

# COMPUTATIONS FOR COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA II: GROUPS OF RANK FIVE AND SIX

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ABSTRACT. In recent papers we have refined a conjecture of Lehrer and Solomon expressing the character of a finite Coxeter group  $W$  acting on the graded components of its Orlik-Solomon algebra as a sum of characters induced from linear characters of centralizers of elements of  $W$ . The refined conjecture relates the character above to a decomposition of the regular character of  $W$  related to Solomon's descent algebra of  $W$ . The refined conjecture has been proved for symmetric and dihedral groups, as well as for finite Coxeter groups of rank three and four. In this paper, we prove the conjecture for finite Coxeter groups of rank five and six. The techniques developed and implemented in this paper provide previously unknown decompositions of the regular and Orlik-Solomon characters of the groups considered.

## 1. INTRODUCTION

Let  $(W, S)$  be a finite Coxeter system. In previous articles [2–4] we proposed a conjecture relating the character  $\omega$  of the Orlik-Solomon algebra of  $W$  to the regular character  $\rho$  of  $W$ . Based on a conjecture of Lehrer and Solomon [9], in this paper we prove the conjecture for the Coxeter groups of type  $B_5$ ,  $B_6$ ,  $D_5$ ,  $D_6$ , and  $E_6$ . These computations, together with the remarks about reducible Coxeter groups following Theorem 2.3 of [2] and the proof of the conjecture for groups of type  $A$  in [3], prove the conjecture for all finite Coxeter groups of rank five and six. Our result is stated for these groups as the following theorem.

**Theorem 1.** *Suppose that  $W$  is a finite Coxeter group of rank five or six and that  $\mathcal{R}$  is a set of conjugacy class representatives of  $W$ . Then for each  $w \in \mathcal{R}$  there exists a linear character  $\varphi_w$  of  $C_W(w)$  such that*

$$\rho = \sum_{w \in \mathcal{R}} \text{Ind}_{C_W(w)}^W \varphi_w \quad \text{and} \quad \omega = \epsilon \sum_{w \in \mathcal{R}} \text{Ind}_{C_W(w)}^W (\alpha_w \varphi_w)$$

where  $\epsilon$  is the sign character of  $W$  and for  $z \in C_W(w)$ ,  $\alpha_w(z)$  denotes the determinant of the restriction of  $z$  to the 1-eigenspace of  $w$  in the complex reflection representation of  $W$ .

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2010 *Mathematics Subject Classification.* 20F55, 20C15, 20C40, 52C35.

*Key words and phrases.* Coxeter group, descent algebra, Orlik-Solomon algebra.

Let  $A(W)$  be the Orlik-Solomon algebra of  $W$ . The strategy for proving [Theorem 1](#) is to decompose the  $\mathbb{C}W$ -modules  $\mathbb{C}W$  and  $A(W)$  into direct sums and prove a refinement of [Theorem 1](#) for each summand. This method is somewhat stronger than directly proving [Theorem 1](#), because it requires the solution to be compatible with the direct sum decompositions of  $\mathbb{C}W$  and  $A(W)$ . This method also has the advantage that it splits the problem into smaller problems and provides additional insight into how the representations of  $W$  on  $\mathbb{C}W$  and on  $A(W)$  are related.

The decomposition of  $\mathbb{C}W$  comes from idempotents  $e_\lambda$  in the descent algebra  $\Sigma(W)$  constructed in [\[1\]](#). These idempotents are indexed by subsets of  $S$  up to conjugacy in  $W$ . A class of conjugate subsets of  $S$  is called a *shape* of  $W$ . Denote the set of shapes of  $W$  by  $\Lambda$ . In [\[1\]](#) it is shown how to construct a quasi-idempotent  $e_L$  for any  $L \subseteq S$  and then  $e_\lambda$  is the sum of the quasi-idempotents  $e_L$  where  $L$  runs over the subsets in the shape  $\lambda$ . It is also shown that  $\{e_\lambda \mid \lambda \in \Lambda\}$  is a complete set of primitive orthogonal idempotents of  $\Sigma(W)$ . Since  $\Sigma(W)$  is a subalgebra of  $\mathbb{C}W$  and  $1 = \sum_{\lambda \in \Lambda} e_\lambda$ , we conclude that  $\mathbb{C}W = \bigoplus_{\lambda \in \Lambda} e_\lambda \mathbb{C}W$  as a  $\mathbb{C}W$ -module. Denoting the character of  $e_\lambda \mathbb{C}W$  by  $\rho_\lambda$  we have

$$(1.1) \quad \rho = \sum_{\lambda \in \Lambda} \rho_\lambda.$$

The corresponding decomposition of  $A(W)$  comes from Brieskorn's Lemma. Let  $T$  be the set of reflections in the complex reflection representation  $V$  of  $W$ . Recall that the Orlik-Solomon algebra  $A(W)$  may be defined as the quotient of the exterior algebra with generators  $\{e_t \mid t \in T\}$  by the ideal generated by elements of the form  $\sum_{i=1}^k (-1)^i e_{t_1} e_{t_2} \cdots \widehat{e_{t_i}} \cdots e_{t_k}$  for all sets  $\{t_1, t_2, \dots, t_k\} \subseteq T$  of linearly dependent reflections. Here, we say that a set of reflections is linearly dependent if the linear forms defining their reflecting hyperplanes are linearly dependent in the dual of  $V$ . For  $t \in T$  we denote the image of the generator  $e_t$  in  $A(W)$  by  $a_t$ . Thus, an arbitrary element of  $A(W)$  can be expressed as a linear combination of monomials  $a_{t_1} a_{t_2} \cdots a_{t_k}$  with  $t_1, t_2, \dots, t_k \in T$ . The algebra  $A(W)$  is a right  $\mathbb{C}W$ -module with  $(a_{t_1} a_{t_2} \cdots a_{t_k}) \cdot w = a_{w^{-1}t_1 w} a_{w^{-1}t_2 w} \cdots a_{w^{-1}t_k w}$  for  $w \in W$ .

