Rank distributions of graphs over the field of two elements

Title	Rank distributions of graphs over the field of two elements
Author(s)	Safarji, Badriah
Publication Date	2026-01-14
Publisher	University of Galway

Rank distributions of graphs over the field of two elements

by

Badriah Safarji

A thesis presented in fulfilment of the requirements for the degree of Doctor of Philosophy

Supervisors:
Dr Rachel Quinlan, Dr Cian O'Brien

School of Mathematical and Statistical Sciences
University of Galway
Galway City, Ireland

January, 2026

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Declaration

I, Badriah Safarji, declare that the work presented in this thesis is my own, and has not been previously submitted for award at another degree granting institution.

Acknowledgements

First and foremost, all praise and thanks are due to God, the Most Merciful and the Most Kind. The completion of this thesis would not have been possible without the support and help of many people, to whom I am deeply grateful.

I owe my greatest thanks to my supervisors, Dr. Rachel Quinlan and Dr. Cian O'Brien, for their continuous guidance, encouragement, and enthusiasm for the project. Their insightful discussions, constructive feedback, and valuable comments greatly improved my work and deepened my understanding of the topic.

In addition, I would also like to thank the other members of my graduate research committee, Dr. Kevin Jennings, Dr. Emil Sköldberg, and Dr. Jim Cruickshank, for their support throughout my postgraduate degree.

My sincere thanks go to everyone in the School of Mathematics, Statistics, and Applied Mathematics at university of Galway for creating such a friendly and supportive environment. I am grateful to all the staff who were always willing to help and made my time here much easier and more enjoyable.

To my fellow postgraduate students past and present, and especially those with whom I shared an office, thank you for the many discussions, both mathematical and otherwise. Your company made the long days much brighter.

I would like to express my heartfelt gratitude to my daughter, Farah Aluqmani, for her patience and understanding of a busy Ph.D. single parent. Thanks to my husband Mohammed Alluqmani for his patience and support. My deepest thanks also go to my parents, Fathiyyah Afif and Abdulkarim Safarji, for their constant prayers, to my sisters Hannen, Afnan, Masarrah, and brother Mohammed for their love and strong support, and to my dear friends in Galway, whose encouragement helped me. Your care and kindness have meant more to me than words can express.

I am grateful for the financial support from the Saudi Arabian Cultural Bureau, Saudi Embassy, and University of Tabuk, without which this research would not have been possible.

Finally, I wish to thank my examiners, Dr. Jane Breen (external) and Dr. Jesse Lansdown (internal), for their time, effort, and valuable feedback during my viva and the thesis corrections.

Abstract

A square matrix M represents a graph Γ if its nonzero off-diagonal entries encode the adjacencies of Γ according to a fixed vertex ordering. Over the field of two elements, we study the distribution of ranks within the affine space of all matrices representing a particular graph. The motivating question is which graphs of order n are represented by more matrices of rank $n-1$ than of rank n. This reflects the fact that the most frequently occurring rank is not n but $n-1$ in the space of all $n \times n$ matrices over $\mathbb{F}_2$, a property which is exceptional to $\mathbb{F}_2$. This thesis focuses on connected graphs that have a path or cycle as a subgraph induced on all but one vertex (called the *extra* vertex).

The path graph P_n serves as the starting point of this study. The path graph is fundamental in the related and widely studied minimum rank problem, and provides a foundation for our later analysis of the set $\mathcal{G}^P$ of graphs containing an induced path on all but one vertex. A main result is a characterisation of all such graphs that are represented by more matrices of rank $n-1$ than rank n over $\mathbb{F}_2$. This is achieved by first examining the vectors in the nullspace of each matrix representing P_n. An expression for the difference $\alpha(\Gamma)$ between rank $n-1$ and rank n representations of a given graph $\Gamma \in \mathcal{G}^P$ is determined in terms of these nullspace vectors. A recurrence is then established, expressing $\alpha(\Gamma)$ in terms of α for graphs in $\mathcal{G}^P$ for which the extra vertex has lower degree than in Γ. We classify all $\Gamma \in \mathcal{G}^P$ satisfying $\alpha(\Gamma) < 0$ by first classifying those for which the extra vertex has degree 1, then using that to simplify and classify the degree 2 case, and continuing like this until it is shown that no such graphs exist for degree $\geqslant 6$.

