3 = 1 + d^{(0)} + d^{(1)} = 11, m_3 = 1 + d^{(0)} + d^{(1)} + d^{(2)} = 13$$

and since $h = 30$ we have:

$$m_8 = 29, m_7 = 23, m_6 = 19, m_5 = 17.$$

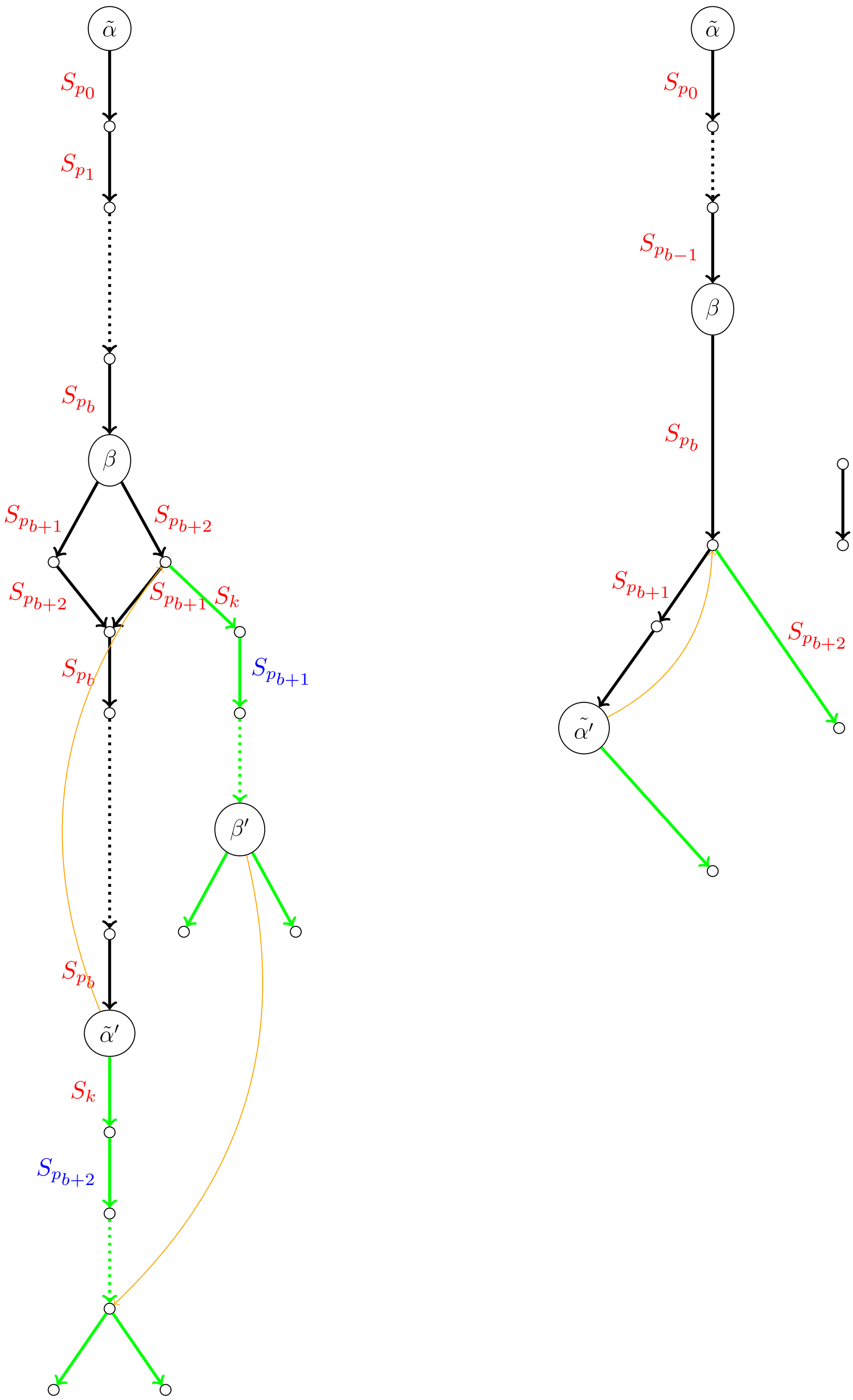

Figure 3.5: Copy of the $\tilde{\alpha}^{(2)}$ string through $\tilde{\alpha}'$

Chapter 4

Maximal parabolic subroot systems

In the first section of this chapter, we extend the results of R. Carles ([12]), described in chapter 3, to all maximal parabolic subroot systems, where the subroot system corresponds to deletion of a node α_i from the Dynkin diagram, and a formula for the cardinality of these subroot systems is obtained at the end of this section.

In the next section, we relate the sum of the positive roots of the parabolic subroot systems to the corresponding quantities of the maximal rank subgroups K_i of G (also maximal when $n_i^{\tilde{\alpha}}$ is one or a prime). The Dynkin diagram of the root system $\Re_{K_i}$ for K_i is obtained from the extended Dynkin diagram of G by deletion of the node α_i.

4.1 Cardinality of parabolic subroot systems

In this section, we first extend Wolf's Theorem 2.0.21 to the cases when $n_i^{\tilde{\alpha}} = 4, 6$.

Lemma 4.1.1. *Let π_v be the irreducible representation of K with highest weight v.*

(i) If $n_i^{\tilde{\alpha}} = 4$, then $\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^* + \pi_{-\beta_i}$, where β_i is the lowest height positive root with $n_i^{\beta_i} = 2$.

(ii) If $n_i^{\tilde{\alpha}} = 6$, then $\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^* + \pi_{-\beta_i} + (\pi_{-\beta_i})^* + \pi_{-\gamma_i}$, where β_i is as above and γ_i is the lowest height positive root with $n_i^{\gamma_i} = 3$.

Proof. We prove the case when $n_i^{\tilde{\alpha}} = 4$, in this case $\Psi = \{\alpha \in \Re^+ \,|\, n_i^\alpha = 1, 2, 3\}$. Suppose that

$$W' = \langle S_{\alpha_1}, \ldots, S_{\alpha_{i-1}}, S_{\alpha_{i+1}}, \ldots, S_{\alpha_r}, S_{-\tilde{\alpha}}\rangle$$

is the Weyl group of K, we consider the orbits of the action of the Weyl group on the set of the complementary roots. If α is a complementary root, then the coefficient of α_i in α is not affected by S_{α_j}, but the action of $S_{-\tilde{\alpha}}$ on $\Re \setminus \Re_K$ has three irreducible components; $P_1 = \{\alpha : n_i^\alpha = -1, 3\}$, $P_2 = \{\alpha : n_i^\alpha = 1, -3\}$ and $P_3 = \{\alpha : n_i^\alpha = -2, 2\}$. We note that in non-simply laced cases, each P_i consists of two orbits including long and short roots. Now we show that $\pi_{-\beta_i}$ is a summand of $\mathsf{ad}_{G/K}$ where β_i is the lowest height positive root with $n_i^{\beta_i} = 2$. By [32], we know that if $v \in \Re \setminus \Re_K$, then π_v is a summand of $\mathsf{ad}_{G/K}$ if and only if $\psi \in \Re_K$, implies $v + \psi \notin \Re$. So in order to prove that $\pi_{-\beta_i}$ is a summand of $\mathsf{ad}_{G/K}$, we show that $-\beta_i + \alpha$ is not a root when $\alpha \in \Re_K$. Let $\alpha \in \Re_K$, then either $n_i^\alpha = 4$ or $n_i^\alpha = 0$. If $n_i^\alpha = 4$ then $-\beta_i + \alpha$ is a positive root with $n_i^{\alpha-\beta_i} = 2$ but $ht(\alpha - \beta_i) < ht(\beta_i)$ which is a contradiction. Now suppose that $n_i^\alpha = 0$, then $n_i^{\alpha-\beta_i} = -2$, it means that $\beta_i - \alpha$ is a positive root with $n_i^{\beta_i-\alpha} = 2$ but $ht(\beta_i - \alpha) < ht(\beta_i)$ which is again a contradiction, therefore $\pi_{-\beta_i}$ is a summand of $\mathsf{ad}_{G/K}$ and P_3 is the set of its weights. By using the same argument we obtain that $\pi_{-\alpha_i}$ and $(\pi_{-\alpha_i})^*$ are also the summands of $\mathsf{ad}_{G/K}$, so it follows that:

$$\mathsf{ad}_{G/K} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^* + \pi_{-\beta_i}.$$

□

Theorem 4.1.2. *Suppose that $\sum_{\{n_i^\alpha = j\}} \alpha = s_j \omega_i,$. Let k be an integer, $0 \leq k < n$ and let $n_i^{\tilde{\alpha}} = n$, then:*

$$s_j = s_{n-(j+k)} \text{ for } k + 1 \leq j \leq \lfloor \tfrac{n-k}{2} \rfloor.$$

Also

$$s_k + s_{n-(j-k)} - s_{n-(j+k)} = 0 \text{ for } 1 \leq k \leq n \text{ and } k \leq j < \lfloor \tfrac{n-k}{2} \rfloor.$$

Proof. For $m \in \mathbb{Z}$, let $\Lambda_m = \{\alpha \in \Re : n_i^\alpha = m\}$, and let $\alpha \in \Lambda_{n-k}$ with $k \geq 1$ (we always choose α to be long in non-simply laces cases). For $\beta \in \Lambda_j$ with $k+1 \leq j$ we have that if $\langle \alpha, \beta \rangle \neq 0$ then $S_\alpha \beta = -\gamma$ where $n_i^\gamma = n - (j+k)$, as $\alpha + \beta$ would be contained in Λ_{n-k+j}, however $n - k + j > n$. The set $\Lambda_j - \Lambda_{n-(j+k)} := \Lambda_j \cup \Lambda_{(j+k)-n}$ is therefore invariant under S_α so that

$$\langle \sum_{\lambda \in \Lambda_j - \Lambda_{n-(j+k)}} \lambda,\, \alpha \rangle = \langle S_\alpha(\sum_{\lambda \in \Lambda_j - \Lambda_{n-(j+k)}} \lambda),\, S_\alpha \alpha \rangle = -\langle \sum_{\lambda \in \Lambda_j - \Lambda_{n-(j+k)}} \lambda,\, \alpha \rangle = 0.$$

As the simple reflections S_l for $l \neq i$ also leave Λ_j and $\Lambda_{n-(j+k)}$ invariant, we similarly have that $\sum_{\lambda \in \Lambda_j - \Lambda_{n-(j+k)}} \lambda = s_j - s_{n-(j+k)}$ is also orthogonal to α_l for $l \neq i$ and is therefore the zero vector. We now consider the cases when $j \leq k$. For $j \leq k$, either $S_\alpha \beta = -\gamma$ where as above $n_i^\gamma = n-(j+k)$, or $S_\alpha \beta = \psi$ with $n_i^\psi = n-(k-j)$. The set $\Lambda_j - \Lambda_{n-(j+k)} + \Lambda_{n-(k-j)} := \Lambda_j - \Lambda_{n-(j+k)} \cup \Lambda_{n-(k-j)}$ is therefore invariant under S_α so that $\langle \sum_{\Lambda_j - \Lambda_{n-(j+k)} + \Lambda_{n-(k-j)}} \lambda,\, \alpha \rangle = 0$. As the simple reflections S_l for $l \neq i$ also leave $\Lambda_j - \Lambda_{n-(j+k)} + \Lambda_{n-(k-j)}$ invariant we have that $\langle \sum_{\Lambda_j - \Lambda_{n-(j+k)} + \Lambda_{n-(k-j)}} \lambda,\, \alpha_l \rangle = 0$, so that $\sum_{\Lambda_j - \Lambda_{n-(j+k)} + \Lambda_{n-(k-j)}} \lambda = s_j - s_{n-(j+k)} + s_{n-(j-k)} = 0$. □

Proposition 4.1.3. *Suppose that $n_i^{\tilde{\alpha}} = n$, then*

$$\tau := s_1 + s_2 + \cdots + s_n = \frac{2k_i g}{n} - s_n.$$

Proof. First we consider the case when n is an odd number. By Proposition 1.2.14, $\sum_{\alpha \in \Re^+} \langle \alpha, \omega_i \rangle \alpha = g\, \omega_i$, also we know that $\sum_{\{n_i^\alpha = j\}} \alpha = s_j \omega_i$ and $\langle \alpha_i, \omega_i \rangle = 1/k_i$. So

$$g\omega_i = \sum_{\alpha \in \Re^+} \langle \alpha,\, \omega_i \rangle \alpha = \sum_{n_i^\alpha = 0} \langle \alpha,\, \omega_i \rangle \alpha + \sum_{n_i^\alpha = 1} \langle \alpha,\, \omega_i \rangle \alpha + \cdots + \sum_{n_i^\alpha = n} \langle \alpha,\, \omega_i \rangle \alpha,$$

which yields $gk_i\omega_i = s_1\omega_i + 2s_2\omega_i + \cdots + ns_n\omega_i$, so

$$gk_i = s_1 + 2s_2 + \cdots + (\frac{n-1}{2}) s_{\frac{n-1}{2}} + (\frac{n-1}{2} + 1) s_{\frac{n-1}{2}+1} + \cdots + (n-1)s_{n-1} + ns_n,$$

it is an immediate consequence of Theorem 4.1.2 that, $s_1 = s_{n-1}, \cdots, s_{\frac{n-1}{2}} = s_{\frac{n-1}{2}+1}$, therefore we get $gk_i = ns_n + ns_{n-1} + \cdots + ns_{\frac{n-1}{2}+1}$, so $\frac{2gk_i}{n} = 2s_n + 2s_{n-1} + \cdots + 2s_{\frac{n-1}{2}+1}$.