Each monomial  $a_{t_1} a_{t_2} \cdots a_{t_k}$  determines a subspace of  $V$ , namely the intersection of the fixed point spaces of the reflections  $t_1, t_2, \dots, t_k$ . If  $X$  is a subspace of  $V$ , then we denote by  $A_X$  the span of all monomials with fixed point space equal to  $X$ . Then taking  $A_\lambda$  to be the sum of the  $A_X$  for which  $X$  is the fixed point set of some conjugate of  $W_L$  for some  $L \in \lambda$ , we have a decomposition  $A(W) = \bigoplus_{\lambda \in \Lambda} A_\lambda$ . Denoting the character of  $A_\lambda$  by  $\omega_\lambda$  we have

$$\omega = \sum_{\lambda \in \Lambda} \omega_\lambda.$$

Finally, we choose a set of conjugacy class representatives compatible with the decompositions above. A conjugacy class in  $W_L$  is called *cuspidal* if the fixed point set in the reflection representation of  $W_L$  of any of its elements is trivial. Now if we choose a fixed representative  $L(\lambda)$  of each shape  $\lambda$  and let  $\mathcal{C}_{L(\lambda)}$  be a set of

representatives of the cuspidal classes in  $W_{L(\lambda)}$ , then by Theorem 3.2.12 of [5]

$$(1.2) \quad \mathcal{R} = \bigcup_{\lambda \in \Lambda} \mathcal{C}_{L(\lambda)} \quad \text{is a set of conjugacy class representatives of } W.$$

Suppose that  $L \subseteq S$ . The homogeneous component of  $A(W_L)$  of highest degree is called the *top component* of  $A(W_L)$ . On the other hand,  $W_L$  is also a Coxeter group and thereby admits a system of quasi-idempotents as in [1], now denoted by  $e_J^\dagger$  for  $J \subseteq L$  to distinguish them from the quasi-idempotents  $e_J$  in  $CW$ . Note that this notation does not agree with that used in [1, §7]. In analogy with  $A(W_L)$  the homogeneous component  $e_L^\dagger CW_L$  of  $CW_L$  is called the *top component* of  $CW_L$ . We denote the characters of  $W_L$  afforded by the top components of  $A(W_L)$  and  $CW_L$  by  $\omega_L$  and  $\rho_L$  respectively.

Now the top components of  $A(W_L)$  and  $CW_L$  are naturally  $W_L$ -stable subspaces of  $A(W)$  and  $CW$  and it turns out that they are  $N_W(W_L)$ -stable by [3, Proposition 4.8]. Thus they afford characters  $\widetilde{\omega}_L$  and  $\widetilde{\rho}_L$  of  $N_W(W_L)$  which are extensions of  $\omega_L$  and  $\rho_L$ . Furthermore, if  $L \in \lambda$  then by the same proposition,

$$(1.3) \quad \omega_\lambda = \text{Ind}_{N_W(W_L)}^W \widetilde{\omega}_L \quad \text{and} \quad \rho_\lambda = \text{Ind}_{N_W(W_L)}^W \widetilde{\rho}_L.$$

Consider the following refinement of Theorem 1. In the statement of this theorem we use the fact, proved in [8, Theorem 3.1], that if  $w$  is cuspidal in  $W_L$ , then  $C_W(w) \subseteq N_W(W_L)$ .

**Theorem 2.** *Suppose that  $(W, S)$  is a finite Coxeter system of rank five or six, that  $L \subseteq S$ , and that  $\mathcal{C}_L$  is a set of representatives of the cuspidal conjugacy classes of  $W_L$ . Then for each  $w \in \mathcal{C}_L$  there exists a linear character  $\varphi_w$  of  $C_W(w)$  such that*

$$\widetilde{\rho}_L = \sum_{w \in \mathcal{C}_L} \text{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w = \alpha_L \epsilon \widetilde{\omega}_L$$

where for  $\mathfrak{n} \in N_W(W_L)$ ,  $\alpha_L(\mathfrak{n})$  denotes the determinant of the restriction of  $\mathfrak{n}$  to the subspace of fixed points of  $W_L$  in  $V$ .

To prove Theorem 1 we prove Theorem 2 for the representative  $L(\lambda)$  of each shape  $\lambda$ . Then the characters  $\varphi_w$  that satisfy Theorem 2 with  $L = L(\lambda)$  as  $\lambda$  varies over all shapes prove the first equality of Theorem 1 because

$$\begin{aligned} \rho &= \sum_{\lambda \in \Lambda} \rho_\lambda && \text{by (1.1)} \\ &= \sum_{\lambda \in \Lambda} \text{Ind}_{N_W(W_{L(\lambda)})}^W \widetilde{\rho}_{L(\lambda)} && \text{by (1.3)} \\ &= \sum_{\lambda \in \Lambda} \sum_{w \in \mathcal{C}_{L(\lambda)}} \text{Ind}_{N_W(W_{L(\lambda)})}^W \text{Ind}_{C_W(w)}^{N_W(W_{L(\lambda)})} \varphi_w && \text{by Theorem 2} \\ &= \sum_{w \in \mathcal{R}} \text{Ind}_{C_W(w)}^W \varphi_w \end{aligned}$$

where the last equality follows from transitivity of induction and (1.2). A similar argument proves the second equality in Theorem 1. We prove Theorem 2 in §3 and §4.

## 2. IMPLEMENTATION

As in [2], we have implemented the calculations for this article in the computer algebra system GAP [11] in conjunction with the CHEVIE [6] and the ZigZag [10] packages. In addition to our comments about the implementation in [2] we make the following remarks about the techniques new to this paper and improvements to old techniques.

**2.1. The Extension  $\widetilde{\rho}_L$ .** In this subsection we develop a formula for the character  $\widetilde{\rho}_L$  of  $N_W(W_L)$  for  $L \subseteq S$ . First we review the definitions of the constructions used in the process.

If  $J \subseteq L$  then the *parabolic transversal* of  $W_J$  in  $W_L$  is the set  $X_J^L$  of elements  $w \in W_L$  satisfying  $\ell(sw) > \ell(w)$  for all  $s \in J$ , where  $\ell$  is the usual length function of  $W$  with respect to  $S$ . Then  $X_J^L$  can be calculated directly from the definition or by using the `ParabolicTransversal` function supplied by the `ZigZag` package.

In order to use the formula for  $\widetilde{\rho}_L$  below, we need to be able to decompose an element of  $N_W(W_L)$  into the product of an element of  $W_L$  and an element of the normalizer complement  $N_L$  of  $W_L$ . Recall that  $N_L$  consists of certain elements of the parabolic transversal  $X_L^S$  of  $W_L$  in  $W$ . Therefore, the decomposition of an element of  $N_W(W_L)$  into a product  $nw$  with  $n \in N_L$  and  $w \in W_L$  is a special case of the more general decomposition of an element of  $W$  into a product of a coset representative in  $X_L^S$  by an element of  $W_L$ . In `ZigZag` this decomposition is implemented as the `ParabolicCoordinates` function.

