An unforeseen phenomenon in soft laminate model under enforced post-buckling wavelengths: model complexity and insights

Mechanics of Soft Materials (2025) 7:6
https://doi.org/10.1007/s42558-025-00068-5

ORIGINAL PAPER

An unforeseen phenomenon in soft laminate model under enforced post-buckling wavelengths: model complexity and insights

Qi Yao[1] · Stephan Rudykh[1,2]

Received: 15 April 2025 / Accepted: 10 November 2025

Abstract
This letter investigates an unforeseen phenomenon in post-buckled soft laminate models when the enforced wavelength deviates from the theoretical value. Through dispersion analysis, we identify the emergence of zero frequencies at post-buckling stretches, which serve as practical diagnostics for detecting significant mismatches in wavelength. Our results demonstrate that these zero modes shift based on the ratio between enforced and theoretical wavelengths and disappear at the Brillouin zone edges when the correct wavelength is applied. We further discuss the sensitivity of post-buckling behavior to the accuracy of imposed wavelengths and discuss the association of zero frequencies with model complexity embedded in these nonlinear models simulating the ideal buckling process. These findings provide new insights and considerations for future studies of buckling behavior in soft composites.

Keywords Soft materials · Laminates · Post-buckling analysis · Dispersion relations · Acoustic waves

1 Motivation

To study the buckling behavior, instability analyses based on the Bloch-Floquet method have been widely adopted [1–15]. In these models, the reinforcement is typically set to be perfect, and analysis is focused on a unit cell [12, 13, 16–19] (see the laminate model in Fig. 1(a) as an example). However, critical stretches and critical wavenumbers, along with their associated wavemodes, provide only a partial picture of material behavior in the highly nonlinear post-buckling regime. To capture structural evolution beyond instability onset, post-buckling models incorporating imperfections have been developed [8, 9, 20, 21]. Figure 1(b) shows the post-buckling mode for laminate.

In such post-buckling models, the critical wavenumber identified from the instability model is often used to determine the unit cell length ($H^{cr} = W/\overline{K}_2^{cr}$). Accurately identifying the accurate critical parameters typically requires a high computational cost to refine the window containing buckling onset in the $\lambda - \overline{K}_2$ parameter space. To see what appears if the accuracy requirement on the critical wavenumber is relaxed, we deliberately enforced a buckling wavelength H that significantly deviated from the theoretical value H^{cr} (i.e., $H = cH^{cr}$ with $c \neq 1$). Then, we observed the emergence of zero frequencies in

Highlights
- Investigated dispersion relations in post-buckling laminate models under enforced wavelengths deviating from the theoretical critical value.
- Exhibited that the appearance of zero frequencies can serve as a practical diagnostic for significant deviated wavelengths imposed in the model.
- Reaffirmed that zero frequencies only appear at the edges of the Brillouin zone when the correct wavelength is applied, supported by two key inferences.
- Emphasized the sensitivity of post-buckling models to the accuracy of imposed wavelengths, underscoring the need for deeper exploration of their nonlinear behavior.

✉ Qi Yao
qyao34@wisc.edu

[1] Department of Mechanical Engineering, University of Wisconsin– Madison, Madison, WI 53706, USA

[2] School of Mathematical and Statistical Sciences, University of Galway, University Road, Galway, Ireland