If we apply Theorem 4.1.2 again, we have

$$\frac{2gk_i}{n} = s_1 + s_2 \cdots + s_{n-1} + 2s_n.$$

Therefore

$$s_1 + s_2 + \cdots + s_n = \frac{2k_i g}{n} - s_n.$$

We use a similar argument to prove the Proposition when n is an even number. By Theorem 4.1.2, $s_{\frac{n}{2}-1} = s_{\frac{n}{2}+1}$, therefore $gk_i = ns_n + ns_{n-1} + \cdots + ns_{\frac{n}{2}+1} + \frac{n}{2}s_{\frac{n}{2}}$, so

$$\frac{2gk_i}{n} = 2s_n + 2s_{n-1} + \cdots + 2s_{\frac{n}{2}+1} + s_{\frac{n}{2}},$$

if we apply Theorem 4.1.2, it follows that

$$\frac{2gk_i}{n} = s_1 + s_2 + \cdots + s_{n-1} + 2s_n,$$

which proves the Proposition. □

Remark 4.1.1. In [9], it is proved that $\sum_{n_i^\alpha > 0} \alpha = \frac{\langle 2\rho, \omega_i \rangle}{\langle \omega_i, \omega_i \rangle}\omega_i$, also as defined above

$$\sum_{n_i^\alpha > 0} \alpha = (s_1 + \cdots + s_n)\omega_i,$$

therefore it follows from Proposition 4.1.3 that $\tau = \frac{2gk_i}{n} - s_n = \frac{\langle 2\rho, \omega_i \rangle}{\langle \omega_i, \omega_i \rangle}$.

Proposition 4.1.4. *Let a_i be the coefficient of α_i in 2ρ, then $a_i = \tau k_i \langle \omega_i, \omega_i \rangle$.*

Proof. The coefficient of α_i in 2ρ and $\sum_{n_i^\alpha > 0} \alpha$ are equal. By Proposition 4.1.3,

$$\sum_{n_i^\alpha > 0} \alpha = (s_1 + \cdots + s_n)\omega_i = (\frac{2k_i g}{n} - s_n)\omega_i,$$

also $n_i^{\omega_i} = \langle \omega_i, \omega_i \rangle k_i$, therefore

$$a_i = (\frac{2k_i g}{n} - s_n)\langle \omega_i, \omega_i \rangle k_i = \tau k_i \langle \omega_i, \omega_i \rangle.$$

□

Theorem 4.1.5. *Let $\Re$ be an irreducible reduced crystallographic root system. Denote by V_{ω_i} the hyperplane perpendicular to ω_i and let $\Re_{\omega_i} = \Re \cap V_{\omega_i}$, then:*

$\Re_{\omega_i}$ is a root system and for $n_i^{\tilde{\alpha}} \geq 2$, card $\Re^+_{\omega_i}$ =

$$card\ \Re^+ - \{\frac{2gk_i}{n(n-1)}[1 + \frac{1}{2} \cdots + \frac{1}{n-1}] - s_n(1 + \frac{1}{n(n-1)})\}k_i\langle\omega_i, \omega_i\rangle.$$

For $n_i^{\tilde{\alpha}} = 1$, card $\Re^+_{\omega_i} =$ card $\Re^+ - (2gk_i - s_1)k_i\langle\omega_i, \omega_i\rangle =$ card $\Re^+ - g\langle\omega_i, \omega_i\rangle$.

Proof. $\Re_{\omega_i} = \Re \cap V_{\omega_i}$, so $\Re_{\omega_i}$ consists of those roots with α_i coefficient equal to zero, and they constitute the root system (not usually reduced) with Dynkin diagram obtained from that of $\mathfrak{g}$ by deletion of the node labelled α_i. We now count the number of roots in the complement (in $\Re^+$) of $\Re^+_{\omega_i}$ i.e. the positive roots with non-zero α_i coefficient. For $j > 0$, the α_i coefficient of $\sum_{\{n_i^\alpha = j\}} \alpha$ can be alternatively written as jn_j or $s_jk_i\langle\omega_i, \omega_i\rangle$ so that $n_j = \frac{s_jk_i}{j}\langle\omega_i, \omega_i\rangle$. When $n_i^{\tilde{\alpha}} \geq 2$ we use the equations in $s_1, \ldots, s_n$ derived from Theorem 4.1.2 together with the additional equation $s_1 + 2s_2 + \cdots + (n-1)s_{n-1} + ns_n = gk_i$ (from Proposition 1.2.14). These $n-1$ equations are easily solved in terms of g, n and s_n using Gaussian elimination (when $n > 2$). All rows of the extended matrix with the exception of the last (coming from Proposition 1.2.14) consist of two or three non zero entries (equal to ± 1) and are essentially in upper echelon form. Killing the entries $1, 2, \ldots, n-1$ in the last row has the effect of making $1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}$ the coefficient of s_{n-1} and s_n in the last row of the reduced extended matrix to give the equation $\frac{n(n-1)}{2}s_{n-1} + \frac{n(n-1)}{2}s_n = gk_i$. Back substitution using $s_{n-1} = \frac{2}{n(n-1)}gk_i - s_n\,(= s_1)$ then gives $s_2 = \cdots = s_{n-2} = \frac{2}{n(n-1)}gk_i$ and recalling that $n_j = \frac{s_jk_i}{j}\langle\omega_i, \omega_i\rangle$ the result follows in these cases. When $n_i^{\tilde{\alpha}} = 1$, the result follows from the equation $s_1 + 2s_2 + \cdots + (n-1)s_{n-1} + ns_n = gk_i$ where $n = 1$ and the fact that $k_i = 1$ as α_i is always long (because $\tilde{\alpha}$ is). Also when $n = 2$ we have that $s_1 = gk_i - 2s_2$. □

As an immediate consequence of Theorem 4.1.5, we obtain the original formula of the cardinality of card $\Re^+_{\omega_i}$ by Carles [12], when ω_i is the highest root.

Corollary 4.1.6. *Let $\Re$ and $\Re_{\omega_i}$ be as in Theorem 4.1.5. Suppose that ω_i is the highest root, then:*

$$card\ \Re^+_{\omega_i} = card\ \Re^+ - 2g + 3.$$

Proof. In this case $n_i^{\omega_i} = n_i^{\tilde{\alpha}} = 2$. By Theorem 4.1.5,

$$\text{card } \Re_{\omega_i}^+ = \text{card } \Re^+ + (\frac{3}{2}s_2 - gk_i)k_i\langle\omega_i,\, \omega_i\rangle.$$

As ω_i is the highest root, so $k_i = 1, \langle\omega_i, \omega_i\rangle = 2$ and $s_2 = 1$, therefore

$$\text{card } \Re_{\omega_i}^+ = \text{card } \Re^+ - 2g + 3.$$

□

4.2 Sums of roots of maximal parabolic and maximal rank subgroups

Proposition 4.2.1. *Suppose that* $2\rho = \sum_{i=1}^r a_i\alpha_i$, *then* $2a_i = 2 - \sum_{j\in N(\alpha_i)} a_j c_{ij}$. *In particular, in simply laced cases* $2a_i = 2 + \sum_{j\in N(\alpha_i)} a_j$.

Proof. We start from this fact that $\langle\alpha_i, \rho\rangle = \langle\alpha_i, \omega_i\rangle = \frac{\langle\alpha_i,\alpha_i\rangle}{2}$, which yields:

$$\langle\alpha_i, \alpha_i\rangle = \langle\alpha_i, 2\rho\rangle = a_i\langle\alpha_i, \alpha_i\rangle + \sum_{j\in N(i)} a_j\langle\alpha_i, \alpha_j\rangle,$$

then we have $2a_i = 2 - \sum_{j\in N(\alpha_i)} a_j c_{ij}$. □

Proposition 4.2.2. *Suppose that* $\omega_j = \sum_{i=1}^r d_i^j\alpha_i$, *for* $1 \leqslant j \leqslant r$, *then*

$$\frac{d_i^j}{d_j^i} = \frac{\langle\alpha_j, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle}.$$

Proof. Let $\omega_i = \sum_{i=1}^r d_j^i\alpha_j$, then

$$\langle\omega_i, \omega_j\rangle = \langle\sum_{i=1}^r d_j^i\alpha_j, \omega_j\rangle = d_j^i\langle\alpha_j, \omega_j\rangle = d_j^i\frac{\langle\alpha_j, \alpha_j\rangle}{2},$$

on the other hand

$$\langle \omega_j, \omega_i \rangle = \langle \sum_{i=1}^{r} d_i^j \alpha_i, \omega_i \rangle = d_i^j \langle \alpha_i, \omega_i \rangle = d_i^j \frac{\langle \alpha_i, \alpha_i \rangle}{2}.$$

Therefore $\frac{d_i^j}{d_j^i} = \frac{\langle \alpha_j, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$. □

Remark 4.2.1. By Proposition 4.2.2, it is obvious that in simply laced cases $d_i^j = d_j^i$. In fact if both α_i and α_j are short or long roots then $d_i^j = d_j^i$, but if one of them, for example α_j, is long and the other one is short then $d_i^j = k_i d_j^i$.

Lemma 4.2.3. *If $\alpha = \sum_{i=1}^{r} n_i^{\alpha} \alpha_i$, then $\alpha^\vee = \sum_{i=1}^{r} (n_i^{\alpha})^\vee \alpha_i^\vee$, where*

$$(n_i^{\alpha})^\vee = \frac{\langle \alpha_i, \alpha_i \rangle}{\langle \alpha, \alpha \rangle} n_i^{\alpha}.$$

Proof. We know that $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, so

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = \frac{2}{(\alpha, \alpha)} \sum_{i=1}^{r} n_i^{\alpha} \alpha_i = \frac{2}{(\alpha, \alpha)} \sum_{i=1}^{r} n_i^{\alpha} \frac{(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i$$

$$= \sum_{i=1}^{r} \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} n_i^{\alpha} \frac{2\alpha_i}{(\alpha_i, \alpha_i)} = \sum_{i=1}^{r} (n_i^{\alpha})^\vee \alpha_i^\vee.$$

Therefore $(n_i^{\alpha})^\vee = \frac{(\alpha_i,\alpha_i)}{(\alpha,\alpha)} n_i^{\alpha} = \frac{\langle \alpha_i,\alpha_i \rangle}{\langle \alpha,\alpha \rangle} n_i^{\alpha}$. □

Corollary 4.2.4. *Let $\tilde{\alpha} = \sum_{i=1}^{r} n_i^{\tilde{\alpha}} \alpha_i$ be the highest root, then the dual Coxeter number $g = 1 + \sum_{i=1}^{r} n_i^{\tilde{\alpha}} / k_i$.*

Proof. If $\tilde{\alpha} = \sum_{i=1}^{r} n_i^{\tilde{\alpha}} \alpha_i$, then Lemma 4.2.3 implies that $\tilde{\alpha}^\vee = \sum_{i=1}^{r} (n_i^{\tilde{\alpha}})^\vee \alpha_i^\vee$, an alternative definition of the dual coxeter number ([19]) is

$$g = 1 + \sum_{i=1}^{r} (n_i^{\tilde{\alpha}})^\vee = 1 + \sum_{i=1}^{r} n_i^{\tilde{\alpha}} / k_i.$$

□

Remark 4.2.2. Although the following Theorem can be derived from Theorem 4.2.5 (see Remark 4.2.3), we give a rather long alternative proof containing some observations of independent interest.