The quasi-idempotents  $e_J^L$  are defined in [1] by means of the matrix  $M = (m_{KJ})$ , whose rows and columns are indexed by the subsets of  $S$  and whose  $(K, J)$ -entry is

$$m_{KJ} = \begin{cases} |\{x \in X_K \mid J^x \subseteq S\}| & \text{if } K \supseteq J \\ 0 & \text{otherwise.} \end{cases}$$

The matrix  $M$  can be calculated directly from the definition or by calling the method `Mu` supplied by the `ZigZag` package. Then putting  $N = M^{-1} = (n_{KJ})$  we have  $e_J^L = \sum_{K \subseteq L} n_{JK} x_K^L$  where  $x_K^L$  is the sum in  $\mathbb{C}W_L$  of the elements of  $X_K^L$ .

In the following discussion let  $w \in W_L$ ,  $n \in N_L$ , and  $x \in \mathbb{C}W_L$ . Observe that  $N_W(W_L)$  acts on  $\mathbb{C}W_L$  on the right by  $x \cdot (wn) = n^{-1}xwn$ . Using this action we define the map

$$\gamma(wn, x) : \mathbb{C}W_L \rightarrow \mathbb{C}W_L \quad \text{by} \quad \gamma(wn, x)(v) = (xv) \cdot (wn) = n^{-1}xvwn$$

for  $v \in \mathbb{C}W_L$ .

The idempotent  $e_L^L \in \mathbb{C}W_L$  determines a  $\mathbb{C}N_W(W_L)$ -stable decomposition of the group algebra of  $W_L$ ,  $\mathbb{C}W_L = e_L^L \mathbb{C}W_L \oplus (1 - e_L^L) \mathbb{C}W_L$ . Calculating the trace of the action of  $\gamma(wn, e_L^L)$  with respect to a basis of  $\mathbb{C}W_L$  adapted to this decomposition,

we find that  $\widetilde{\rho}_L(\mathbf{wn}) = \text{Tr}(\gamma(\mathbf{wn}, \mathbf{e}_L^1))$  since  $\gamma(\mathbf{wn}, \mathbf{e}_L^1)$  sends  $(1 - \mathbf{e}_L^1)\mathbb{C}W_L$  to  $\mathbf{e}_L^1\mathbb{C}W_L$ . Using the linearity of  $\gamma$  in its second argument we can further refine this to

$$\widetilde{\rho}_L(\mathbf{wn}) = \sum_{\mathbf{y} \in W_L} \mathbf{a}_y \text{Tr}(\gamma(\mathbf{wn}, \mathbf{y}))$$

where the numbers  $\mathbf{a}_y$  are such that  $\mathbf{e}_L^1 = \sum_{\mathbf{y} \in W_L} \mathbf{a}_y \mathbf{y}$ .

For fixed  $\mathbf{n} \in \mathbf{N}_L$  we define a right action  $\cdot_{\mathbf{n}}$  of  $W_L$  on  $W_L$  by

$$\mathbf{y} \cdot_{\mathbf{n}} \mathbf{z} = \mathbf{n} \mathbf{z}^{-1} \mathbf{n}^{-1} \mathbf{y} \mathbf{z}$$

for  $\mathbf{y}, \mathbf{z} \in W_L$ . Then the stabilizer  $Z_{\mathbf{n}}(\mathbf{y})$  of  $\mathbf{y} \in W_L$  under this action is  $C_{W_L}(\mathbf{n}^{-1}\mathbf{y})$ . We denote the orbit of  $\mathbf{y}$  by  $\mathcal{O}_{\mathbf{n}}(\mathbf{y}) = \{\mathbf{n} \mathbf{z}^{-1} \mathbf{n}^{-1} \mathbf{y} \mathbf{z} \mid \mathbf{z} \in W_L / C_{W_L}(\mathbf{n}^{-1}\mathbf{y})\}$ . Now

$$\begin{aligned} \text{Tr}(\gamma(\mathbf{wn}, \mathbf{y})) &= |\{z \in W_L \mid \mathbf{w}^{-1} = \mathbf{y} \cdot_{\mathbf{n}} z\}| \\ &= \begin{cases} 0 & \text{if } \mathbf{w}^{-1} \notin \mathcal{O}_{\mathbf{n}}(\mathbf{y}) \\ |Z_{\mathbf{n}}(\mathbf{y})| & \text{if } \mathbf{w}^{-1} \in \mathcal{O}_{\mathbf{n}}(\mathbf{y}) \end{cases} \\ &= \begin{cases} 0 & \text{if } \mathbf{y} \notin \mathcal{O}_{\mathbf{n}}(\mathbf{w}^{-1}) \\ |C_{W_L}(\mathbf{wn})| & \text{if } \mathbf{y} \in \mathcal{O}_{\mathbf{n}}(\mathbf{w}^{-1}), \end{cases} \end{aligned}$$

where in the last equality we have used the fact that  $\mathbf{w}^{-1} \in \mathcal{O}_{\mathbf{n}}(\mathbf{y})$  if and only if  $\mathbf{y} \in \mathcal{O}_{\mathbf{n}}(\mathbf{w}^{-1})$ , and if so, then the value of  $\text{Tr}(\gamma(\mathbf{wn}, \mathbf{y}))$  is  $|Z_{\mathbf{n}}(\mathbf{w}^{-1})| = |C_{W_L}(\mathbf{n}^{-1}\mathbf{w}^{-1})| = |C_{W_L}(\mathbf{wn})|$  by the calculation above. In conclusion, we obtain the following formula.

$$\begin{aligned} \widetilde{\rho}_L(\mathbf{wn}) &= \sum_{\mathbf{y} \in \mathcal{O}_{\mathbf{n}}(\mathbf{w}^{-1})} \mathbf{a}_y |C_{W_L}(\mathbf{wn})| \\ &= |C_{W_L}(\mathbf{wn})| \sum_{\mathbf{y} \in \mathcal{O}_{\mathbf{n}}(\mathbf{w}^{-1})} \sum_{\substack{J \subseteq L \\ \mathcal{D}(\mathbf{y}) \subseteq L \setminus J}} \mathbf{n}_{LJ} \\ &= |C_{W_L}(\mathbf{wn})| \sum_{J \subseteq L} \mathbf{n}_{LJ} |\mathcal{O}_{\mathbf{n}}(\mathbf{w}^{-1}) \cap X_J^L|. \end{aligned}$$

Here we have used the descent set  $\mathcal{D}(\mathbf{y}) = \{s \in L \mid \ell(\mathbf{s}\mathbf{y}) < \ell(\mathbf{y})\}$  to derive the formula  $\sum_{\mathcal{D}(\mathbf{y}) \subseteq L \setminus J} \mathbf{n}_{LJ}$  for  $\mathbf{a}_y$ .