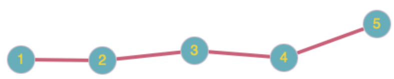
We then turn to the analogous problem for the cycle graph. We show that half of all $\mathbb{F}_2$-matrices representing C_n have rank $n-1$, approximately one-third have rank n, and approximately one-sixth have rank $n-2$. We then investigate the set of graphs containing an induced cycle on all but one vertex, denoted by $\mathcal{G}^C$. Our analysis reveals essential structural contrasts between the classes $\mathcal{G}^P$ and $\mathcal{G}^C$: while the degree of the extra vertex is bounded in the path case, it can be arbitrarily large in the cycle case. An infinite family of graphs called alternating wheel graphs demonstrate this contrast, as there exists an alternating wheel graph $\Gamma \in \mathcal{G}^C$ with an extra vertex of any even degree $d \geqslant 4$ satisfying $\alpha(\Gamma) < 0$.

Chapter 1

Introduction

Given a field $\mathbb{F}$ and a simple undirected graph Γ with vertices $x_1, x_2, \ldots, x_n$, a symmetric matrix M with entries in $\mathbb{F}$ *represents* Γ (with respect to this ordering of the vertices) if, when $i \neq j$, the (i,j)-entry of M is 0 if and only if there is no edge between x_i and x_j in Γ. The diagonal entries of M are not subject to any constraints, and therefore there are many matrices representing Γ over $\mathbb{F}$. Since the non-zero off-diagonal entries may vary through the non-zero elements of $\mathbb{F}$, subject to the constraint of symmetry, and the diagonal entries are freely chosen from $\mathbb{F}$, the set $S(\mathbb{F}, \Gamma)$ of all symmetric matrices over $\mathbb{F}$ representing Γ is a union of affine spaces of dimension n.

Example 1.0.1. *Figure 1.1 shows the path graph P_5 together with the general form of a matrix representing P_5 over a field $\mathbb{F}$.*



Path graph P_5

$$M = \begin{bmatrix} d_1 & * & 0 & 0 & 0 \\ * & d_2 & * & 0 & 0 \\ 0 & * & d_3 & * & 0 \\ 0 & 0 & * & d_4 & * \\ 0 & 0 & 0 & * & d_5 \end{bmatrix}$$

$d_1, \ldots, d_5 \in \mathbb{F}$, each $*$ denotes a nonzero element of $\mathbb{F}$, and M is symmetric.

Figure 1.1: P_5 and the general form of a matrix M representing P_5.

1.1 The minimum rank problem

Definition 1.1.1. *The* rank of a matrix *is the number of linearly independent rows (equivalently, columns).*

Lemma 1.1.2. *Let A be a symmetric $n \times n$ matrix of rank $r < n$, with entries in a field $\mathbb{F}$. Then one may alter a single diagonal entry in A to another element of $\mathbb{F}$ to obtain a symmetric matrix of rank $r + 1$.*

Proof. Since $\text{rank}(A) = r < n$, some row of A is a linear combination of the others. Without loss of generality, suppose that the first row has this property. Let d be the entry in the $(1, 1)$-position of A, let v be the vector with entries equal to the last $n - 1$ of column 1 of A, and let A' be the submatrix obtained by deleting the first row and column of A. Then

$$A = \left(\begin{array}{c|c} d & v^\top \\ \hline v & A' \end{array}\right).$$

Since the first row is a linear combination of the others, there exists some $w \in \mathbb{F}^{n-1}$ with $w^\top(v|A') = (d|v^\top)$. Suppose $u \in \mathbb{F}^{n-1}$ satisfies $u^\top A' = v^\top$. Then $(w^\top - u^\top)A' = 0$, and since v is a linear combination of the rows of A', this implies that $(w^\top - u^\top)v = 0$. Therefore $u^\top v = w^\top v = d$, and so $(d'|v^\top)$ is not a linear combination of the rows of $(v|A')$ for any $d' \neq d$. So, for any $d' \neq d$,

$$\left(\begin{array}{c|c} d' & v^\top \\ \hline v & A' \end{array}\right)$$

is symmetric and has rank $r + 1$. □

Since changing a diagonal entry does not change the graph represented by a symmetric matrix, it follows from Lemma 1.1.2 that for every graph Γ the set of all ranks of matrices in $S(\mathbb{F}, \Gamma)$ is an interval of the form $\{r \in \mathbb{N} : m \leqslant r \leqslant n\}$. The positive integer m is called the *minimum rank* of Γ over $\mathbb{F}$. The problem of determining m is known as the *minimum rank problem for graphs*.