Theorem 4.2.5. *Let $\Re_{K_i}$ and $\Re_{\omega_i}$ be the root systems defined in the previous section. Suppose that $2\rho_i'$ and $2\rho_{\omega_i}$ are respectively the sum of positive roots of $\Re_{K_i}$ and $\Re_{\omega_i}$, then:*

$$2\rho_i' = 2\rho - \frac{2k_i g}{n_i^{\tilde{\alpha}}}\,\omega_i,$$

and

$$2\rho_{\omega_i} = 2\rho - (\frac{2k_i g}{n_i^{\tilde{\alpha}}} - s_{n_i^{\tilde{\alpha}}})\,\omega_i.$$

Proof. By the observation after Lemma 1.2.15, we know that $\sum_{\alpha\in\Re_{i,x}}\alpha = c_x\omega_i$ for some $c_x \in \mathbb{R}$ where $(i, x) \in \{1, \ldots, r\} \times \mathbb{R}$, and $\Re_{i,x} = \{\alpha \in \Re : \langle\omega_i, \alpha\rangle = x\}$. Summing over all positive values of x we see that $\sum_{\langle\alpha,\omega_i\rangle>0}\alpha = c\omega_i$. Therefore we must determine this multiple c. Now $-\tilde{\alpha} = -\omega_{p_0}$ is a pendant node (which we will denote by α_p) with neighbouring node α_{p_0} of the extended Dynkin Diagram with the node α_i removed. By Proposition 4.2.1, the coefficients a_i defined by $2\rho = \sum_{i=1}^r a_i\alpha_i$ satisfy $2a_i = \sum_{k\in N(\alpha_i)} a_k + 2$. Therefore $2a_p' = a_{p_0}' + 2$ by our recurrence relations. Since $\tilde{\alpha} = \omega_{p_0}$ has length two, it has α_{p_0} coefficient two also. The α_{p_0} coefficient of $2\rho_i'$ is therefore equal to

$$a_p'(-2) + a_{p_0}' = -a_{p_0}' - 2 + a_{p_0}' = -2.$$

On the other hand because $d_i^j = d_j^i$ (Remark 4.2.1) and $\tilde{\alpha} = \sum_{i=1}^r n_i^{\tilde{\alpha}}\alpha_i = \sum_{i=1}^r d_i^{p_0}\alpha_i$ has height $h-1$, the α_{p_0} coefficient of 2ρ is $2(h-1)$, so that the α_{p_0} coefficient of $2\rho - 2\rho_i'$ is $2h$. The α_{p_0} coefficient in $c\omega_i$ is therefore equal to $2h$ and $cd_{p_0}^i = cd_i^{p_0} = cn_i^{\tilde{\alpha}} = 2h$.

To prove the theorem for non-simply laced cases, we argue as before that $2\rho - 2\rho_i' = c\omega_i$. In order to find this multiple c, it is sufficient to compare the coefficient of α_{p_0} in both sides. α_p is not a short root in $\Re'$, so by Proposition 4.2.1, $2a_p' = a_{p_0}' + 2$, also the coefficient of α_{p_0} in $\tilde{\alpha}$ is 2, so the coefficient of α_{p_0} in $2\rho_i'$ is

$$(-2)a_p' + a_{p_0}' = -a_{p_0}' - 2 + a_{p_0}' = -2.$$

Also we know that $\rho = \sum_{i=1}^{r} \omega_i$, so (relabelling $\alpha_i^{,}s$)

$$2\rho = 2\sum_{i=1}^{r} \omega_i = 2(\sum_{i=1}^{r}\sum_{j=1}^{r} d_j^i \alpha_j) = 2(\sum_{i=1}^{r}(d_{p_0}^i \alpha_{p_0} + \cdots + d_{p_{r-1}}^i \alpha_{p_{r-1}}))$$

Therefore the coefficient of α_{p_0} in 2ρ equals

$$2(\sum_{i=1}^{r} d_{p_0}^i) = 2(d_{p_0}^1 + \cdots + d_{p_0}^r).$$

Now α_{p_0} can be either a short root (C_r) or a long root (F_4, B_r, G_2). First we consider the case where α_{p_0} is a long root. Suppose that α_s is the short root which is connected to the long root α_l. We can suppose that $\alpha_i^{,}s \quad (i \geq s)$ are short roots and $\alpha_i^{,}s \quad (i < s)$ are long roots. Now based on the fact that α_{p_0} is a long root and Remark 4.2.1, the coefficient of α_{p_0} in 2ρ equals

$$2(d_1^{p_0} + \cdots + d_{s-1}^{p_0} + \frac{1}{k_s} d_s^{p_0} + \cdots + \frac{1}{k_r} d_r^{p_0}).$$

The length of all short roots in a root system are equal, $k_s = \cdots = k_r$, and we denote this quantity by k, so the coefficient of α_{p_0} in 2ρ is

$$2(d_1^{p_0} + \cdots + d_{s-1}^{p_0} + \frac{1}{k} d_s^{p_0} + \cdots + \frac{1}{k} d_r^{p_0}).$$

Now by applying Corollary 4.2.4 and using the fact that α_{p_0} is a long root, and $\tilde{\alpha} = \omega_{p_0}$, we can easily find the coefficient of α_{p_0} in 2ρ which is

$$2(d_1^{p_0} + \cdots + d_{s-1}^{p_0} + \frac{1}{k} d_s^{p_0} + \cdots + \frac{1}{k} d_r^{p_0}) = 2((d_1^{p_0})^\vee + \cdots + (d_{s-1}^{p_0})^\vee + (d_s^{p_0})^\vee + \cdots + (d_r^{p_0})^\vee)$$

$$= 2(g-1).$$

Now we consider the case when α_{p_0} is a short root (C_r), like the previous argument the coefficient of α_{p_0} in 2ρ is $2(\sum_{i=1}^{r} d_{p_0}^i) = 2(d_{p_0}^1 + \cdots + d_{p_0}^r)$. Again suppose that for $i \geq s$, $\alpha_i^{,}s$ are short roots and for $i < s$, $\alpha_i^{,}s$ are long roots, therefore for $i \geq s$, $d_{p_0}^i = d_i^{p_0}$(because both α_{p_0} and α_i are short roots), also for $i < s$, $d_{p_0}^i = 2d_i^{p_0}$ so the coefficient of α_{p_0} in 2ρ is equal to

$$2(2d_1^{p_0} + \cdots + d_{s-1}^{p_0} + d_s^{p_0} + \cdots + d_r^{p_0})$$

In the case of C_r, $d_i^{p_0} \neq n_i^{\tilde{\alpha}}$ because $\tilde{\alpha} \neq \omega_{p_0}$, in fact $\tilde{\alpha} = 2\omega_{p_0}$, it means $\sum_{i=1}^r n_i^{\tilde{\alpha}} \alpha_i = \sum_{i=1}^r 2d_i^{p_0} \alpha_i$, so $n_i^{\tilde{\alpha}} = 2d_i^{p_0}$, therefore the coefficient of α_{p_0} in 2ρ is

$$\begin{aligned} 2(2d_1^{p_0} + \cdots + 2d_{s-1}^{p_0} + d_s^{p_0} + \cdots + d_r^{p_0}) &= 2(n_1^{\tilde{\alpha}} + \cdots + n_{s-1}^{\tilde{\alpha}} + \frac{1}{k} n_s^{\tilde{\alpha}} + \cdots + \frac{1}{k} n_r^{\tilde{\alpha}}) \\ &= 2((n_1^{\tilde{\alpha}})^\vee + \cdots + (n_{s-1}^{\tilde{\alpha}})^\vee + (n_s^{\tilde{\alpha}})^\vee + \cdots + (n_r^{\tilde{\alpha}})^\vee) \\ &= 2(\sum_{i=1}^r (n_i^{\tilde{\alpha}})^\vee) = 2(g-1). \end{aligned}$$

So in all cases the coefficient of α_{p_0} in 2ρ is $2(g-1)$ and this coefficient in $2\rho_i'$ is -2, so the coefficient of α_{p_0} in $2\rho - 2\rho_i'$ equals $2g - 2 + 2 = 2g$.

Now we calculate the coefficient of α_{p_0} in $c\omega_i$. There are again two cases, first when α_{p_0} is long the other one when α_{p_0} is short, investigating both cases depends on whether α_i is short or long. First we consider the case where α_{p_0} and α_i are both long roots so $d_{p_0}^i = d_i^{p_0} = n_i^{\tilde{\alpha}}$, also the coefficient of α_{p_0} in ω_i is $d_{p_0}^i$ so the coefficient of α_{p_0} in $c\omega_i$ is $cn_i^{\tilde{\alpha}}$ therefore $cn_i^{\tilde{\alpha}} = 2g$, so $c = \frac{2g}{n_i^{\tilde{\alpha}}}$. Now suppose that α_{p_0} is long and α_i is short, then $\frac{1}{k} d_i^{p_0} = d_{p_0}^i$, so the coefficient of α_{p_0} in $c\omega_i$ equals $\frac{c}{k} n_i^{\tilde{\alpha}}$, therefore $\frac{c}{k} n_i^{\tilde{\alpha}} = 2g$ so

$$c = \frac{2kg}{n_i^{\tilde{\alpha}}}.$$

Now we consider the case when α_{p_0} is a short root, it happens when $\Re = C_r$, in this case $n_i^{\tilde{\alpha}} = 2d_i^{p_0}$. First suppose that α_i is a long root, then $d_{p_0}^i = 2d_i^{p_0} = n_i^{\tilde{\alpha}}$, so the coefficient of α_{p_0} in $c\omega_i$ is $cn_i^{\tilde{\alpha}}$ and therefore

$$c = \frac{2g}{n_i^{\tilde{\alpha}}}.$$

The last case is when α_i is a short root. In this case $d_{p_0}^i = d_i^{p_0} = (1/2) n_i^{\tilde{\alpha}}$ so $c = \frac{2kg}{n_i^{\tilde{\alpha}}}$. As it is observed in all cases $c = \frac{2k_i g}{n_i^{\tilde{\alpha}}}$.

To prove the expression for $2\rho_{\omega_i}$, we first note that in this case $\Re_{\omega_i}$ does not contain the roots whose α_i coefficient equals $n_i^{\tilde{\alpha}}$, so $2\rho_{\omega_i} = 2\rho_i' + s_{n_i^{\tilde{\alpha}}} \omega_i$, then the formula follows.

$\square$

Remark 4.2.3. We can also have an alternative proof of Theorem 4.2.5. By the description of the root system $\Re_{\omega_i}$, it is obvious that $\Re_{\omega_i} = \{\alpha \in \Re : n_i^{\alpha} = 0\}$. Therefore:

$$2\rho_{\omega_i} = 2\rho - (\sum_{n_i^{\alpha}=1} \alpha + \cdots + \sum_{n_i^{\alpha}=n} \alpha),$$

where $n_i^{\tilde{\alpha}} = n$. Then by Proposition 4.1.3 it follows that

$$2\rho_{\omega_i} = 2\rho - (\frac{2k_i g}{n_i^{\tilde{\alpha}}} - s_{n_i^{\tilde{\alpha}}})\,\omega_i.$$

In the following Corollary we show that Carles formula [12] is obtained by applying Theorem 4.2.5.

Corollary 4.2.6. *Let ω_i be the highest root and $2\rho_{\omega_i}$ be as above, then:*

$$2\rho_{\omega_i} = 2\rho - (g-1)\omega_i.$$

Proof. By Theorem 4.2.5,

$$2\rho_{\omega_i} = 2\rho - (\frac{2k_i g}{n_i^{\tilde{\alpha}}} - s_{n_i^{\tilde{\alpha}}})\,\omega_i.$$

We note that $k_i = 1$, $n_i^{\tilde{\alpha}} = 2$ and $s_{n_i^{\tilde{\alpha}}} = 1$, therefore $2\rho_{\omega_i} = 2\rho - (g-1)\omega_i$. $\square$

Chapter 5

Applications to topological invariants, nef value and the defect of projective varieties

In this chapter, we consider some topological and geometric applications of the results of previous chapters. In particular, we use the results of Chapter 2 to calculate a topological invariant introduced in [24] for compact homogeneous spaces G/K, where G is a compact, simple, centerless Lie group and K is a maximal closed subgroup of maximal rank.

We apply the results of Chapter 4 to give uniform formulae for the first Chern class $c_1(X)$ of a flag manifold $X = G^{\mathbb{C}}/P$, where P is a maximal parabolic subgroup. We will also use another interpretation of the first Chern class $c_1(X)$ in terms of nef values of line bundles on X to give formulae for the defect $def(X, L)$ where L is an ample line bundle on X, when $def(X, L) > 0$.