**2.2. The Extension  $\widetilde{\omega}_L$ .** In this subsection we discuss the calculation of  $\widetilde{\omega}_L$  for  $L \subseteq S$ . As this calculation is almost identical to the calculation of  $\omega$ , we begin with  $\omega$  and discuss the minor modifications needed to calculate  $\widetilde{\omega}_L$  at the end.

For computational purposes, rather than working with the set  $T$  of reflections in  $W$ , it is simpler to work with the positive roots of  $W$ . The positive roots are stored in CHEVIE as vectors in the `roots` component of a Coxeter group record, the first half being the positive roots and the second half being the negative of the first half. This means that whenever a calculation involving roots results in a negative root, we need to replace the negative root with its positive counterpart.

With this convention the generator  $\mathbf{a}_t$  of  $\mathbf{A}(W)$  is denoted by  $\mathbf{a}_r$ , where  $r$  is the positive root orthogonal to hyperplane fixed by  $t$ . To simplify the notation, we will denote  $\mathbf{a}_r$  simply by  $r$ . This also reflects the way one implements  $\mathbf{A}(W)$  on a computer. Namely, the elements of  $\mathbf{A}(W)$  are represented by linear combinations of sequences  $r_1 r_2 \cdots r_q$  of positive roots. We will also assume that any element  $r_1 r_2 \cdots r_q$  satisfies  $r_1 < r_2 < \cdots < r_q$ , explicitly sorting the factors and inserting the appropriate sign  $\pm 1$  whenever the factors become unsorted. Here  $<$  denotes a fixed total order on the positive roots, which can be simply be taken to be the order in which the roots appear in the `roots` component of the record for  $W$ .

Now since CHEVIE implements the element  $w$  of  $W$  as a permutation  $\sigma_w$  of the roots in  $V$ , it follows that if  $t$  is the reflection defined by the root  $r$ , then the conjugate  $w^{-1}tw$  is the reflection defined by  $r \cdot \sigma_w$ , which we simplify to  $r \cdot w$ . Therefore, the action of  $W$  on  $\mathbf{A}(W)$  is given by  $(r_1 r_2 \cdots r_q) \cdot w = (r_1 \cdot w) (r_2 \cdot w) \cdots (r_q \cdot w)$ .

We use the non-broken circuit basis  $\mathcal{B}$  of  $\mathbf{A}(W)$  described in [2] to calculate its character  $\omega$ . While this works exactly as in [2], we briefly describe some improvements to the algorithm that make the calculations in this paper possible. Let  $n = |S|$  be the rank of  $W$  and recall that the non-broken circuit basis of  $\mathbf{A}(W)$  consists of elements of the form  $r_1 r_2 \cdots r_n$  not containing certain sequences called *broken circuits* as subsequences. A broken circuit  $r_{i_1} r_{i_2} \cdots r_{i_q}$  has the property that there exists a positive root  $r$  with  $r > r_{i_q}$  for which  $r_{i_1} r_{i_2} \cdots r_{i_q} r$  is dependent, so the defining relation for  $\mathbf{A}(W)$  implies that

$$(2.1) \quad (-1)^q r_{i_1} r_{i_2} \cdots r_{i_q} = \sum_{k=1}^q (-1)^k r_{i_1} r_{i_2} \cdots \widehat{r_{i_k}} \cdots r_{i_q} r.$$

Therefore, any element *not* in  $\mathcal{B}$  can be expressed as a linear combination of lexicographically larger elements of  $\mathbf{A}(W)$  by applying (2.1) to a broken circuit subsequence. This observation is the rationale for the procedure for expressing an arbitrary element of  $\mathbf{A}(W)$  in terms of the non-broken circuit basis, but it also leads to a significant improvement in the calculation of  $\omega$ .

Namely, to calculate the value of  $\omega$  at an element  $w \in W$ , one in principle runs through all basis elements  $\mathbf{b} \in \mathcal{B}$ , expressing  $\mathbf{b} \cdot w$  as a linear combination of elements of  $\mathcal{B}$  using (2.1) and storing the coefficients of the result into the rows of a matrix  $\mathbf{m}$ . Then  $\mathbf{m}$  represents the linear transformation  $w$  of  $\mathbf{A}(W)$  and  $\omega(w)$  is the trace of  $\mathbf{m}$ . We observe that if at any point in the calculation of  $\mathbf{b} \cdot w$  we arrive at a monomial lexicographically larger than  $\mathbf{b}$ , then this monomial cannot contribute to the trace of  $\mathbf{m}$ . Such calculations can therefore be terminated. Furthermore, the matrix  $\mathbf{m}$  itself exists only in concept. In practice we need only its diagonal entries. Therefore, we use the following algorithm.