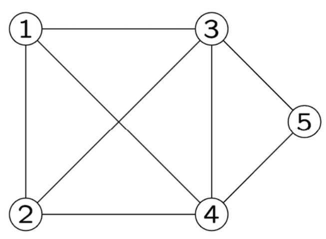
An informative survey of the extensive literature on the minimum rank problem for graphs (until 2007) is provided by Fallat and Hogben in [5]. A second survey by Hogben in 2010 [9] situates the minimum rank problem for graphs in a wider combinatorial context. The minimum rank generally depends on the choice of field $\mathbb{F}$. The case $\mathbb{F} = \mathbb{R}$ has been the subject of particularly concentrated and sustained attention, but the case of finite fields offers scope to determine more detailed information, due to the availability of combinatorial and enumerative methods. For every positive integer k and prime power q, a description of the structure of all graphs whose minimum rank over $\mathbb{F}_q$ is at most k is given in [8]. Every simple graph whose minimum rank over $\mathbb{F}_q$ is at most k is a blowup of one of a finite family of graphs, determined by congruence classes of symmetric $k \times k$ matrices over $\mathbb{F}_q$ (possibly with some isolated vertices added). A blowup of a graph is obtained by replacing every vertex with a finite collection of copies so that the copies of two vertices are adjacent if and only if the originals are. Simple undirected graphs with minimum rank 2 over a finite field are characterized in [2]. For finite fields of characteristic different from 2, Theorem 11 of this paper characterizes such graphs as those that are free of nine specified induced subgraphs. For finite fields of characteristic 2, the analagous theorem has a list of nine forbidden induced subgraphs, which reduces to seven in the special case of the field of two elements. It is shown in [4] that analogous characterizations via forbidden induced subgraphs exist for higher values of the minimum rank. It is demonstrated that for any finite field F and any integer k, the set of graphs whose minimal rank over F is at most k can be described via a list of finitely many forbidden induced subgraphs. Each of these forbidden subgraphs has at most $(|F| \cdot k/2 + 1)^2$ vertices. The average minimum rank of all labelled graphs of order n over a finite field is investigated in [7]. This work also studies the minimum rank of undirected simple graphs that contain a clique of size k over any finite non-prime fields of characteristic p. The paper shows that when $k = n - 2$ and the order of the graph is at least 4, this class of graphs has minimum rank at most 3 over $\mathbb{F}_q$ with $q \geqslant 4$. In contrast, over the field $\mathbb{F}_3$, the minimum rank is greater than 3 for every graph of order at least 10. The paper also considers the case $k = n - 3$ and shows that this class of graphs has minimum rank at most 4 over finite fields $\mathbb{F}_q$ with $q \geqslant 4$.

A number of papers focus specifically on minimum rank problems over the field $\mathbb{F}_2$ of two elements, which is exceptional in many ways and is the main focus of this thesis. For example, the relationship between the minimum rank of a simple graph over $\mathbb{F}_2$ and the minimum cardinality of a *subgraph complementation system* of the graph is studied in [3]. A *subgraph complementation system* in a graph Γ with vertex set V is a collection $\mathcal{C}$ of subsets of V with the property that the vertices u and v are adjacent in Γ if and only if they occur together in an odd number of sets in $\mathcal{C}$. The analysis shows that the minimum cardinality of such a collection equals the minimum rank of the graph over $\mathbb{F}_2$ when the minimum rank of the graph over $\mathbb{F}_2$ is odd or the graph is a forest. Otherwise, the difference between these two parameters is at most 1. Graphs with minimum rank at most 3 over $\mathbb{F}_2$ are characterized in [1], via a list of 62 minimal forbidden subgraphs of order at most 8. The paper also explores how this list of all minimal forbidden subgraphs for this class of graphs extends to arbitrary fields.

We denote the minimum rank of a simple graph Γ over a field $\mathbb{F}$ by

$$\mathrm{mr}(\mathbb{F}, \Gamma) = \min\{\operatorname{rank} A \mid A \in S(\mathbb{F}, \Gamma)\}.$$

Example 1.1.3. *Figure 1.2 shows a graph Γ and corresponding matrix A.*



$$\Longleftrightarrow \begin{bmatrix} d_1 & * & * & * & 0 \\ * & d_2 & * & * & 0 \\ * & * & d_3 & * & * \\ * & * & * & d_4 & * \\ 0 & 0 & * & * & d_5 \end{bmatrix}$$

Figure 1.2: A graph Γ and the general form of $A \in S(\mathbb{F}, \Gamma)$, adapted from [8].