5.1 Topology of G/K

In this section, we relate our formulae in Chapter 2 to a topological invariant (introduced in [24]) of a compact oriented manifold of dimension $2n$, with Poincare polynomial $P(t)$. Poincare duality implies that $P'(-1) = -n\chi$, where a prime denotes differentiation w.r.t. t and $P(-1) = \chi$ is the Euler characteristic. From the study of hyper-Kähler structures in [24], the quantity $P''(-1)$ was seen to be of geometrical significance. In [17] and [16] the normalized expression

$$\phi_2 = \frac{P''(-1)}{2\chi} - n^2/2,$$

(for $\chi \neq 0$.) which is additive w.r.t. products of manifolds was studied. When G/K is an Hermitian symmetric space ϕ_2 is related to identities for the Chern classes of the Kähler structure on G/K. Using our strange formulae we obtain some alternative descriptions from [17] for $\phi_2(M)$ when M is a symmetric space and we give new formulae in the non-symmetric cases ([10]).

Proposition 5.1.1. *([17, p. 280]) Let $M = G/K$ be an irreducible symmetric space (corresponding to the deletion of a node α_i from the extended Dynkin diagram).*

(i) If G is simply-laced or if M is an Hermitian symmetric space,

$$\phi_2(M) = \frac{(h-2)n}{6}.$$

(ii) If G is of type C_r (and M is non-Hermitian),

$$\phi_2(M) = \frac{(h-1)n}{6} = \frac{(h-2)n + 2i(r-i)}{6}.$$

(iii) If G is of type B_r (and M is non-Hermitian),

$$\phi_2(M) = \frac{(h-2)n + i(i+1) - 2r}{6}.$$

(iv) If M is a Wolf space,

$$\phi_2(M) = \frac{(h-2)n + 2(h-g)}{6} = \frac{(h-2)(g-2) + (h-g)}{3}.$$

(v) If G is of type F_4 and M is not a Wolf space,

$$\phi_2(M) = \frac{(h-2)n + 2h}{6}.$$

Proof. From [17, **Lemma 2.1 and Proposition 1.4**]:

$$\phi_2(G/K) = \frac{1}{3}[ht_{\mathfrak{g}}(2\rho) - ht_{\mathfrak{k}}(2\rho_K) - n],$$

where for an integral linear combination $\gamma = \sum_{i=1}^r c_i\alpha_i$, its height $ht_{\mathfrak{g}}(\gamma) = \sum_{i=1}^r c_i$. Suppose that $2\rho = \sum_{i=1}^r a_i\alpha_i = \sum_{i=1}^r 2\omega_i$, then in the simply laced cases $ht_{\mathfrak{g}}(2\rho) = \frac{1}{2}\langle 2\rho, 2\rho\rangle$ since

$$\langle 2\rho, 2\rho\rangle = \langle \sum_{i=1}^r a_i\alpha_i, \sum_{i=1}^r 2\omega_i\rangle = 2\sum_{i=1}^r a_i = 2ht_{\mathfrak{g}}(2\rho).$$

By the same argument $ht_{\mathfrak{k}}(2\rho_K) = \frac{1}{2}\langle 2\rho_K, 2\rho_K\rangle$, where the height of $2\rho_K$ is measured in K, therefore:

$$ht_{\mathfrak{g}}(2\rho) - ht_{\mathfrak{k}}(2\rho_K) = \frac{1}{2}(\langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle) = \frac{g}{2}n$$

by Corollary 2.1.4. As $g = h$ when $\mathfrak{g}$ is simply laced, the result follows for these cases. In the two remaining cases $n_i^{\tilde{\alpha}} = 1$, implies $i = 1$ for B_r and $i = r$ for C_r. For Hermitian symmetric spaces, we have $\phi_2 = \frac{1}{3}[ht_{\mathfrak{m}}(2\rho_\Psi) - n]$, where as we have seen, $2\rho_\Psi = g\omega_i$. This implies that $n = \langle 2\rho_\Psi, \omega_i\rangle = g\langle\omega_i, \omega_i\rangle$ so that

$$hn = 2rg\langle\omega_i, \omega_i\rangle = 2\, ht_{\mathfrak{g}}(g\omega_i).$$

In the non-Hermitian, non-simply laced cases, we calculate the difference between $ht_{\mathfrak{g}}(2\rho) - ht_{\mathfrak{k}}(2\rho_K)$ and $\frac{1}{2}(\langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle) = \frac{g}{2}n$. For the C_r cases we observe that all α_i, $i \neq r$ are short, so that if $2\rho = \sum_{i=1}^r a_i\alpha_i$, then $ht_{\mathfrak{g}}(2\rho) = \sum_{i=1}^r a_i$, whereas $\langle 2\rho, \rho\rangle = \sum_{i=1}^{r-1} \frac{a_i}{2} + a_r$. Since $\Re_K$ is of type $C_i \oplus C_{r-i}$ and $a_r = \frac{r(r+1)}{2}$ (see [7]), we have that:

$$\begin{aligned}\phi_2(M) &= \frac{1}{3}\{gn - n - \frac{r(r+1)}{2} + \frac{i(i+1)}{2} + \frac{(r-i)(r-i+1)}{2}\}\\ &= \frac{1}{3}\{gn - n - i(i-r)\}\\ &= \frac{1}{3}\{gn - n - \frac{n}{2}\}\\ &= \frac{(2g-3)n}{6}\\ &= \frac{h-1}{6}.\end{aligned}$$

The other cases are similar with the exception of the Wolf spaces where it is easier to compute $ht_{\mathfrak{g}}(2\rho) - ht_{\mathfrak{k}}(2\rho_K)$. Using [9, **Proposition 2.2**], but compensating for the fact that in this note we are deleting from the extended Dynkin diagram, we have $ht_{\mathfrak{g}}(2\rho) - ht_{\mathfrak{k}}(2\rho_K) = g\, ht_{\mathfrak{g}}(\tilde{\alpha}) - h$ so that $\phi_2(M) = \frac{g(h-1)-h-n}{3}$. Now using the fact that for these spaces $n = 2g-4$ ([9, **Proposition 2.2**] or [27, **Proposition 1**]), we have $\phi_2(M) = \frac{(h-2)n+2(h-g)}{6}$. Similarly the result for the non-Wolf space in the case of F_4 follows from [9, **Proposition 2.2**] applied to the highest short root. □

Theorem 5.1.2. *Let $M = G/K$ be non-symmetric (corresponding to the deletion of a node α_i from the extended Dynkin diagram).*

(i) For $n_i^{\tilde{\alpha}} = 3$, we have

$$\phi_2(M) = \frac{(\frac{8}{9}h - 2)n}{6}$$

in the simply-laced cases, otherwise

$$\phi_2(M) = \frac{(\frac{8}{9}h - 2)n + 4(h-g)}{6}.$$

(ii) For $n_i^{\tilde{\alpha}} = 5$, we have

$$\phi_2(M) = \frac{(\frac{4}{5}h - 2)n}{6}.$$

Proof. We give the proof only for the simple laced cases, where

$$\phi_2(G/K) = \frac{1}{3}[ht_{\mathfrak{g}}(2\rho) - ht_{\mathfrak{k}}(2\rho_K) - n] = \frac{1}{3}[\frac{1}{2}(\langle 2\rho, 2\rho\rangle - \langle 2\rho_K, 2\rho_K\rangle) - n].$$

We now apply Corollary 2.2.2. □

5.2 Geometry of flag manifolds

All the background material for this section can be found in [25] and [2].

A flag manifold X is a homogeneous space $G/C(S)$, where G is a compact Lie group and $C(S)$ is the centralizer of a torus $S \subseteq G$, or equivalently they are the orbits of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Flag manifolds have an alternative description of the form $G^{\mathbb{C}}/P$, where $G^{\mathbb{C}}$ is the complexification of G and P is a parabolic subgroup of $G^{\mathbb{C}}$ ([2]), the definition of which we now recall.

The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha<0} \mathfrak{g}_\alpha$ is a maximal solvable subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let B be the closed connected solvable subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{b}$, then any conjugate of B is called a *Borel* subgroup.

Definition 5.2.1. A connected subgroup P of $G^{\mathbb{C}}$ containing a Borel subgroup is called a *parabolic* subgroup.

The Lie algebra of P is given by

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha>0} \mathfrak{g}_{-\alpha} + \sum_{\alpha\in\Re_P^+} \mathfrak{g}_\alpha,$$

where $\Re_P^+$ is a closed subset (under addition) of positive roots. As $\Re_P := \Re^- \cup \Re_P^+$ is also a closed set of roots containing all negative simple roots $-\alpha_i$, it follows that $\Re_P^+$ is generated by a set of positive simple roots $\{\alpha_i : i \in I\}$, where $I \subseteq \{1, \cdots, r\}$.

The set of complementary roots $\Re^+ \setminus \Re_P^+$ is denoted $\Re_X$ and is called the set of roots of X. In particular if $|I| = r-1$, we call P a maximal parabolic subgroup.

The Chern classes of the tangent bundle of $X = G^{\mathbb{C}}/P$ (or simply the Chern classes of X) can be given in terms of the roots of X (see [25] , [6]): The total Chern class $c(X)$ has a description as

$$c(X) = \prod_{\alpha\in\Re_X} (t+\alpha) = \sum_{q=0}^{r} c_q t^{r-q},$$

so that the first Chern class $c_1(X) = \sum_{\alpha\in\Re_X} \alpha$. The results from Chapter 4 and Chapter 2 concerning 2ρ and $2\rho_{\omega_i}$ now have a geometric interpretation in terms of the first Chern class of $X = G^{\mathbb{C}}/P_{\alpha_i}$, where P_{α_i} in the maximal parabolic corresponding to $I = \{1, \cdots, \hat{i}, \cdots, r\}$, and the space of full flags $X = G^{\mathbb{C}}/B$, where B is a Borel subgroup and:

$$c_1(G^{\mathbb{C}}/B) = 2\rho = 2\rho_{\omega_i} + c_1(G^{\mathbb{C}}/P_{\alpha_i}),$$

also by 4.2.5,

$$c_1(G^{\mathbb{C}}/P_{\alpha_i}) = (\frac{2k_i g}{n_i^{\tilde{\alpha}}} - s_{n_i^{\tilde{\alpha}}})\omega_i.$$

Remark 5.2.1. By [26], the first Chern class $c_1(G^{\mathbb{C}}/P)$ of any parabolic subgroup P can be obtained from $c_1(G^{\mathbb{C}}/P_{\alpha_i})$, where P_{α_i} are the maximal parabolic subgroups.

Alternatively if $\tau := 2\rho - 2\rho_{\omega_i}$, using the natural isomorphism between $\mathfrak{h}^*$ and $\Omega^2(X)^G$ (the G-invariant 2 forms on X) we see that

$$\tau \longleftrightarrow \frac{i}{2\pi}d\tau = \frac{i}{4\pi}\sum_{\alpha\in\Re_X^+}\langle\tau,\alpha\rangle dx_\alpha \wedge d\overline{x}_\alpha,$$

represents the first Chern class of X ([1]).

We now consider another interpretation of $c_1(X)$. Let P be a paraboilc subgroup defined by a subset I with corresponding roots $\Re_P$ and let

$$\Lambda_X = \{\lambda \in \Lambda : \langle\lambda, \alpha\rangle = 0, \quad \forall\alpha \in \Re_P\},$$

which is generated by $\{\omega_i : i \notin I\}$. Any line bundle on X is homogeneous and is determined by a character $\lambda \in \Lambda_X$ which gives a character $\tilde{\lambda} : P \to \mathbb{C}^*$ so that

$$L = G^{\mathbb{C}} \times_P \mathbb{C}^{\tilde{\lambda}} = G^{\mathbb{C}} \times \mathbb{C}^{\tilde{\lambda}}/(g, z) \sim (gp^{-1}, \tilde{\lambda}(p)z) \quad \forall p \in P.$$

As above the first Chern class of $L = L_\lambda$ is:

$$c_1(L_\lambda) = \frac{i}{4\pi}\sum_{\alpha\in\Re_\lambda}\langle\lambda,\alpha\rangle dx \wedge d\overline{x}_\alpha,$$

and we say that L_λ is nef (numerically effective) if $\int_c c_1(L_\lambda) \geq 0$, for all (effective) curves in X (L_λ is nef $\Leftrightarrow n_i \geq 0, \lambda = \sum_{i\notin I} n_i\omega_i$). We denote the (holomorphic) sections of L_λ by:

$$\Gamma := H^0(X, \mathbb{C}^{\tilde{\lambda}}) := \{s : G^{\mathbb{C}} \to \mathbb{C}^{\tilde{\lambda}} : s(gp^{-1}) = \tilde{\lambda}(p)s(g) \quad \forall p \in P\}.$$

Γ is a vector space on which $G^{\mathbb{C}}$ acts via:

$$g.s(g\prime) = s(g^{-1}g\prime).$$

Definition 5.2.2. A line bundle L_λ on X is said to be *ample* if some power L_λ^m embeds X via its sections in $\mathbb{P}(\Gamma^*)$.