**COEFF** (*Individual coefficient with respect to  $\mathcal{B}$* ) With respect to the non-broken circuit basis  $\mathcal{B}$  of  $\mathbf{A}(W)$  this algorithm takes as input a monomial  $\mathbf{a} = r_1 r_2 \cdots r_n \in \mathbf{A}(W)$  and a basis element  $\mathbf{b} \in \mathcal{B}$ . It returns the coefficient of  $\mathbf{b}$  when  $\mathbf{a}$  is expressed with respect to  $\mathcal{B}$ .

```

if  $a > b$  then
  return 0
else if  $a \in \mathcal{B}$  then
  if  $a = b$  then
    return 1
  else
    return 0
else
  find a subsequence  $r_{i_1} r_{i_2} \cdots r_{i_q}$  of  $\mathbf{a}$  which is a broken circuit
  find a root  $r$  for which  $r_{i_1} r_{i_2} \cdots r_{i_q} r$  is dependent
  return  $\sum_{j=1}^m (-1)^j \mathbf{COEFF}(r_1 r_2 \cdots \widehat{r_{i_j}} \cdots r_n r, \mathbf{b})$ 

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Observe that in the last line of the algorithm, we have inserted  $r$  at the end of the first argument of **COEFF** for notational convenience. Moving the factor to its proper position will introduce a sign  $\pm 1$ . Then to calculate  $\omega(w)$  we simply calculate  $\sum_{b \in \mathcal{B}} \mathbf{COEFF}(b.w, b)$ .

Finally, to calculate the character  $\widetilde{\omega}_L$  for  $L \subseteq S$  we calculate the non-broken circuit basis of the top component of  $A(W_L)$ . Observe that an element  $w \in N_W(W_L)$  is implemented as a permutation  $\sigma_w$  of the roots in  $V$ , so to apply  $w$  to an element  $r_1 r_2 \cdots r_q$  of  $A(W_L)$  each  $r_i$  must be replaced with its corresponding root in  $V$ . In CHEVIE this can be accomplished with the **rootInclusion** component of the  $W_L$  record. Then the permutation  $\sigma_w$  can be applied directly, followed by replacing each root with the corresponding root in the reflection representation of  $W_L$  using the **rootRestriction** component of the  $W_L$  record. With this modification, we proceed exactly as in the calculation of  $\omega$  above.

### 3. PROOF OF THEOREM 2 WHEN $L = S$

Observe that if  $L = S$  then  $\widetilde{\rho}_S = \rho_S$ ,  $\widetilde{\omega}_S = \omega_S$ , and  $N_W(W_L) = W$ . Observe also that  $\alpha_S(w) = 1$  for all  $w \in W$  since the space of fixed points of  $W$  is the zero subspace of  $V$ . Therefore, to verify Theorem 2 we need to find a character  $\varphi_w$  of  $C_W(w)$  for each  $w \in \mathcal{C}_S$  such that

$$(3.1) \quad \rho_S = \sum_{w \in \mathcal{C}_S} \text{Ind}_{C_W(w)}^W \varphi_w = \epsilon \omega_S.$$

In this section we exhibit these characters for each irreducible Coxeter group  $W$  of rank five or six. Once the characters  $\varphi_w$  are specified, one verifies (3.1) by routine calculations, so we limit ourselves to displaying the characters  $\text{Ind}_{C_W(w)}^W \varphi_w$  (denoted simply by  $\varphi_w$ ),  $\epsilon$ ,  $\rho_S$ , and  $\omega_S$  only for the group  $W = W(E_6)$ .

Because each character  $\varphi_w$  is one-dimensional, it suffices to list its values on a generating set for the group  $C_W(w)$ . For the group  $W(E_6)$  we have constructed generating sets for the groups  $C_W(w)$  ad hoc. In type B generating sets for  $C_W(w)$

are known, while in type  $D$  generating sets for  $C_W(\mathbf{w})$  can be determined as described below. We use the notation for these generating sets from [2], which we now briefly review.

The cuspidal classes of  $W(B_n)$  are indexed by partitions of  $n$ . We always display partitions in non-decreasing order without punctuation. With the labeling of the elements of  $S$  given by the diagram  $\bullet_1 \leftarrow \bullet_2 \text{---} \bullet_3 \cdots \text{---} \bullet_{n-1} \text{---} \bullet_n$  we define the following elements of  $W(B_n)$ , where we denote the elements of  $S$  by  $1, 2, \dots, n$  rather than  $s_1, s_2, \dots, s_n$  to improve legibility. If  $\lambda = \lambda_1 \lambda_2 \cdots \lambda_k$  is a partition of  $n$  then for each  $1 \leq i \leq k$  we define a negative  $\lambda_i$ -cycle

$$(3.2) \quad \mathbf{c}_i = (j+1)j(j-1)\cdots 212\cdots(j+\lambda_i) \quad \text{where} \quad j = \sum_{k=1}^{i-1} \lambda_k.$$

Then each  $\mathbf{c}_i$  centralizes the element  $w_\lambda = \mathbf{c}_1 \mathbf{c}_2 \cdots \mathbf{c}_k$ , which we take to be the representative of the cuspidal class labeled by  $\lambda$ . Whenever  $\lambda_i = \lambda_{i+1}$  the element

$$(3.3) \quad \mathbf{x}_i = \prod_{k=1}^{\lambda_i} (j+k)(j+k-1)\cdots(j+k-\lambda_i+1) \quad \text{where} \quad j = \sum_{k=1}^i \lambda_k$$

also centralizes  $w_\lambda$ . In fact, if  $m(j) = \min\{k \mid \lambda_k = j\}$  then  $C_W(w_\lambda)$  is generated by the elements  $\mathbf{x}_i$  for all  $i$  satisfying  $\lambda_i = \lambda_{i+1}$ , together with the elements  $\mathbf{c}_{m(j)}$  for all  $j$  appearing as parts of  $\lambda$ . We remark that the elements defined in (3.2) and (3.3) coincide with the elements  $\mathbf{c}_i$  and  $\mathbf{x}_i$  defined in [2]. The character  $\varphi_{\mathbf{c}_\lambda}$  of  $C_W(w_\lambda)$  is denoted simply by  $\varphi_\lambda$ .