Published online: 06 December 2025

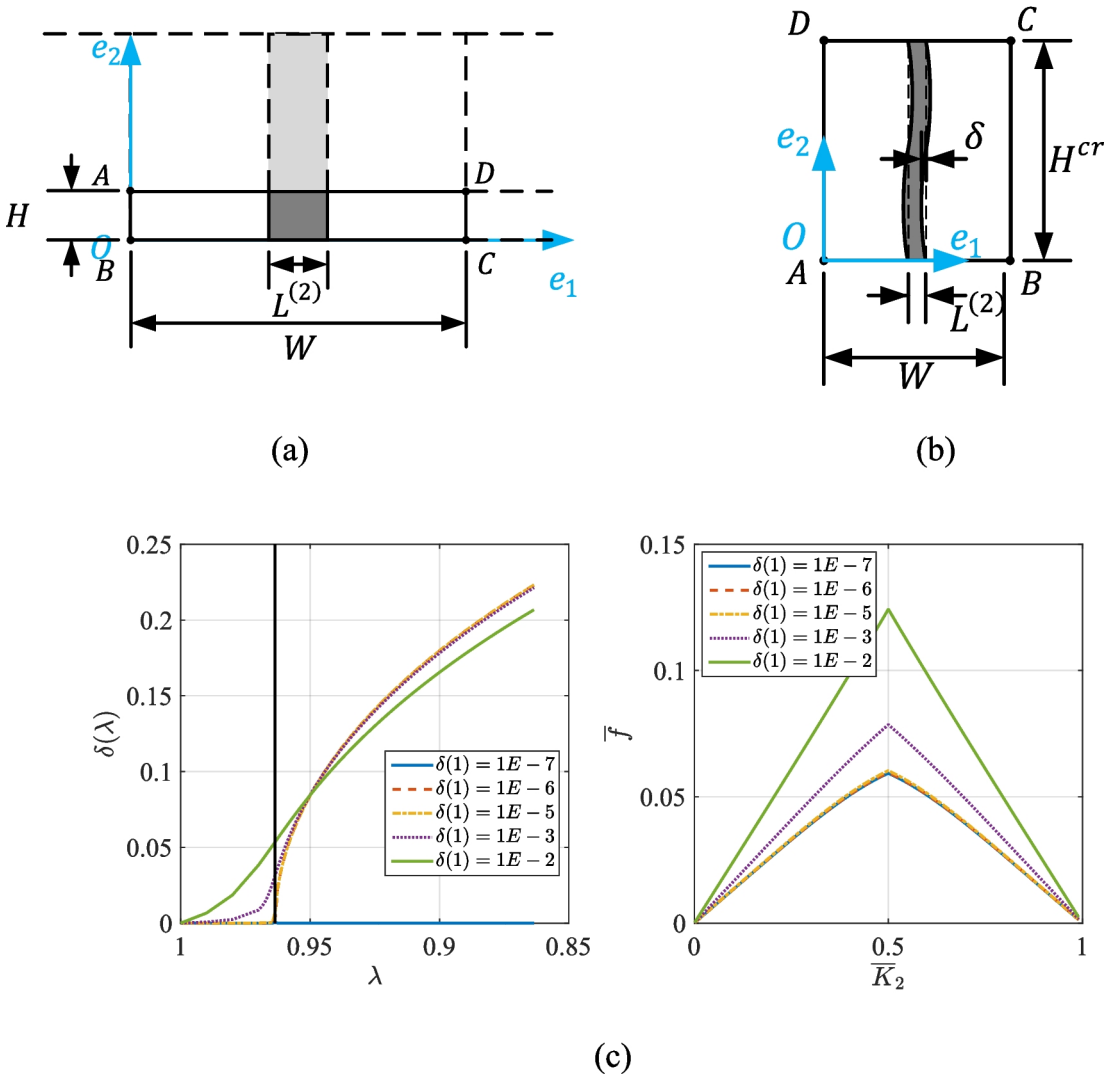

Fig. 1 (a) Schematic of instability analysis model for laminate; (b) schematic of the post-buckling model for laminate; (c) layer magnitude cures $\delta(\lambda)$ and corresponding dispersion curves $\bar{f}\left(\overline{K}_2\right)$ at the critical stretch under varying initial magnitudes

the post-buckling domain (see Fig. 2). Such values are not seen when $H \approx H^{cr}$ (Fig. 1(c)). This raises a major question that motivates this study: do the zero frequencies reflect the discrepancy between the imposed wavelength and the theoretical one, and how do they evolve as H changes?