In this setting, ampleness is equivalent to the condition that $\lambda = \sum_{i \notin I} n_i \omega_i$, with all $n_i > 0$.

An important line bundle on X is the canonical bundle $K_X = \wedge^{dim(X)} TX^*$, where TX^* is the cotangent bundle of X and $c_1(K_X) = -\sum_{\alpha \in \Re_X} \alpha$, so that it is never nef.

Definition 5.2.3. Let X be a projective manifold whose canonical bundle is not nef and let L be an ample line bundle on X. The nef value of L denoted

$$\tau(X, L) = inf\{p/q \in \mathbb{Q} : K_X^q \otimes L^p \quad is \quad nef\}.$$

In the case that $X = G/P$, where P is the maximal parabolic subgroup corresponding to $I = \{1, \cdots, \hat{i}, \cdots, r\}$ and $L = L_{\omega_i}$, it can be shown that $\tau(X, L) = c_1(X)$ ([25]), and therefore by Proposition 4.1.3, we have:

Proposition 5.2.1.

$$\tau(X, L) = c_1(X) = (\frac{2k_i g}{n_i^{\tilde{\alpha}}} - s_{n_i^{\tilde{\alpha}}}).$$

5.2.1 Nef values and dual varieties

For a smooth projective variety X imbedded in $\mathbb{P}(\Gamma^*)$ via the global sections of a very ample line bundle L, there is a connection between the nef value $\tau(X, L)$ and the codimension of the variety $X' \subset \mathbb{P}^{\mathbb{N}}$ of hyperplanes tangent to X, known as the *dual* or *discriminant* variety of X. The *defect* of (X, L) is defined to be

$$def(X, L) = \operatorname{codim} X' - 1,$$

and most smooth varieties have defect 0.

If $def(X, L) > 0$, then ([28])

$$def(X, L) = 2(\tau(X, L) - 1) - \dim X.$$

Essentially the only known examples of smooth varieties with positive defect are homogeneous. An important role in finding them is played by the homogeneous space $X = G^{\mathbb{C}}/P$ where P is a maximal parabolic. In these cases our results from Chapter 4 give formulae for $def(X, L)$ and therefore the quantity $2(\tau(X, L) - 1) - \dim X$ can be evaluated to determine in which cases $def(X, L)$ is positive. From Theorem 4.1.5, we have:

Theorem 5.2.2. *(i) If $n_i^{\tilde{\alpha}} = 1$, then*

$$dim(X) = (2gk_i - s_1)k_i\langle\omega_i, \, \omega_i\rangle$$

(ii) For $n_i^{\tilde{\alpha}} \geq 2$, then:

$$dim(X) = \{\frac{2gk_i}{n(n-1)}[1 + \frac{1}{2} \cdots + \frac{1}{n-1}] - s_n(1 + \frac{1}{n(n-1)})\}k_i\langle\omega_i, \, \omega_i\rangle.$$

In particular:

Corollary 5.2.3. *(i) If $n_i^{\tilde{\alpha}} = 1$, then $2(\tau(X, L) - 1) - dim\, X = (2 - \langle\omega_i, \omega_i\rangle)g - 2$.*

(ii) If ω_i is the highest root, then $2(\tau(X, L) - 1) - dim\, X = -1$

Proof. (i) We note that when $n_i^{\tilde{\alpha}} = 1$, then $k_i = 1$ and $s_1 = g$. (ii) By Theorem 4.1.5, $\dim(X) = 2g - 3$, therefore

$$2(\tau(X, L) - 1) - \dim X = 2(g - 2) - (2g - 3) = -1.$$

In the case where $\omega_i = \tilde{\alpha}$, then $k_i = 1$ and $s_2 = 1$. Using Theorem 4.1.5, the formulae follows. $\square$

Appendix A

In this chapter, we provide the root tree for D_4 and D_5. The labelling on each node is its coefficients when expressed with respect to the fundamental weights and the number on each edge is the simple reflection used to generate each node.

A.1 Root tree for D_4

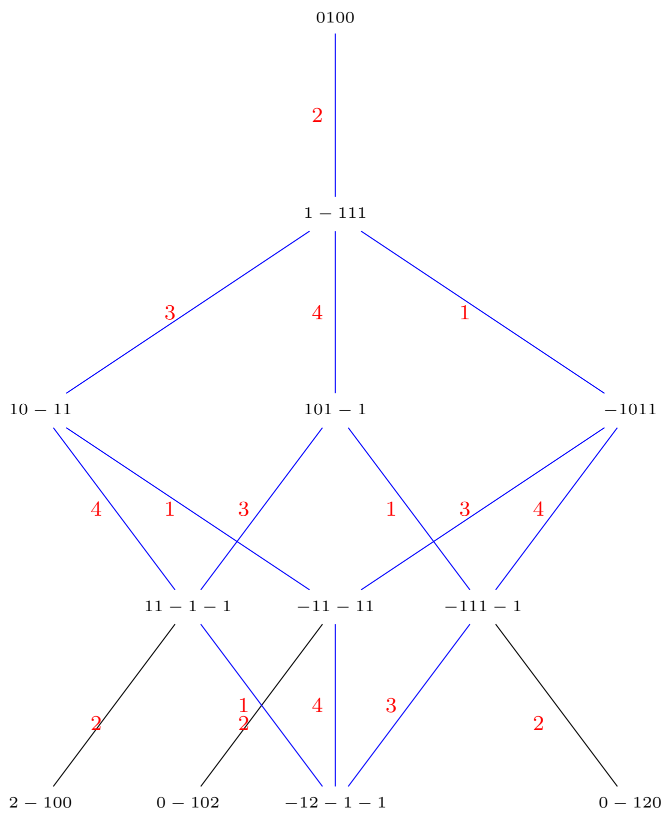

Figure A.1: The Root tree for D_4

A.2 Root tree for D_5

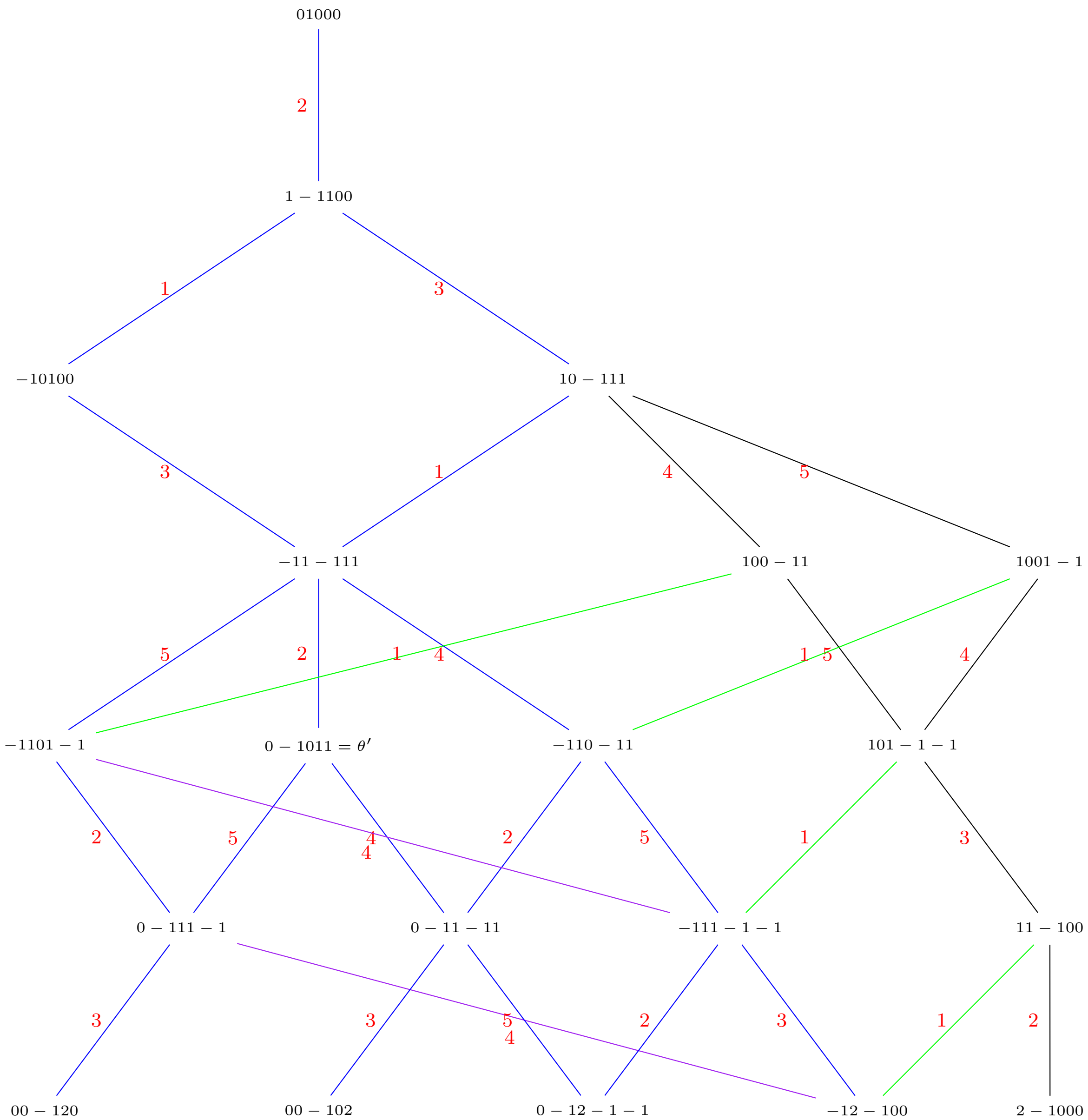

Figure A.2: The Root tree for D_5

Appendix B

We now provide tables of the positive roots, along with some useful information, such as the dual coxeter and coxeter numbers. These tables can be an aid in seeing the various theorems in action.

B.1 Positive roots of A_4

Given A_n then $h = g = n + 1$.

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1, 0, 0, 0)$	$(2, -1, 0, 0)$
2	1	$(0, 1, 0, 0)$	$(-1, 2, -1, 0)$
3	1	$(0, 0, 1, 0)$	$(0, -1, 2, -1)$
4	1	$(0, , 0, 0, 1)$	$(0, 0, -1, 2)$
5	2	$(1, 1, 0, 0)$	$(1, 1, -1, 0)$
6	2	$(0, 1, 1, 0)$	$(-1, 1, 1, -1)$
7	2	$(0, 0, 1, 1)$	$(0, -1, 1, 1)$
8	3	$(1, , 1, 1, 0)$	$(1, 0, 1, -1)$
9	3	$(0, 1, 1, 1)$	$(-1, 1, 0, 1)$
10	4	$(1, 1, 1, 1)$	$(1, 0, 0, 1)$

Table B.1: Positive roots of A_4

B.2 Positive roots of B_3

Given B_n then $h = 2n$ and $g = 2n - 1$

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1, 0, 0)$	$(2, -1, 0)$
2	1	$(0, 1, 0)$	$(-1, 2, -2)$
3	1	$(0, 0, 1)$	$(0, -1, 2)$
4	2	$(0, 1, 1)$	$(-1, 1, 0)$
5	2	$(1, 1, 0)$	$(1, 1, -2)$
6	3	$(0, 1, 2)$	$(-1, 0, 1)$
7	3	$(1, 1, 1)$	$(1, 0, 0)$
8	4	$(1, 1, 2)$	$(-1, -1, 2)$
9	5	$(1, 2, 2)$	$(0, 1, 0)$