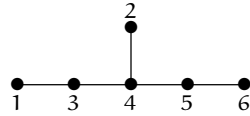
We view  $W(D_n)$  as a reflection subgroup of  $W(B_n)$  generated by the reflections  $1' = 121$  and  $2, 3, \dots, n$ . Then  $w_\lambda \in W(D_n)$  whenever  $\lambda$  has an even number of parts. In fact, such elements  $w_\lambda$  are representatives of the cuspidal classes of  $W(D_n)$  and the centralizer  $C_{W(D_n)}(w_\lambda)$  is the intersection of  $W(D_n)$  with  $C_{W(B_n)}(w_\lambda)$ . We observe that the last factor  $j+k-\lambda_i+1$  of (3.3) is at least  $\lambda_i+1$  and that the other factors are greater than  $j+k-\lambda_i+1$ . This means that  $1$  is never a factor of  $\mathbf{x}_i$  so that  $\mathbf{x}_i \in W(D_n)$ . However, (3.2) shows that  $1$  occurs as a factor of  $\mathbf{c}_i$  exactly once, making the replacement of  $121$  by  $1'$  impossible. This shows that  $\mathbf{c}_i \notin W(D_n)$ . Nevertheless, generators of  $C_{W(D_n)}(w_\lambda)$  can often be found among products of an even number of the elements  $\mathbf{c}_i$ .

In each of the following subsections we present the results of our calculations for the finite irreducible Coxeter groups of rank five and six. For each cuspidal class representative  $w$  we display a generating set of  $C_W(w)$ , where the generators are written as words in the Coxeter generators. At each generator, we display the value of the character  $\varphi_w$ . If  $\zeta$  is an eigenvalue of  $w$  on  $V$ , we denote the determinant of the representation of  $C_W(w)$  on the  $\zeta$ -eigenspace of  $w$  in  $V$  by  $\det|_\zeta$ . If  $\varphi_w$  is a power of  $\det|_\zeta$  for some  $\zeta$ , then we also indicate this. By Springer's theory of regular elements [12], the centralizer  $C_W(w)$  is a complex reflection group when  $w$  is a regular element. When this is the case, we identify  $C_W(w)$  as such a group.

For  $n \geq 1$  we denote the  $n^{\text{th}}$  root of unity  $e^{2\pi i/n}$  by  $\zeta_n$ , the cyclic group of size  $n$  by  $Z_n$ , and the symmetric group on  $n$  letters by  $S_n$ .

3.1.  $W = W(E_6)$ . We begin with  $W(E_6)$  and present the calculations that lead to the proof of [Theorem 2](#) for this group. For the other groups of rank five and six we present only the basic information described above.

Define the characters  $\varphi_d = \varphi_{w_d}$  in the following table, where the conjugacy classes of  $W$  are labeled by their Carter diagrams  $d$ . Here the elements of  $S$  are labeled as in the Coxeter graph



and  $r$  denotes the reflection defined by the highest root of  $W$ .

$d$	$w_d$	Gen	$\varphi_d$	$C_W(w_d)$	Det
$A_2^3$	12356r	24542314	$\zeta_3$	$G_{25}$	$\det _{\zeta_3}$
		13	$\zeta_3$		
		56	$\zeta_3$		
$E_6(a_2)$	$w_{E_6}^2$	$w_{E_6}$	1	$G_5$	
		234543	$\zeta_3$		
$A_5A_1$	13456r	$w_{A_5A_1}$	$\zeta_3$		
		2345432	-1		
		$r$	-1		
$E_6(a_1)$	$34w_{E_6}$	$w_{E_6(a_1)}$	$\zeta_9$	$Z_9$	$\det _{\zeta_9}$
$E_6$	123456	$w_{E_6}$	-1	$Z_{12}$	$(\det _{\zeta_{12}})^6$

Finally, the values of the characters  $\varphi_d^W$  together with  $\rho_S$  and  $\omega_S$  are shown in the following table.

$W$	$\emptyset$	$A_1^4$	$A_1^2$	$A_2^3$	$A_2$	$A_2^2$	$D_4(a_1)$	$A_3A_1$	$A_4$	$E_6(a_2)$	$D_4$	$A_5A_1$	$A_2A_1^2$
$\varphi_{A_2^3}$	80	16	.	-10	-4	2	8	.	.	-2	-2	-2	.
$\varphi_{E_6(a_2)}$	720	16	.	-18	-12	-6	8	.	.	-2	-2	-2	.
$\varphi_{A_5A_1}$	1440	32	.	-36	12	-3	.	.	.	-4	2	-1	.
$\varphi_{E_6(a_1)}$	5760	.	.	-72	.	.	.	.	.	.	.	.	.
$\varphi_{E_6}$	4320	96	.	108	.	.	-16	.	.	12	.	.	.
$\epsilon$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\rho_S$	12320	160	.	-28	-4	-7	.	.	.	4	-2	-5	.
$\omega_S$	12320	160	.	-28	-4	-7	.	.	.	4	-2	-5	.

$W$	$E_6(\alpha_1)$	$E_6$	$A_1$	$A_1^3$	$A_3A_1^2$	$A_3$	$A_2A_1$	$A_2^2A_1$	$A_5$	$D_5$	$A_4A_1$	$D_5(\alpha_1)$
$\varphi_{A_1^3}$	-1	2	.	.	.	.	.	.	.	.	.	.
$\varphi_{E_6(\alpha_2)}$	.	2	.	.	.	.	.	.	.	.	.	.
$\varphi_{A_1A_5}$	.	.	-120	-8	.	.	.	3	1	.	.	.
$\varphi_{E_6(\alpha_1)}$	.	.	.	.	.	.	.	.	.	.	.	.
$\varphi_{E_6}$	.	-4	.	.	.	.	.	.	.	.	.	.
$\epsilon$	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\rho_S$	-1	.	-120	-8	.	.	.	3	1	.	.	.
$\omega_S$	-1	.	120	8	.	.	.	-3	-1	.	.	.

3.2.  $W = W(B_5)$ . The characters defined in the following table satisfy [Theorem 2](#) for  $W = W(B_5)$  when  $L = S$ .

$\lambda$	Gen	Word	$\varphi_\lambda$	$C_W(w_\lambda)$	Det
$1^5$	S		$\epsilon$	$W$	$\det _{-1}$
$1^32$	$c_1$	1	-1		
	$c_4$	43212345	-1		
	$x_1$	2	-1		
	$x_2$	3	-1		
$12^2$	$c_1$	1	-1		
	$c_2$	2123	-1		
	$x_2$	4354	-1		
$1^23$	$c_1$	1	-1		
	$c_3$	3212345	$\zeta_6$		
	$x_1$	2	-1		
23	$c_1$	12	-1		
	$c_2$	3212345	$\zeta_6$		
14	$c_1$	1	-1		
	$c_2$	212345	-1		
5	$c_1$	12345	$\zeta_{10}$	$Z_{10}$	$\det _{\zeta_{10}}$

3.3.  $W = W(B_6)$ . The characters defined in the following table satisfy [Theorem 2](#) for  $W = W(B_6)$  when  $L = S$ .