To approach this area, we study the fully neo-Hookean laminate with $c^{(2)} = 0.06, k_\mu = 100, \lambda^{cr} = 0.96352$, and accurately calculated critical wavenumber $\overline{K}_2^{cr} = 0.89$. We start by analyzing the post-buckling models by changing the initial magnitude of layer $\delta(1)$. Then, we impose various enforced wavelengths, which are expressed as a ratio c relative to the critical wavelength H^{cr} (i.e., $H = cH^{cr}$), in the post-buckling model and exhibit the appearance of zero values in dispersion relations. Finally, the symmetric dispersion relations with $\overline{K}_2$ ranging from 0 to 1 are analyzed to show the implications of wavelength deviation on the post-buckling response.

2 Results

Unlike the instability analysis model for a laminate with perfectly straight interfaces, the post-buckling models incorporate predefined periodic structures, say the slightly curved stiff layers (see Fig. 1(b)). These layers amplify their deformation magnitude near the critical stretch, mimicking the ideal buckling process, wherein straight stiff layers curve upon passing

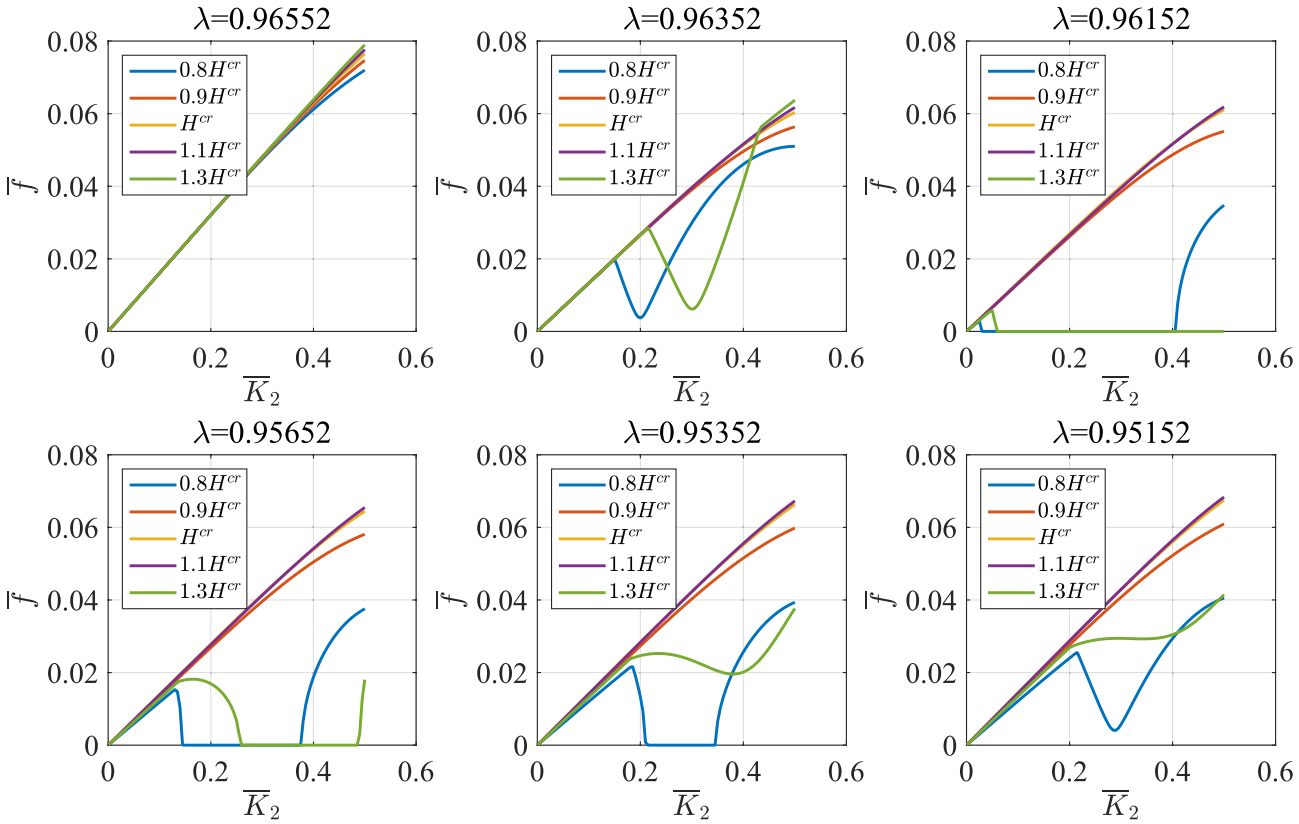

Fig. 2 Dispersion relations versus deformation in the post-buckling models with various $H = cH^{cr}$

critical stretches. If we prescribe an extremely small initial curvature (e.g., $\delta(1) = 1E-7$), the layer does not undergo significant curving at the critical stretch, indicating a failure to adequately replicate the buckling behavior.