Table B.2: Positive roots of B_3

B.3 Positive roots of B_5

Given B_n then $h = 2n$ and $g = 2n - 1$

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1, 0, 0, 0, 0)$	$(2, -1, 0, 0, 0)$
2	1	$(0, 1, 0, 0, 0)$	$(-1, 2, -1, 0, 0)$
3	1	$(0, 0, 1, 0, 0)$	$(0, -1, 2, -1, 0)$
4	1	$(0, 0, 0, 1, 0)$	$(0, 0, -1, 2, -2)$
5	1	$(0, 0, 0, 0, 1)$	$(0, 0, 0, -1, 2)$
6	2	$(1, 1, 0, 0, 0)$	$(1, 1, -1, 0, 0)$
7	2	$(0, 1, 1, 0, 0)$	$(-1, 1, 1, -1, 0)$
8	2	$(0, 0, 1, 1, 0)$	$(0, -1, 1, 1, -2)$
9	2	$(0, 0, 0, 1, 1)$	$(0, 0, -1, 1, 0)$
10	3	$(1, 1, 1, 0, 0)$	$(1, 0, 1, -1, 0)$
11	3	$(0, 1, 1, 1, 0)$	$(-1, 1, 0, 1, -2)$
12	3	$(0, 0, 1, 1, 1)$	$(0, -1, 1, 0, 0)$
13	3	$(0, 0, 0, 1, 2)$	$(0, 0, -1, 0, 2)$
14	4	$(1, 1, 1, 1, 0)$	$(1, 0, 0, 1, -2)$
15	4	$(0, 1, 1, 1, 1)$	$(-1, 1, 0, 0, 0)$
16	4	$(0, 0, 1, 1, 2)$	$(0, -1, 1, -1, 2)$
17	5	$(1, 1, 1, 1, 1)$	$(1, 0, 0, 0, 0)$
18	5	$(0, 1, 1, 1, 2)$	$(-1, 1, 0, -1, 2)$
19	5	$(0, 0, 1, 2, 2)$	$(0, -1, 0, 1, 0)$
20	6	$(1, 1, 1, 1, 2)$	$(1, 0, 0, -1, 2)$
21	6	$(0, 1, 1, 2, 2)$	$(-1, 1, -1, 1, 0)$
22	7	$(1, 1, 1, 2, 2)$	$(1, 0, -1, 1, 0)$
23	7	$(0, 1, 2, 2, 2)$	$(-1, 0, 1, 0, 0)$
24	8	$(1, 1, 2, 2, 2)$	$(1, -1, 1, 0, 0)$
25	9	$(1, 2, 2, 2, 2)$	$(0, 1, 0, 0, 0)$

Table B.3: Positive roots of B_5

B.4 Positive roots of C_3

Given C_n then $h = 2n$ and $g = n + 1$

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0,0)$	$(2,-1,0)$
2	1	$(0,1,0)$	$(-1,2,-1)$
3	1	$(0,0,1)$	$(0,-2,2)$
4	2	$(1,1,0)$	$(1,1,-1)$
5	2	$(0,1,1)$	$(-1,0,1)$
6	3	$(1,1,1)$	$(1,-1,1)$
7	3	$(0,2,1)$	$(-2,2,0)$
8	4	$(1,2,1)$	$(0,1,0)$
9	5	$(2,2,1)$	$(2,0,0)$

Table B.4: Positive roots of C_3

B.5 Positive roots of C_4

Given C_n then $h = 2n$ and $g = n + 1$

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0,0,0)$	$(2,-1,0,0)$
2	1	$(0,1,0,0)$	$(-1,2,-1,0)$
3	1	$(0,0,1,0)$	$(0,-1,2,-1)$
4	1	$(0,0,0,1)$	$(0,0,-2,2)$
5	2	$(1,1,0,0)$	$(1,1,-1,0)$
6	2	$(0,1,1,0)$	$(-1,1,1,-1)$
7	2	$(0,0,1,1)$	$(0,-1,0,1)$
8	3	$(1,1,1,0)$	$(1,0,1,-1)$
9	3	$(0,1,1,1)$	$(-1,1,-1,1)$
10	3	$(0,0,2,1)$	$(0,-2,2,0)$
11	4	$(1,1,1,1)$	$(1,0,-1,1)$
12	4	$(0,1,2,1)$	$(-1,0,1,0)$
13	5	$(1,1,2,1)$	$(1,-1,1,0)$
14	5	$(0,2,2,1)$	$(-2,2,0,0)$
15	6	$(1,2,2,1)$	$(0,1,0,0)$
16	7	$(2,2,2,1)$	$(2,0,0,0)$

Table B.5: Positive roots of C_4

B.6 Positive roots of D_4

Given D_n then $h = g = 2n - 2$.

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1, 0, 0, 0)$	$(2, -1, 0, 0)$
2	1	$(0, 1, 0, 0)$	$(-1, 2, -1, -1)$
3	1	$(0, 0, 1, 0)$	$(0, -1, 2, 0)$
4	1	$(0, 0, 0, 1)$	$(0, -1, 0, 2)$
5	2	$(0, 1, 1, 0)$	$(-1, 1, 1, -1)$
6	2	$(0, 1, 0, 1)$	$(-1, 1, -1, 1)$
7	2	$(1, 1, 0, 0)$	$(1, 1, -1, -1)$
8	3	$(0, 1, 1, 1)$	$(-1, 0, 1, 1)$
9	3	$(1, 1, 1, 0)$	$(1, 0, 1, -1)$
10	3	$(1, 1, 0, 1)$	$(1, 0, -1, 1)$
11	4	$(1, 1, 1, 1)$	$(1, -1, 1, 1)$
12	5	$(1, 2, 1, 1)$	$(0, 1, 0, 0)$

Table B.6: Positive roots of D_4

B.7 Positive roots of D_5

Given D_n then $h = g = 2n - 2$.

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0,0,0,0)$	$(2,-1,0,0,0)$
2	1	$(0,1,0,0,0)$	$(-1,2,-1,0,0)$
3	1	$(0,0,1,0,0)$	$(0,-1,2,-1,-1)$
4	1	$(0,0,0,1,0)$	$(0,0,-1,2,0)$
5	1	$(0,0,0,0,1)$	$(0,0,-1,0,2)$
6	2	$(1,1,0,0,0)$	$(1,1,-1,0,0)$
7	2	$(0,1,1,0,0)$	$(-1,1,1,-1,-1)$
8	2	$(0,0,1,1,0)$	$(0,-1,1,1,-1)$
9	2	$(0,0,1,0,1)$	$(0,-1,1,-1,1)$
10	3	$(1,1,1,0,0)$	$(1,0,1,-1,-1)$
11	3	$(0,1,1,1,0)$	$(-1,1,0,1,-1)$
12	3	$(0,1,1,0,1)$	$(-1,1,0,-1,1)$
13	3	$(0,0,1,1,1)$	$(0,-1,0,1,1)$
14	4	$(1,1,1,1,0)$	$(1,0,0,1,-1)$
15	4	$(1,1,1,0,1)$	$(1,0,0,-1,1)$
16	4	$(0,1,1,1,1)$	$(-1,1,-1,1,1)$
17	5	$(1,1,1,1,1)$	$(1,0,-1,1,1)$
18	5	$(0,1,2,1,1)$	$(-1,0,1,0,0)$
19	6	$(1,1,2,1,1)$	$(1,-1,1,0,0)$
20	7	$(1,2,2,1,1)$	$(0,1,0,0,0)$

Table B.7: Positive roots of D_5

B.8 Positive roots of E_6

Given E_6 then $h = g = 12$.

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0,0,0,0,0)$	$(2,0,-1,0,0,0)$
2	1	$(0,1,0,0,0,0)$	$(0,2,0,-1,0,0)$
3	1	$(0,0,1,0,0,0)$	$(-1,0,2,-1,0,0)$
4	1	$(0,0,0,1,0,0)$	$(0,-1,-1,2,-1,0)$
5	1	$(0,0,0,0,1,0)$	$(0,0,0,-1,2,-1)$
6	1	$(0,0,0,0,0,1)$	$(0,0,0,0,-1,2)$
7	2	$(1,0,1,0,0,0)$	$(1,0,1,-1,0,0)$
8	2	$(0,1,0,1,0,0)$	$(0,1,-1,1,-1,0)$
9	2	$(0,0,1,1,0,0)$	$(-1,-1,1,1,-1,0)$
10	2	$(0,0,0,1,1,0)$	$(0,-1,-1,1,1,-1)$
11	2	$(0,0,0,0,1,1)$	$(0,0,0,-1,1,1)$
12	3	$(1,0,1,1,0,0)$	$(1,-1,0,1,-1,0)$
13	3	$(0,1,1,1,0,0)$	$(-1,1,1,0,-1,0)$
14	3	$(0,1,0,1,1,0)$	$(0,1,-1,0,1,-1)$
15	3	$(0,0,1,1,1,0)$	$(-1,-1,1,0,1,-1)$
16	3	$(0,0,0,1,1,1)$	$(0,-1,-1,1,0,1)$
17	4	$(1,1,1,1,0,0)$	$(1,1,0,0,-1,0)$
18	4	$(1,0,1,1,1,0)$	$(1,-1,0,0,1,-1)$
19	4	$(0,1,1,1,1,0)$	$(-1,1,1,-1,1,-1)$
20	4	$(0,1,0,1,1,1)$	$(0,1,-1,0,0,1)$
21	4	$(0,0,1,1,1,1)$	$(-1,-1,1,0,0,1)$
22	5	$(1,1,1,1,1,0)$	$(1,1,0,-1,1,-1)$
23	5	$(1,0,1,1,1,1)$	$(1,-1,0,0,0,1)$
24	5	$(0,1,1,2,1,0)$	$(-1,0,0,1,0,-1)$
25	5	$(0,1,1,1,1,1)$	$(-1,1,1,-1,0,1)$
26	6	$(1,1,1,2,1,0)$	$(1,0,-1,1,0,-1)$
27	6	$(1,1,1,1,1,1)$	$(1,1,0,-1,0,1)$
28	6	$(0,1,1,2,1,1)$	$(-1,0,0,1,-1,1)$
29	7	$(1,1,2,2,1,0)$	$(0,0,1,0,0,-1)$
30	7	$(1,1,1,2,1,1)$	$(1,0,-1,1,-1,1)$
31	7	$(0,1,1,2,2,1)$	$(-1,0,0,0,1,0)$
32	8	$(1,1,2,2,1,1)$	$(0,0,1,0,-1,1)$
33	8	$(1,1,1,2,2,1)$	$(1,0,-1,0,1,0)$
34	9	$(1,1,2,2,2,1)$	$(0,0,1,-1,1,0)$
35	10	$(1,1,2,3,2,1)$	$(0,-1,0,1,0,0)$
36	11	$(1,2,2,3,2,1)$	$(0,1,0,0,0,0)$

Table B.8: Positive roots of E_6

B.9 Positive roots of E_7

Given E_7 then $h = g = 18$.

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0,0,0,0,0,0)$	$(2,0,-1,0,0,0,0)$
2	1	$(0,1,0,0,0,0,0)$	$(0,2,0,-1,0,0,0)$
3	1	$(0,0,1,0,0,0,0)$	$(-1,0,2,-1,0,0,0)$
4	1	$(0,0,0,1,0,0,0)$	$(0,-1,-1,2,-1,0,0)$
5	1	$(0,0,0,0,1,0,0)$	$(0,0,0,-1,2,-1,0)$
6	1	$(0,0,0,0,0,1,0)$	$(0,0,0,0,-1,2,-1)$
7	1	$(0,0,0,0,0,0,1)$	$(0,0,0,0,0,-1,2)$
8	2	$(1,0,1,0,0,0,0)$	$(1,0,1,-1,0,0,0)$
9	2	$(0,1,0,1,0,0,0)$	$(0,1,-1,1,-1,0,0)$
10	2	$(0,0,1,1,0,0,0)$	$(-1,-1,1,1,-1,0,0)$
11	2	$(0,0,0,1,1,0,0)$	$(0,-1,-1,1,1,-1,0)$
12	2	$(0,0,0,0,1,1,0)$	$(0,0,0,-1,1,1,-1)$
13	2	$(0,0,0,0,0,1,1)$	$(0,0,0,0,-1,1,1)$
14	3	$(1,0,1,1,0,0,0)$	$(1,-1,0,1,-1,0,0)$
15	3	$(0,1,1,1,0,0,0)$	$(-1,1,1,0,-1,0,0)$
16	3	$(0,1,0,1,1,0,0)$	$(0,1,-1,0,1,-1,0)$
17	3	$(0,0,1,1,1,0,0)$	$(-1,-1,1,0,1,-1,0)$
18	3	$(0,0,0,1,1,1,0)$	$(0,-1,-1,1,0,1,-1)$
19	3	$(0,0,0,0,1,1,1)$	$(0,0,0,-1,1,0,1)$
20	4	$(1,1,1,1,0,0,0)$	$(1,1,0,0,-1,0,0)$
21	4	$(1,0,1,1,1,0,0)$	$(1,-1,0,0,1,-1,0)$
22	4	$(0,1,1,1,1,0,0)$	$(-1,1,1,-1,1,-1,0)$
23	4	$(0,1,0,1,1,1,0)$	$(0,1,-1,0,0,1,-1)$
24	4	$(0,0,1,1,1,1,0)$	$(-1,-1,1,0,0,1,-1)$
25	4	$(0,0,0,1,1,1,1)$	$(0,-1,-1,1,0,0,1)$
26	5	$(1,1,1,1,1,0,0)$	$(1,1,0,-1,1,-1,0)$
27	5	$(1,0,1,1,1,1,0)$	$(1,-1,0,0,0,1,-1)$
28	5	$(0,1,1,2,1,0,0)$	$(-1,0,0,1,0,-1,0)$
29	5	$(0,1,1,1,1,1,0)$	$(-1,1,1,-1,0,1,-1)$
30	5	$(0,1,0,1,1,1,1)$	$(0,1,-1,0,0,0,1)$
31	5	$(0,0,1,1,1,1,1)$	$(-1,-1,1,0,0,0,1)$
32	6	$(1,1,1,2,1,0,0)$	$(1,0,-1,1,0,-1,0)$
33	6	$(1,1,1,1,1,1,0)$	$(1,1,0,-1,0,1,-1)$
34	6	$(1,0,1,1,1,1,1)$	$(1,-1,0,0,0,0,1)$