$\lambda$	Gen	Word	$\varphi_\lambda$	$C_W(w_\lambda)$	Det
$1^6$	S		$\epsilon$	$W$	$\det _{-1}$
$1^4 2$	$c_1$	1	-1		
	$c_5$	5432123456	-1		
	$x_1$	2	-1		
	$x_2$	3	-1		
	$x_3$	4	-1		
$1^2 2^2$	$c_1$	1	-1		
	$c_3$	321234	-1		
	$x_1$	2	-1		
	$x_3$	5465	-1		
$2^3$	$c_1$	12	-1	$Z_4 \wr S_3$	
	$x_1$	3243	-1		
	$x_2$	5465	-1		
$1^3 3$	$c_1$	1	-1		
	$c_4$	432123456	$\zeta_6$		
	$x_1$	2	-1		
	$x_2$	3	-1		
$123$	$c_1$	1	-1		
	$c_2$	2123	-1		
	$c_3$	432123456	$\zeta_6$		
$3^2$	$c_1$	123	$\zeta_6$	$Z_6 \wr S_2$	$\det _{\zeta_6}$
	$x_1$	432543654	-1		
$1^2 4$	$c_1$	1	-1		
	$c_2$	32123456	-1		
	$x_1$	2	-1		
$24$	$c_1$	12	-1		
	$c_2$	32123456	-1		
$15$	$c_1$	1	-1		
	$c_2$	2123456	$\zeta_{10}$		
$6$	$c_1$	123456	$\zeta_6$	$Z_{12}$	$(\det _{\zeta_{12}})^2$

3.4.  $W = W(D_5)$ . The characters defined in the following table satisfy [Theorem 2](#) for  $W = W(D_5)$  when  $L = S$ .

$\lambda$	Gen	Word	$\varphi_\lambda$	$C_W(w_\lambda)$	Det
$1^3 2$	$w_{1^3 2}$	$1'2321'3431'2345$	$\zeta_4$		
	$c_1$	$1'$	$-1$		
	$x_1$	$2$	$-1$		
	$x_2$	$3$	$-1$		
$23$	$w_{23}$	$1'3w_{14}$	$\zeta_{12}$		
$14$	$w_{14}$	$1'2345$	$\zeta_8$	$Z_8$	$\det _{\zeta_8}$

3.5.  $W = W(D_6)$ . The characters defined in the following table satisfy [Theorem 2](#) for  $W = W(D_6)$  when  $L = S$ .

$\lambda$	Gen	Word	$\varphi_\lambda$	$C_W(w_\lambda)$	Det
$1^6$	$S$		$\epsilon$	$W$	$\det _{-1}$
$1^2 2^2$	$c_1 c_3$	$31'234$	$\zeta_4$		
	$x_1$	$2$	$-1$		
	$x_3$	$5465$	$-1$		
$1^3 3$	$c_1 c_4$	$43w_{15}$	$\zeta_3$		
	$x_1$	$2$	$-1$		
	$x_2$	$232$	$-1$		
$3^2$	$w_{3^2}$	$1'343w_{15}$	$\zeta_3$	$G(6, 2, 2)$	$\det _{\zeta_6}$
	$c_1^2$	$1'232$	$\zeta_3$		
	$x_1$	$432543654$	$-1$		
$24$	$w_{24}$	$1'3w_{15}$	$\zeta_8^3$		
	$c_2^2$	$(3w_{15})^2$	$\zeta_4$		
$15$	$w_{15}$	$1'23456$	$\zeta_5$	$Z_{10}$	$(\det _{\zeta_{10}})^2$

4. PROOF OF [THEOREM 2](#) WHEN  $L$  IS A PROPER SUBSET OF  $S$

Recall that the normalizer in  $W$  of  $W_L$  factors as the semidirect product of  $W_L$  and a normalizer complement [7]. When the semidirect product is a direct product,  $W_L$  is called *bulky*. It is shown in [4] that [Theorem 2](#) holds if either  $W_L$  is bulky or the rank of  $W_L$  is two or less. Also, it is shown in [3] that [Theorem 2](#) holds if  $W_L$  is a direct product of Coxeter groups of type  $A$ . Thus, to prove [Theorem 1](#) it suffices to prove [Theorem 2](#) for all pairs  $W, W_L$  where the rank of  $W$  is five or six and  $L$  is a proper subset of  $S$  for which the following hold.

- (1)  $W_L$  is not bulky in  $W$ ,
- (2)  $W_L$  has rank at least three, and
- (3)  $W_L$  is not a direct product of Coxeter groups of type  $A$ .

After consulting the table of bulky parabolic subgroups in [2], it remains to consider the pairs shown in [Table 1](#).

$W$	$W_L$
$B_5$	$A_2B_2$
$B_6$	$A_2B_2, A_2B_3, A_3B_2, A_1^2B_2$
$D_5$	$D_4$
$D_6$	$D_4, D_5$
$E_6$	$D_4$

TABLE 1. List of pairs  $W, W_L$  to be considered for [Theorem 2](#)

We consider each such pair  $W, W_L$  separately in the following subsections. For each pair we indicate representatives of the cuspidal conjugacy classes of  $W_L$ , generators of the centralizers of these representatives, and linear characters of the centralizers that satisfy the conclusion of [Theorem 2](#). Additionally, we also give the values of  $\widetilde{\rho}_L, \widetilde{\omega}_L, \alpha_L$ , and  $\epsilon$  for the pair  $W(B_5), W(A_2B_2)$  and the pair  $W(E_6), W(D_4)$ .

In the following sections we use the symbol  $w_n$  to denote a representative of the  $n^{\text{th}}$  conjugacy class of a group in the list of conjugacy classes returned by the command `ConjugacyClasses` in `GAP`. We denote the longest element of  $W$  by  $w_0$  and the longest element in  $W_L$  by  $w_L$ . As in [§3](#) the symbols  $1, 2, \dots, n$  denote the elements of  $S$ .

4.1.  $W = W(B_5)$ . As an illustration, we provide somewhat more detail in the case where  $W = W(B_5)$  and  $L = \{1, 2, 4, 5\}$ . The cuspidal conjugacy classes in  $W_L$  are represented by  $w_{13}$  and  $w_{15}$ . The centralizer of  $w_{15}$  is  $\langle w_{15} \rangle \times \langle w_0 \rangle = Z_{12} \times Z_2$ . The centralizer of  $w_{13}$  is generated by  $C_W(w_{15})$  and 1. We define the characters  $\varphi_{13}$  and  $\varphi_{15}$  by supplying their values at these generators shown in the following table.

L	Type	Characters
$\{1, 2, 4, 5\}$	$A_2B_2$	$\varphi_{13} : (w_{15}, w_0, 1) \mapsto (\zeta_3, 1, -1)$ $\varphi_{15} : (w_{15}, w_0) \mapsto (\zeta_6, 1)$

Then  $\varphi_{13}^{N_W(W_L)} + \varphi_{15}^{N_W(W_L)} = \widetilde{\rho}_L$ . This character is shown in Table 2 together with  $\widetilde{\omega}_L$ ,  $\alpha_L$ , and  $\epsilon$ . The conjugacy classes of  $N_W(W_L)$  are listed in the order determined by GAP where  $W$  is constructed using the command `W:=CoxeterGroup("B",5)` and  $N_W(W_L)$  is constructed using the command

`Normalizer(W,ReflectionSubgroup(W,[1,2,4,5]))`.