Interestingly, regardless of the initial curvature magnitude, zero-frequency modes only appear at the trivial wavenumbers $\overline{K}_2 = 0$ or 1, corresponding to the edges of the Brillouin zone, due to the periodic nature of the post-buckling model. However, when we enforce a non-ideal buckling wavelength by adjusting the unit cell height to cH^{cr}, zero-frequency modes appear at stretches $\lambda < \lambda^{cr}$, as shown in Fig. 2. This behavior contrasts with the observations under the theoretical wavelength H^{cr}, highlighting the sensitivity of the model response to deviations in imposed wavelength.

To study the influence of varying enforced wavelengths, we examined the changes in dispersion relations for different values of the ratio c. The dispersion curves in the post-buckling domain for $c = 0.85, 0.9$, and 0.94 are presented in Fig. 3(a), accompanied by a subplot showing the corresponding layer magnitude curves $\delta(\lambda)$ versus with vertical lines indicating the critical stretch. Additionally, we analyzed three special cases with more pronounced deviations from the critical wavelength, namely $c = 0.4, 0.5$, and 1.5, which are illustrated in Fig. 3(b) along with their respective layer magnitude curves.

By extending the dispersion curves to a symmetric form, zero frequencies appear around $\overline{K}_2 = c$, as marked by the black dots in Fig. 3(a). Due to the symmetry of the dispersion relations in the post-buckling models, similar values also appear around $1 - c$; however, these are artifacts of the symmetry and should not be misinterpreted. The special cases in Fig. 3(b) further support this pattern: under $H = 0.4H^{cr}$, zeros appear around $\overline{K}_2 = 0.4$ and 0.6; for $H = 0.5H^{cr}$, they cluster solely around $\overline{K}_2 = 0.5$. Interestingly, the emergence of these zeros appears to indicate that the laminate is not buckling at the correct wavelength. For $H = 1.5H^{cr}$, zeros once again appear around 0.5. Based on the prior observations, one might expect them to appear at $\overline{K}_2 = 1.5$; however, because the dispersion curves repeat periodically over each integer interval (i.e., $\overline{f}\left(\overline{K}_2\right) = \overline{f}\left(\overline{K}_2 - K\right) for \overline{K}_2 \in [K, K+1], K \in \mathbb{Z}$), the zeros fold back into the interval [0,1], explaining their position near $\overline{K}_2 = 0.5$.

As the value of c increases from 0.85 to 0.94, the zero-frequency points shift toward the edges of the Brillouin zone, i.e., $\overline{K}_2 = 0$ or 1. Combined with the observations in Fig. 3(b), this supports the inference that as $c \to 1$, the zeros converge to the boundary points:

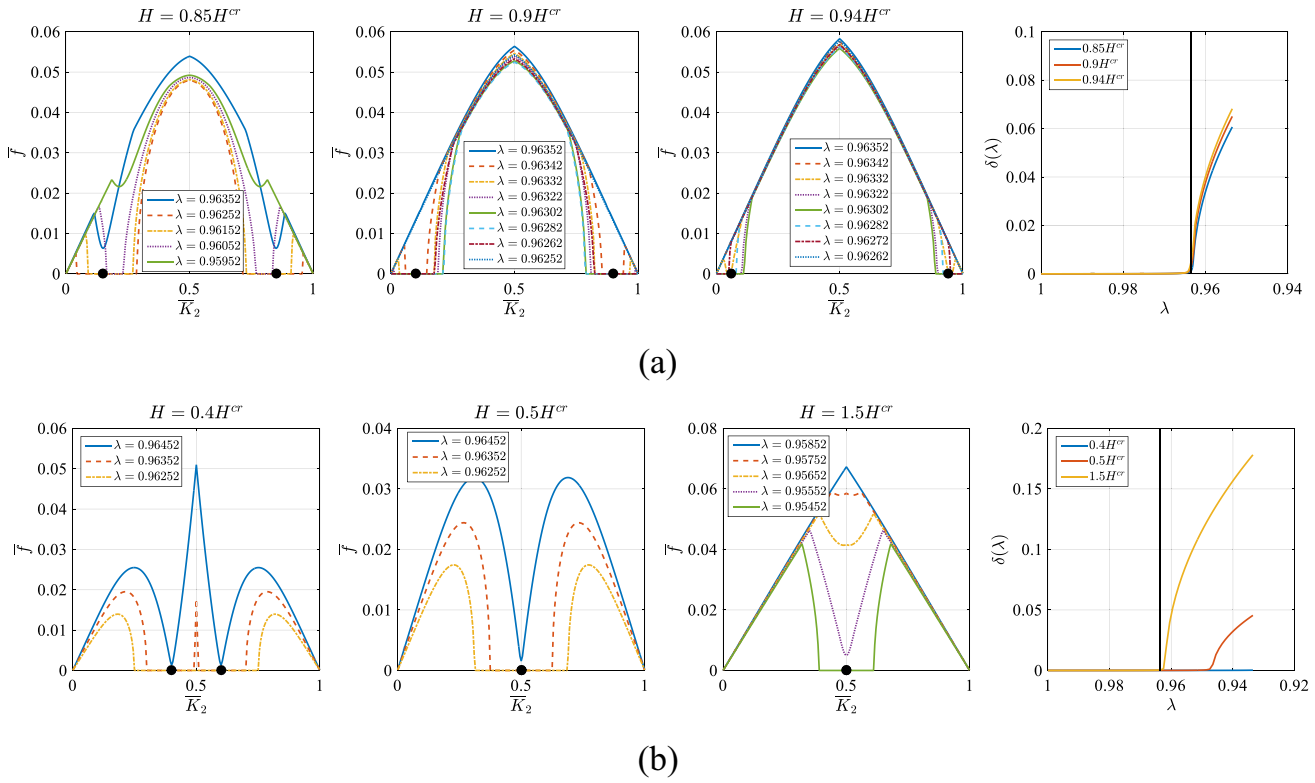

Fig. 3 (a) Dispersion curves in the post-buckling domain for c=0.85,0.9, and 0.94 and their layer magnitude curves; (b) Dispersion curves in the post-buckling domain for c=0.4,0.5, and 1.5 and their layer magnitude curves.

$$\lim_{H \to H^{cr}} \overline{f}_{zero} = 0 \text{ or } 1$$

As stated, the zeros naturally appear at $\overline{K}_2 = 0,1$ when $H \approx H^{cr}$ due to the periodicity of models.

Additionally, the nonlinear post-buckling models cannot imitate the ideal buckling reliably when the enforced wavelength deviates significantly from the theoretical one. For example, in Fig. 3(b), under $H = 0.5H^{cr}$ the layer magnitude starts to increase at a deformation level much higher than the critical stretch. The magnitude of the stiff layer under $H = 0.4H^{cr}$ remains close to flat regardless of increasing deformation indicating that the curving (i.e., buckling) process is difficult to trigger when the wavelength is much smaller than the ideal. For $H = 1.5H^{cr}$, zero frequencies are observed even prior to the critical stretch, highlighting the need for further investigation into these anomalies.