Table B.9: Positive roots of E_7

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
35	6	$(0,1,1,2,1,1,0)$	$(-1,0,0,1,-1,1,-1)$
36	6	$(0,1,1,1,1,1,1)$	$(-1,1,1,-1,0,0,1)$
37	7	$(1,1,2,2,1,0,0)$	$(0,0,1,0,0,-1,0)$
38	7	$(1,1,1,2,1,1,0)$	$(1,0,-1,1,-1,1,-1)$
39	7	$(1,1,1,1,1,1,1)$	$(1,1,0,-1,0,0,1)$
40	7	$(0,1,1,2,2,1,0)$	$(-1,0,0,0,1,0,-1)$
41	7	$(0,1,1,2,1,1,1)$	$(-1,0,0,1,-1,0,1)$
42	8	$(1,1,2,2,1,1,0)$	$(0,0,1,0,-1,1,-1)$
43	8	$(1,1,1,2,2,1,0)$	$(1,0,-1,0,1,0,-1)$
44	8	$(1,1,1,2,1,1,1)$	$(1,0,-1,1,-1,0,1)$
45	8	$(0,1,1,2,2,1,1)$	$(-1,0,0,0,1,-1,1)$
46	9	$(1,1,2,2,2,1,0)$	$(0,0,1,-1,1,0,-1)$
47	9	$(1,1,2,2,1,1,1)$	$(0,0,1,0,-1,0,1)$
48	9	$(1,1,1,2,2,1,1)$	$(1,0,-1,0,1,-1,1)$
49	9	$(0,1,1,2,2,2,1)$	$(-1,0,0,0,0,1,0)$
50	10	$(1,1,2,3,2,1,0)$	$(0,-1,0,1,0,0,-1)$
51	10	$(1,1,2,2,2,1,1)$	$(0,0,1,-1,1,-1,1)$
52	10	$(1,1,1,2,2,2,1)$	$(1,0,-1,0,0,1,0)$
53	11	$(1,2,2,3,2,1,0)$	$(0,1,0,0,0,0,-1)$
54	11	$(1,1,2,3,2,1,1)$	$(0,-1,0,1,0,-1,1)$
55	11	$(1,1,2,2,2,2,1)$	$(0,0,1,-1,0,1,0)$
56	12	$(1,2,2,3,2,1,1)$	$(0,1,0,0,0,-1,1)$
57	12	$(1,1,2,3,2,2,1)$	$(0,-1,0,1,-1,1,0)$
58	13	$(1,2,2,3,2,2,1)$	$(0,1,0,0,-1,1,0)$
59	13	$(1,1,2,3,3,2,1)$	$(0,-1,0,0,1,0,0)$
60	14	$(1,2,2,3,3,2,1)$	$(0,1,0,-1,1,0,0)$
61	15	$(1,2,2,4,3,2,1)$	$(0,0,-1,1,0,0,0)$
62	16	$(1,2,3,4,3,2,1)$	$(-1,0,1,0,0,0,0)$
63	17	$(2,2,3,4,3,2,1)$	$(1,0,0,0,0,0,0)$

Table B.10: Positive roots of E_7

B.10 Positive roots of E_8

Given E_8 then $h = g = 30$.

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0,0,0,0,0,0,0)$	$(2,0,-1,0,0,0,0,0)$
2	1	$(0,1,0,0,0,0,0,0)$	$(0,2,0,-1,0,0,0,0)$
3	1	$(0,0,1,0,0,0,0,0)$	$(-1,0,2,-1,0,0,0,0)$
4	1	$(0,0,0,1,0,0,0,0)$	$(0,-1,-1,2,-1,0,0,0)$
5	1	$(0,0,0,0,1,0,0,0)$	$(0,0,0,-1,2,-1,0,0)$
6	1	$(0,0,0,0,0,1,0,0)$	$(0,0,0,0,-1,2,-1,0)$
7	1	$(0,0,0,0,0,0,1,0)$	$(0,0,0,0,0,-1,2,-1)$
8	1	$(0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,-1,2)$
9	2	$(1,0,1,0,0,0,0,0)$	$(1,0,1,-1,0,0,0,0)$
10	2	$(0,1,0,1,0,0,0,0)$	$(0,1,-1,1,-1,0,0,0)$
11	2	$(0,0,1,1,0,0,0,0)$	$(-1,-1,1,1,-1,0,0,0)$
12	2	$(0,0,0,1,1,0,0,0)$	$(0,-1,-1,1,1,-1,0,0)$
13	2	$(0,0,0,0,1,1,0,0)$	$(0,0,0,-1,1,1,-1,0)$
14	2	$(0,0,0,0,0,1,1,0)$	$(0,0,0,0,-1,1,1,-1)$
15	2	$(0,0,0,0,0,0,1,1)$	$(0,0,0,0,0,-1,1,1)$
16	3	$(1,0,1,1,0,0,0,0)$	$(1,-1,0,1,-1,0,0,0)$
17	3	$(0,1,1,1,0,0,0,0)$	$(-1,1,1,0,-1,0,0,0)$
18	3	$(0,1,0,1,1,0,0,0)$	$(0,1,-1,0,1,-1,0,0)$
19	3	$(0,0,1,1,1,0,0,0)$	$(-1,-1,1,0,1,-1,0,0)$
20	3	$(0,0,0,1,1,1,0,0)$	$(0,-1,-1,1,0,1,-1,0)$
21	3	$(0,0,0,0,1,1,1,0)$	$(0,0,0,-1,1,0,1,-1)$
22	3	$(0,0,0,0,0,1,1,1)$	$(0,0,0,0,-1,1,0,1)$
23	4	$(1,1,1,1,0,0,0,0)$	$(1,1,0,0,-1,0,0,0)$
24	4	$(1,0,1,1,1,0,0,0)$	$(1,-1,0,0,1,-1,0,0)$
25	4	$(0,1,1,1,1,0,0,0)$	$(-1,1,1,-1,1,-1,0,0)$
26	4	$(0,1,0,1,1,1,0,0)$	$(0,1,-1,0,0,1,-1,0)$
27	4	$(0,0,1,1,1,1,0,0)$	$(-1,-1,1,0,0,1,-1,0)$
28	4	$(0,0,0,1,1,1,1,0)$	$(0,-1,-1,1,0,0,1,-1)$
29	4	$(0,0,0,0,1,1,1,1)$	$(0,0,0,-1,1,0,0,1)$
30	5	$(1,1,1,1,1,0,0,0)$	$(1,1,0,-1,1,-1,0,0)$
31	5	$(1,0,1,1,1,1,0,0)$	$(1,-1,0,0,0,1,-1,0)$

Table B.11: Positive roots of E_8

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
32	5	$(0,1,1,2,1,0,0,0)$	$(-1,0,0,1,0,-1,0,0)$
33	5	$(0,1,1,1,1,1,0,0)$	$(-1,1,1,-1,0,1,-1,0)$
34	5	$(0,1,0,1,1,1,1,0)$	$(0,1,-1,0,0,0,1,-1)$
35	5	$(0,0,1,1,1,1,1,0)$	$(-1,-1,1,0,0,0,1,-1)$
36	5	$(0,0,0,1,1,1,1,1)$	$(0,-1,-1,1,0,0,0,1)$
37	6	$(1,1,1,2,1,0,0,0)$	$(1,0,-1,1,0,-1,0,0)$
38	6	$(1,1,1,1,1,1,0,0)$	$(1,1,0,-1,0,1,-1,0)$
39	6	$(1,0,1,1,1,1,1,0)$	$(1,-1,0,0,0,0,1,-1)$
40	6	$(0,1,1,2,1,1,0,0)$	$(-1,0,0,1,-1,1,-1,0)$
41	6	$(0,1,1,1,1,1,1,0)$	$(-1,1,1,-1,0,0,1,-1)$
42	6	$(0,1,0,1,1,1,1,1)$	$(0,1,-1,0,0,0,0,1)$
43	6	$(0,0,1,1,1,1,1,1)$	$(-1,-1,1,0,0,0,0,1)$
44	7	$(1,1,2,2,1,0,0,0)$	$(0,0,1,0,0,-1,0,0)$
45	7	$(1,1,1,2,1,1,0,0)$	$(1,0,-1,1,-1,1,-1,0)$
46	7	$(1,1,1,1,1,1,1,0)$	$(1,1,0,-1,0,0,1,-1)$
47	7	$(1,0,1,1,1,1,1,1)$	$(1,-1,0,0,0,0,0,1)$
48	7	$(0,1,1,2,2,1,0,0)$	$(-1,0,0,0,1,0,-1,0)$
49	7	$(0,1,1,2,1,1,1,0)$	$(-1,0,0,1,-1,0,1,-1)$
50	7	$(0,1,1,1,1,1,1,1)$	$(-1,1,1,-1,0,0,0,1)$
51	8	$(1,1,2,2,1,1,0,0)$	$(0,0,1,0,-1,1,-1,0)$
52	8	$(1,1,1,2,2,1,0,0)$	$(1,0,-1,0,1,0,-1,0)$
53	8	$(1,1,1,2,1,1,1,0)$	$(1,0,-1,1,-1,0,1,-1)$
54	8	$(1,1,1,1,1,1,1,1)$	$(1,1,0,-1,0,0,0,1)$
55	8	$(0,1,1,2,2,1,1,0)$	$(-1,0,0,0,1,-1,1,-1)$
56	8	$(0,1,1,2,1,1,1,1)$	$(-1,0,0,1,-1,0,0,1)$
57	9	$(1,1,2,2,2,1,0,0)$	$(0,0,1,-1,1,0,-1,0)$
58	9	$(1,1,2,2,1,1,1,0)$	$(0,0,1,0,-1,0,1,-1)$
59	9	$(1,1,1,2,2,1,1,0)$	$(1,0,-1,0,1,-1,1,-1)$
60	9	$(1,1,1,2,1,1,1,1)$	$(1,0,-1,1,-1,0,0,1)$
61	9	$(0,1,1,2,2,2,1,0)$	$(-1,0,0,0,0,1,0,-1)$
62	9	$(0,1,1,2,2,1,1,1)$	$(-1,0,0,0,1,-1,0,1)$
63	10	$(1,1,2,3,2,1,0,0)$	$(0,-1,0,1,0,0,-1,0)$

Table B.12: Positive roots of E_8

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
64	10	$(1,1,2,2,2,1,1,0)$	$(0,0,1,-1,1,-1,1,-1)$
65	10	$(1,1,2,2,1,1,1,1)$	$(0,0,1,0,-1,0,0,1)$
66	10	$(1,1,1,2,2,2,1,0)$	$(1,0,-1,0,0,1,0,-1)$
67	10	$(1,1,1,2,2,1,1,1)$	$(1,0,-1,0,1,-1,0,1)$
68	10	$(0,1,1,2,2,2,1,1)$	$(-1,0,0,0,0,1,-1,1)$
69	11	$(1,2,2,3,2,1,0,0)$	$(0,1,0,0,0,0,-1,0)$
70	11	$(1,1,2,3,2,1,1,0)$	$(0,-1,0,1,0,-1,1,-1)$
71	11	$(1,1,2,2,2,2,1,0)$	$(0,0,1,-1,0,1,0,-1)$
72	11	$(1,1,2,2,2,1,1,1)$	$(0,0,1,-1,1,-1,0,1)$
73	11	$(1,1,1,2,2,2,1,1)$	$(1,0,-1,0,0,1,-1,1)$
74	11	$(0,1,1,2,2,2,2,1)$	$(-1,0,0,0,0,0,1,0)$
75	12	$(1,2,2,3,2,1,1,0)$	$(0,1,0,0,0,-1,1,-1)$
76	12	$(1,1,2,3,2,2,1,0)$	$(0,-1,0,1,-1,1,0,-1)$
77	12	$(1,1,2,3,2,1,1,1)$	$(0,-1,0,1,0,-1,0,1)$
78	12	$(1,1,2,2,2,2,1,1)$	$(0,0,1,-1,0,1,-1,1)$
79	12	$(1,1,1,2,2,2,2,1)$	$(1,0,-1,0,0,0,1,0)$
80	13	$(1,2,2,3,2,2,1,0)$	$(0,1,0,0,-1,1,0,-1)$
81	13	$(1,2,2,3,2,1,1,1)$	$(0,1,0,0,0,-1,0,1)$
82	13	$(1,1,2,3,3,2,1,0)$	$(0,-1,0,0,1,0,0,-1)$
83	13	$(1,1,2,3,2,2,1,1)$	$(0,-1,0,1,-1,1,-1,1)$
84	13	$(1,1,2,2,2,2,2,1)$	$(0,0,1,-1,0,0,1,0)$
85	14	$(1,2,2,3,3,2,1,0)$	$(0,1,0,-1,1,0,0,-1)$
86	14	$(1,2,2,3,2,2,1,1)$	$(0,1,0,0,-1,1,-1,1)$
87	14	$(1,1,2,3,3,2,1,1)$	$(0,-1,0,0,1,0,-1,1)$
88	14	$(1,1,2,3,2,2,2,1)$	$(0,-1,0,1,-1,0,1,0)$
89	15	$(1,2,2,4,3,2,1,0)$	$(0,0,-1,1,0,0,0,-1)$
90	15	$(1,2,2,3,3,2,1,1)$	$(0,1,0,-1,1,0,-1,1)$
91	15	$(1,2,2,3,2,2,2,1)$	$(0,1,0,0,-1,0,1,0)$