The classes are labeled by the orders of their elements and an additional letter to distinguish them from one another. We see that  $\widetilde{\rho}_L = \alpha_L \epsilon \widetilde{\omega}_L$  so that Theorem 2 holds for the pair  $W, W_L$ . This completes the proof of Theorem 1 for  $W$ .

$N_W(W_L)$	1a	2a	3a	2b	2c	6a	2d	2e	6b	2f	2g	6c	2h	2i	6d
$\widetilde{\rho}_L$	6	.	-3	6	.	-3	-2	.	1	-2	.	1	-2	.	1
$\widetilde{\omega}_L$	6	.	-3	6	.	-3	2	.	-1	2	.	-1	2	.	-1
$\alpha_L$	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1
$\epsilon$	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1

$N_W(W_L)$	2j	2k	6e	4a	4b	12a	4c	4d	12b	2l	2m	6f	2n	2o	6g
$\widetilde{\rho}_L$	-2	.	1	-2	.	1	-2	.	1	6	.	-3	6	.	-3
$\widetilde{\omega}_L$	2	.	-1	-2	.	1	-2	.	1	6	.	-3	6	.	-3
$\alpha_L$	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1
$\epsilon$	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1

TABLE 2. Characters of  $N_W(W_L)$  where  $W = W(B_5)$  and  $L = \{1, 2, 4, 5\}$

4.2.  $W = W(B_6)$ . The characters defined in the following table satisfy [Theorem 2](#) for  $W = W(B_6)$  and  $W_L$  as in [Table 1](#). For notational convenience, set  $M = \{1, 2, 3, 4, 5\}$  and  $r$  is the reflection corresponding to the highest long root of  $W$ .

L	Type	Characters
$\{1, 2, 4, 5\}$	$A_2B_2$	$\varphi_{13} : (w_{15}, w_0, w_M, 1) \mapsto (\zeta_3, 1, 1, -1)$ $\varphi_{15} : (w_{15}, w_0, w_M) \mapsto (\zeta_6, 1, 1)$
$\{1, 2, 3, 5, 6\}$	$A_3B_2$	$\varphi_{12} : (1, 2, 3, 56, w_0) \mapsto (-1, -1, -1, \zeta_3, -1)$ $\varphi_{24} : (w_{24}, 1, w_0) \mapsto (\zeta_3^2, -1, -1)$ $\varphi_{30} : (123, 56, w_0) \mapsto (-\zeta_3, \zeta_3^2, -1)$
$\{1, 2, 4, 5, 6\}$	$A_2B_3$	$\varphi_{23} : (1, 2, 456, w_0) \mapsto (-1, -1, \zeta_4, -1)$ $\varphi_{25} : (12, 456, w_0) \mapsto (-1, \zeta_4, -1)$
$\{1, 2, 4, 6\}$	$A_1^2B_2$	$\varphi_{12} : (1, 2, 4, 6, 5465, r) \mapsto (-1, -1, -1, -1, 1, 1)$ $\varphi_{20} : (12, 4, 6, 5465, r) \mapsto (-1, -1, -1, 1, 1)$

4.3.  $W = W(D_5)$ . The characters defined in the following table satisfy [Theorem 2](#) for  $W = W(D_5)$  and  $W_L$  as in [Table 1](#).

L	Type	Characters
$\{1', 2, 3, 4\}$	$D_4$	$\varphi_3 = \epsilon$ $\varphi_9 : (x_1, 1'w_0) \mapsto (-1, \zeta_4)$ $\varphi_{11} : (w_{11}, w_0) \mapsto (\zeta_3, 1)$

4.4.  $W = W(D_6)$ . The characters defined in the following table satisfy [Theorem 2](#) for  $W = W(D_6)$  and  $W_L$  as in [Table 1](#). For notational convenience, set  $M = \{1', 2, 3, 4, 5\}$  and  $x_1 = 3243$ .

L	Type	Characters
$\{1', 2, 3, 4\}$	$D_4$	$\varphi_3 : (1', 2, 3, 4, 6, w_M) \mapsto (-1, -1, -1, -1, 1, 1)$ $\varphi_9 : (6, x_1, 2w_M) \mapsto (1, -1, \zeta_4)$ $\varphi_{11} : (w_{11}, 6, w_M) \mapsto (\zeta_3, 1, 1)$
$\{1', 2, 3, 4, 5\}$	$D_5$	$\varphi_7 : (w_7, w_0, 1', 2, 3) \mapsto (\zeta_4, -1, -1, -1, -1)$ $\varphi_{15} : (w_{15}, w_0) \mapsto (\zeta_{12}, -1)$ $\varphi_{17} : (w_{17}, w_0) \mapsto (\zeta_8, -1)$

4.5.  $W = W(E_6)$ . Let  $L = \{2, 3, 4, 5\}$ . The cuspidal conjugacy classes in  $W_L$  are represented by  $w_3, w_9,$  and  $w_{11}$ . The class containing  $w_{11}$  is also labeled by the partition  $13$  in the notation used in type  $D_4$ . It is convenient to take  $y_{13} = 2354$  as a representative of this class instead of  $w_{11}$ . The centralizer of  $y_{13}$  is generated by  $y_{13}, w_M, w_N$  where  $M = \{2, 3, 4, 5, 6\}$  and  $N = \{1, 2, 3, 4, 5\}$ . Then  $W_M$  and  $W_N$  both are of type  $D_5$ . Notice that conjugation by  $w_M$  exchanges 2 with 3 while conjugation by  $w_N$  exchanges 2 with 5. Set  $x_1 = 4354$ . Define the following characters.

L	Type	Characters
$\{2, 3, 4, 5\}$	$D_4$	$\varphi_3 : (3, 4, w_M, w_0) \mapsto (-1, -1, 1, 1)$ $\varphi_9 : (x_1, 2w_M, 243w_0) \mapsto (-1, -\zeta_4, \zeta_4)$ $\varphi_{11} : (y_{13}, w_M, w_N) \mapsto (\zeta_3, 1, 1)$

In this case,  $N_W(W_L) \cong W(F_4)$ . Then using the notation from [5, Table C.3] for the irreducible characters of  $W(F_4)$  (which is identical to the notation used in CHEVIE), we have

$$\begin{aligned}
\varphi_3^{N_W(W_L)} &= \chi''_{(1,12)} \\
\varphi_9^{N_W(W_L)} &= \chi'_{(6,6)} + \chi''_{(6,6)} \\
\varphi_{11}^{N_W(W_L)} &= \chi''_{(2,4)} + \chi_{(9,2)} + \chi''_{(9,6)} + \chi_{(12,4)} \\
\widetilde{\rho}_L &= \chi''_{(1,12)} + \chi''_{(2,4)} + \chi_{(9,2)} + \chi''_{(9,6)} + \chi'_{(6,6)} + \chi''_{(6,6)} + \chi_{(12,4)} \\
\widetilde{\omega}_L &= \chi'_{(1,12)} + \chi'_{(2,16)} + \chi_{(9,10)} + \chi'_{(9,6)} + \chi'_{(6,6)} + \chi''_{(6,6)} + \chi_{(12,4)}.
\end{aligned}$$

Now since  $\alpha_L \epsilon = \chi_{(1,24)}$  is the sign character of  $W(F_4)$ , the calculations above together with [5, Table C.3] show that the characters  $\varphi_3, \varphi_9, \varphi_{11}$  satisfy [Theorem 2](#) for the pair  $W, W_L$ .

**Acknowledgments:** The authors would like to acknowledge support from the DFG-priority program SPP1489 “Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory”.

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