Furthermore, we studied the resolution required in the deformation step size $\Delta\lambda$ to detect the onset of zero-frequency modes. Under $H = 0.85H^{cr}$, a step size of $\Delta\lambda = 0.001$ is sufficient to find the occurrence of zeros within a deformation interval of approximately 0.004 ($\Delta\lambda_{zero} \approx 0.004$). As the wavelength increased to $0.9H^{cr}$ and $0.94H^{cr}$, a finer resolution at $\Delta\lambda = 0.0001$ is needed, and the corresponding zero-frequency intervals narrow to 0.001 and 0.0008, respectively. This supports a second inference:

$$\lim_{H \to H^{cr}} \Delta\lambda_{zero} = 0$$

which aligns with the previous conclusion that zero-frequency points converge to the Brillouin zone boundaries when the imposed wavelength matches the theoretical value. Hence, the onset of zero-frequency modes can serve as a diagnostic signal (a "red flag") for significant deviation from the correct buckling wavelength. This analysis reaffirms the importance of high accuracy in determining the critical wavenumber.

3 Conclusions

By studying the forced buckling of laminates under wavelengths significantly deviating from the theoretical value, we identified that the emergence of "abnormal" zero frequencies in the dispersion relations of post-buckling models at $\lambda < \lambda^{cr}$ is informing. These values absent in post-buckling models with $H = H^{cr}$ cluster around the ratio determined by the fraction between the enforced and theoretical wavelengths. From the exhibited cases, we infer that when the wavelength is accurately selected, the zeros shall appear at trial wavenumbers ($\overline{K}_2 = 0,1$), consistent with the built-in periodicity of post-buckling models. Therefore, the appearance of zeros can serve as a practical diagnostic signal indicating that the selected wavelength significantly deflected deviates from the theoretical one. Based on our results, the wavenumber error should be kept below 6% to avoid observing the zero values with a step $\Delta\lambda = 0.001$.

We would also like to emphasize the importance of depth and rigor in studying post-buckling models. The ideal post-buckling model should imitate the transformation of perfectly straight stiff layers buckling into a curved configuration (straight→ curved). We can also say it is a transformation of linear systems to nonlinear ones. In practice, however, post-buckling models are built with slightly curved stiff layers, which then evolve into significantly waved patterns (small→ large magnitude). As such, these models are nonlinear in both pre- and post-buckling domains. As demonstrated in this study, when the prescribed wavelength deviates substantially from the theoretical one, the model may fail to properly simulate the buckling transition. This built-in nonlinearity and complexity introduce additional challenges in analyzing the inherently intriguing post-buckling behavior.

Accurately defining the wavelength, imperfections, and other modeling parameters is critical for the study of buckling in soft composites [8, 9]. In future work, particularly the 3D post-buckling problem of fiber-reinforced composites (FCs), the mismatch between the enforced and the natural wavelengths will similarly introduce obstacles. Given the wide application and high complexity of these models, we invite our colleagues to engage deeply with these challenges, follow our ongoing research, and contribute their expertise to further advance this field in both depth and breadth.

Author contribution Qi Yao: writing —review and editing, writing —original draft, visualization, validation, investigation, formal analysis, data curation, conceptualization. Stephan Rudykh: writing —review and editing, writing —original draft, supervision, resources, investigation, funding acquisition, conceptualization.

Funding No funding was received for this work.

Data availability The data supporting the findings of this study are openly available and can be provided upon request.

Declarations

Ethics approval and consent to participate Qi Yao and Stephan Rudykh confirm their participation in this research. Ethical approval was not required for this study.

Consent for publication Both authors consent to the publication of this work.

Competing interests The authors declare no competing interests.

References

1. Kochmann, D.M., Bertoldi, K.: Exploiting microstructural instabilities in solids and structures: from metamaterials to structural transitions. Applied Mechanics Reviews **69**, 050801 (2017)
2. Li, J., Arora, N., Rudykh, S.: Elastic instabilities, microstructure transformations, and pattern formations in soft materials. Curr. Opin. Solid State Mater. Sci. **25**, 100898 (2021)