Table B.13: Positive roots of E_8

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
92	15	$(1,1,2,3,3,2,2,1)$	$(0,-1,0,0,1,-1,1,0)$
93	16	$(1,2,3,4,3,2,1,0)$	$(-1,0,1,0,0,0,0,-1)$
94	16	$(1,2,2,4,3,2,1,1)$	$(0,0,-1,1,0,0,-1,1)$
95	16	$(1,2,2,3,3,2,2,1)$	$(0,1,0,-1,1,-1,1,0)$
96	16	$(1,1,2,3,3,3,2,1)$	$(0,-1,0,0,0,1,0,0)$
97	17	$(2,2,3,4,3,2,1,0)$	$(1,0,0,0,0,0,0,-1)$
98	17	$(1,2,3,4,3,2,1,1)$	$(-1,0,1,0,0,0,-1,1)$
99	17	$(1,2,2,4,3,2,2,1)$	$(0,0,-1,1,0,-1,1,0)$
100	17	$(1,2,2,3,3,3,2,1)$	$(0,1,0,-1,0,1,0,0)$
101	18	$(2,2,3,4,3,2,1,1)$	$(1,0,0,0,0,0,-1,1)$
102	18	$(1,2,3,4,3,2,2,1)$	$(-1,0,1,0,0,-1,1,0)$
103	18	$(1,2,2,4,3,3,2,1)$	$(0,0,-1,1,-1,1,0,0)$
104	19	$(2,2,3,4,3,2,2,1)$	$(1,0,0,0,0,-1,1,0)$
105	19	$(1,2,3,4,3,3,2,1)$	$(-1,0,1,0,-1,1,0,0)$
106	19	$(1,2,2,4,4,3,2,1)$	$(0,0,-1,0,1,0,0,0)$
107	20	$(2,2,3,4,3,3,2,1)$	$(1,0,0,0,-1,1,0,0)$
108	20	$(1,2,3,4,4,3,2,1)$	$(-1,0,1,-1,1,0,0,0)$
109	21	$(2,2,3,4,4,3,2,1)$	$(1,0,0,-1,1,0,0,0)$
110	21	$(1,2,3,5,4,3,2,1)$	$(-1,-1,0,1,0,0,0,0)$
111	22	$(2,2,3,5,4,3,2,1)$	$(1,-1,-1,1,0,0,0,0)$
112	22	$(1,3,3,5,4,3,2,1)$	$(-1,1,0,0,0,0,0,0)$
113	23	$(2,3,3,5,4,3,2,1)$	$(1,1,-1,0,0,0,0,0)$
114	23	$(2,2,4,5,4,3,2,1)$	$(0,-1,1,0,0,0,0,0)$
115	24	$(2,3,4,5,4,3,2,1)$	$(0,1,1,-1,0,0,0,0)$
116	25	$(2,3,4,6,4,3,2,1)$	$(0,0,0,1,-1,0,0,0)$
117	26	$(2,3,4,6,5,3,2,1)$	$(0,0,0,0,1,-1,0,0)$
118	27	$(2,3,4,6,5,4,2,1)$	$(0,0,0,0,0,1,-1,0)$
119	28	$(2,3,4,6,5,4,3,1)$	$(0,0,0,0,0,0,1,-1)$
120	29	$(2,3,4,6,5,4,3,2)$	$(0,0,0,0,0,0,0,1)$

Table B.14: Positive roots of E_8

B.11 Positive roots of F_4

Given F_4 then $h = 12$ and $g = 9$

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0,0,0)$	$(2,-1,0,0)$
2	1	$(0,1,0,0)$	$(-1,2,-2,0)$
3	1	$(0,0,1,0)$	$(0,-1,2,-1)$
4	1	$(0,0,0,1)$	$(0,0,-1,2)$
5	2	$(1,1,0,0)$	$(1,1,-2,0)$
6	2	$(0,1,1,0)$	$(-1,1,0,-1)$
7	2	$(0,0,1,1)$	$(0,-1,1,1)$
8	3	$(0,1,1,1)$	$(-1,1,-1,1)$
9	3	$(1,1,1,0)$	$(1,0,0,-1)$
10	4	$(1,1,1,1)$	$(1,0,-1,1)$
11	3	$(0,1,2,0)$	$(-1,0,2,-2)$
12	4	$(1,1,2,0)$	$(1,-1,2,-2)$
13	4	$(0,1,2,1)$	$(-1,0,1,0)$
14	5	$(1,2,2,0)$	$(0,1,0,-2)$
15	5	$(1,1,2,1)$	$(1,-1,1,0)$
16	5	$(0,1,2,2)$	$(-1,0,0,2)$
17	6	$(1,2,2,1)$	$(0,1,-1,0)$
18	6	$(1,1,2,2)$	$(1,-1,0,2)$
19	7	$(1,2,3,1)$	$(0,0,1,-1)$
20	7	$(1,2,2,2)$	$(0,1,-2,2)$
21	8	$(1,2,3,2)$	$(0,0,0,1)$
22	9	$(1,2,4,2)$	$(0,-1,2,0)$
23	10	$(1,3,4,2)$	$(-1,1,0,0)$
24	11	$(2,3,4,2)$	$(1,0,0,0)$

Table B.15: Positive roots of F_4

B.12 Positive roots of G_2

Given G_2 then $h = 6$ and $g = 4$.

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
1	1	$(1,0)$	$(2,-1)$
2	1	$(0,1)$	$(-3,2)$
3	2	$(1,1)$	$(-1,1)$
4	3	$(2,1)$	$(1,0)$
5	4	$(3,1)$	$(3,-1)$
6	5	$(3,2)$	$(0,1)$

Table B.16: Positive roots of G_2

Bibliography

[1] D. V. Alekseevsky and A. M. Perelomov. Invariant Kähler-Einstein metrics on compact homogeneous spaces. *Funct. Analysis Applic*, pages 171–182, 20 (1986). (Cited on page 58.)

[2] A. Arvanitogeōrgos. *An introduction to Lie groups and the geometry of homogeneous spaces*, volume 22. American Mathematical Soc, 2003. (Cited on pages 2, 13, and 57.)

[3] W. M. Boothby. *An introduction to differentiable manifolds and Riemannian geometry*, volume 120. Gulf Professional Publishing, 2003. (Cited on page 13.)

[4] A. Borel. On the curvature tensor of the hermitian symmetric manifolds. *Annals of Mathematics*, pages 508–521, 1960. (Cited on page 18.)

[5] A. Borel and J. de Siebenthal. Les sous-groupes fermés de rang maximum des groupes de lie clos. *Commentarii Mathematici Helvetici*, pages 200–221, 23 (1949). (Cited on page 15.)

[6] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces, I. *American Journal of Mathematics*, pages 458–538, 80(1958). (Cited on page 57.)

[7] N. Bourbaki. *Group et algèbres de Lie*. Ch. 4, 5 et 6, Hermann, Paris, 1968. (Cited on pages 14, 21, and 55.)

[8] J. M. Burns. An elementary proof of the strange formula of Freudenthal and de Vries. *The Quarterly Journal of Mathematics*, pages 295–297, 51 (2000). (Cited on page 12.)

[9] J. M. Burns and M. J. Clancy. Weight sum formulae in Lie algebra representations. *Journal of Algebra*, pages 1–12, 257(2002). (Cited on pages 1, 18, 29, 31, 45, and 56.)

[10] J. M. Burns and M. A. Makrooni. Compact homogeneous spaces with positive Euler characteristic and their strange formulae. *The Quarterly Journal of Mathematics*, pages 507–516, 66 (2015). (Cited on pages 12 and 54.)

[11] F. E. Burstall and J. H. Rawnsley. *Twistor theory for Riemannian symmetric spaces: with applications to harmonic maps of Riemann surfaces*, volume 1424. Springer, 2006. (Cited on page 16.)

[12] R. Carles. Méthode récurrente pour la classification des systèmes de racines réduits et irréductibles. *CR Acad. Sci. Paris*, pages 355–358, 276 (1973). (Cited on pages 1, 12, 27, 36, 42, 46, and 52.)

[13] E. B. Dynkin. Semisimple subalgebras of semisimple Lie algebras. *Selected Papers*, pages 175–308, 1972. (Cited on page 28.)

[14] H. D. Fegan. The heat equation and modular forms. *Journal of Differential Geometry*, pages 589–602, 13 (1978). (Cited on page 12.)

[15] H. D. Fegan and B. Steer. On the strange formula of Freudenthal and de Vries. In *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 249–252. Cambridge Univ Press, 105 (1989). (Cited on page 12.)

[16] A. Fino. Systems of roots and topology of complex flag manifolds. *Geometriae Dedicata*, pages 299–308, 71 (1998). (Cited on page 54.)

[17] A Fino and S Salamon. Observations on the topology of symmetric spaces, in geometry and physics (Aarhus, 1995), 275-286. *Lecture Notes in Pure and Appl. Math*, 184. (Cited on pages 17, 54, and 55.)

[18] H. Freudenthal and H. de Vries. *Linear Lie Groups*. Academic Press, New York, 1969. (Cited on page 12.)

[19] J. Fuchs and C. Schweigert. *Symmetries, Lie algebras and representations: A graduate course for physicists*. Cambridge University Press, 2003. (Cited on page 48.)

[20] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80. Academic press, 1979. (Cited on page 27.)

[21] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 1972. (Cited on page 2.)

[22] R. Kane. *Reflection groups and invariant theory*. Springer Science & Business Media, 2013. (Cited on pages 2 and 27.)

[23] I. G. Macdonald. Affine root systems and Dedekind's η-function. *Inventiones mathematicae*, pages 91–143, 15 (1971). (Cited on page 2.)

[24] S. M. Salamon. On the cohomology of Kähler and hyper-Kähler manifolds. *Topology*, pages 137–155, 35 (1996). (Cited on pages 2, 53, and 54.)

[25] D. M. Snow. Homogeneous vector bundles, 2014. (`http://www3.nd.edu/ snow/Papers/HomogVB.pdf`). (Cited on pages 57 and 59.)

[26] D. M. Snow. Nef value of homogeneous line bundles and related vanishing theorems. In *Forum Mathematicum*, pages 385–392, 7 (1995). (Cited on page 58.)

[27] R. Suter. Coxeter and dual Coxeter numbers. *Communications in Algebra*, pages 147–153, 26 (1998). (Cited on page 56.)

[28] A. E. Tevelev. *Projective duality and homogeneous spaces*, volume 133. Springer Science & Business Media, 2006. (Cited on page 60.)

[29] H. C. Wang. Homogeneous spaces with non-vanishing Euler characteristics. *Annals of Mathematics*, pages 925–953, 1949. (Cited on page 14.)

[30] M. Y. Wang and W. Ziller. On normal homogeneous Einstein manifolds. In *Annales scientifiques de l'Ecole normale supérieure*, number 4, pages 563–633, 18 (1985). (Cited on page 23.)

[31] J. A. Wolf. Complex homogeneous contact manifolds and quaternionic symmetric spaces. *J. Math. Mech*, pages 1033–1047, 14 (1965). (Cited on page 17.)

[32] J. A. Wolf. *Spaces of constant curvature*, volume 372. American Mathematical Soc, 2011. (Cited on pages 15 and 43.)