3. Geymonat, G., Müller, S., Triantafyllidis, N.: Homogenization of nonlinearly elastic materials, microscopic bifurcation and macroscopic loss of rank-one convexity. Arch. Ration. Mech. Anal. **122**, 231–290 (1993)
4. Bertoldi, K., Boyce, M.C.: Wave propagation and instabilities in monolithic and periodically structured elastomeric materials undergoing large deformations. Phys. Rev. B Condens. Matter Mater. Phys. (2008). https://doi.org/10.1103/PhysRevB.78.184107
5. Krishnan, D., Johnson, H.T.: Optical properties of two-dimensional polymer photonic crystals after deformation-induced pattern transformations. J Mech Phys Solids **57**, 1500–1513 (2009)
6. Li, J., Pallicity, T.D., Slesarenko, V., Goshkoderia, A., Rudykh, S.: Domain formations and pattern transitions via instabilities in soft heterogeneous materials. Adv Mater **31**, 1807309 (2019)
7. Li, J., Rudykh, S.: Tunable microstructure transformations and auxetic behavior in 3D-printed multiphase composites: the role of inclusion distribution. Compos. Part B Eng. **172**, 352–362 (2019)
8. Chen, D., Arora, N., Xiang, Y., Li, J., Slesarenko, V., Rudykh, S.: Instability-induced patterns and their post-buckling development in soft particulate composites. Mech. Mater. (2022). https://doi.org/10.1016/j.mechmat.2022.104482
9. Chen, D., Xiang, Y., Arora, N., Yao, Q., Li, J., Rudykh, S.: Post-buckling development in soft particulate composites. Compos. Struct. **322**, 117337 (2023)
10. Combescure, C., Elliott, R., Elliott, R.S.: Hierarchical honeycomb material design and optimization: beyond linearized behavior. International Journal of Solids and Structures (2017). https://doi.org/10.1016/j.ijsolstr.2017.03.011
11. Li, J., Slesarenko, V., Galich, P.I., Rudykh, S.: Oblique shear wave propagation in finitely deformed layered composites. Mech. Res. Commun. **87**, 21–28 (2018)
12. Galich, P.I., Slesarenko, V., Li, J., Rudykh, S.: Elastic instabilities and shear waves in hyperelastic composites with various periodic fiber arrangements. International Journal of Engineering Science **130**, 51–61 (2018)
13. Bertoldi, K., Lopez-Pamies, O.: Some remarks on the effect of interphases on the mechanical response and stability of fiber-reinforced elastomers. Journal of Applied Mechanics **79**, 031023 (2012)
14. Li, G.Y., Zheng, Y., Cao, Y., Feng, X.Q., Zhang, W.: Controlling elastic wave propagation in a soft bilayer system: via wrinkling-induced stress patterns. Soft Matter **12**, 4204–4213 (2016)
15. Liu, J., Bertoldi, K.: Bloch wave approach for the analysis of sequential bifurcations in bilayer structures. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences (2015). https://doi.org/10.1098/rspa.2015.0493
16. Yao, Q., Arora, N., Chen, D., Xiang, Y., Rudykh, S.: Elastic instabilities of soft laminates with stiffening behavior. Appl Math Model **130**, 658–675 (2024)
17. Xiang, Y., Chen, D., Arora, N., Yao, Q., Rudykh, S.: Towards understanding the role of viscoelasticity in microstructural buckling in soft particulate composites. Compos. Part B Eng. **263**, 110850 (2023)
18. Arora, N., Li, J., Slesarenko, V., Rudykh, S.: Microscopic and longwave instabilities in 3D fiber composites with non-Gaussian hyperelastic phases. International Journal of Engineering Science (2020). https://doi.org/10.1016/j.ijengsci.2020.103408
19. Slesarenko, V., Rudykh, S.: Microscopic and macroscopic instabilities in hyperelastic fiber composites. J. Mech. Phys. Solids **99**, 471–482 (2017)
20. Lapina, T., Xiang, Y., Yao, Q., Chen, D., Li, J., Steinmann, P., Rudykh, S.: Microstructural buckling in soft visco-hyperelastic laminates. Int J Solids Struct **311**, 1–9 (2025)
21. Li, J., Slesarenko, V., Rudykh, S.: Microscopic instabilities and elastic wave propagation in finitely deformed laminates with compressible hyperelastic phases. European Journal of Mechanics **73**, 126–136 (2019)