Suppose we replace each $*$ in the off-diagonal of A in Figure 1.2 with a nonzero element from $\mathbb{F}_2$ (i.e $\mathbb{F} = \mathbb{F}_2$). Then we notice that any $A \in S(\mathbb{F}_2, \Gamma)$ has the form

$$\begin{bmatrix} d_1 & 1 & 1 & 1 & 0 \\ 1 & d_2 & 1 & 1 & 0 \\ 1 & 1 & d_3 & 1 & 1 \\ 1 & 1 & 1 & d_4 & 1 \\ 0 & 0 & 1 & 1 & d_5 \end{bmatrix}$$

and the rank of any such $A \geqslant 3$ *[8]. This is because the matrix* A *contains at least three linearly independent rows. For instance, we can see rows* $2, 4,$ *and* 5 *are linearly independent since the* 3×3 *submatrix occupying these rows and columns 1,3,5 has determinant 1 for either choice of* d_5*. There are matrices in* $S(\mathbb{F}_2, \Gamma)$ *with rank exactly 3. For example with* $(d_1, d_2, d_3, d_4, d_5) = (0, 0, 1, 1, 1)$*, we have*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Here, row 3 is equal to row 4. Additionally, row 1 is the sum of rows 2, 4, and 5. Hence, the rank is 3.

The minimum rank of the path P_n over any field is $n - 1$, and this is the unique graph of order n with this property. The minimum rank of the cycle C_n over any field is $n - 2$. See [5] for a discussion of these points.

Since the maximum rank of a graph is always the order of the graph, there is not much interest in the analogous maximum rank problem. However, comparing the number of symmetric matrices of each rank from minimum rank to full rank over a given finite field is wider problem that can be studied. Over any infinite field, the number of symmetric matrices representing a particular graph is infinite, and the number of matrices of different ranks can not be compared. If the field $\mathbb{F}$ is finite, then there are a finite number of matrices that may be used to describe a certain graph over $\mathbb{F}$ and the rank distribution can be calculated in principle. This opens the door to a more thorough investigation of the distribution of the matrices' ranks over finite fields.

1.2 Rank distributions

We begin by counting the number of matrices representing a particular graph over a given finite field.

Theorem 1.2.1. *Let* Γ *be a simple graph with* n *vertices and* $E(\Gamma)$ *edges. The number of symmetric matrices over a finite field* $\mathbb{F}$ *with* $|\mathbb{F}| = q$ *that represent* Γ *is*

$$S(q, \Gamma) = q^n (q - 1)^{|E(\Gamma)|}.$$

Proof. There are q^n possible ways to select the diagonal entries (each of the n positions can be any of the q elements). For the entries above the main diagonal corresponding to edges of Γ, there are $(q-1)^{|E(\Gamma)|}$ possible choices, since each such entry must be a nonzero element of $\mathbb{F}$. Entries above the main diagonal corresponding to non-edges must be 0. By symmetry, the entries below the main diagonal are determined. Thus, the total number of symmetric matrices representing Γ is $q^n(q-1)^{|E(\Gamma)|}$. □

For the finite field $\mathbb{F}_q$ of order q, one can investigate the number of matrices of each rank in the finite set $S(\mathbb{F}_q, \Gamma)$. For $k \leqslant n$, we write $R_{k,q}(\Gamma)$ for the number of matrices of rank k in $S(\mathbb{F}_q, \Gamma)$.

Definition 1.2.2. *Let Γ be a graph of order n. The* rank distribution *of Γ over the finite field $\mathbb{F}_q$ is the list $R_{m,q}(\Gamma), \ldots, R_{n,q}(\Gamma)$, where m is the minimum rank of Γ over $\mathbb{F}_q$.*

Example 1.2.3. *The following table demonstrates that the rank distribution of a graph over $\mathbb{F}_2$ may or may not be a strictly increasing sequence. These rank counts were obtained using SageMath by generating all symmetric $\mathbb{F}_2$matrices representing P_5 and C_5.*

Rank	3	4	5
P_5		11	21
C_5	5	16	11

Table 1.1: Rank distribution of P_5 and C_5 over $\mathbb{F}_2$.

Example 1.2.4. *The following table demonstrates that the rank distribution of a particular graph may or may not be a strictly increasing sequence over different fields. These rank counts were obtained using SageMath by generating all symmetric $\mathbb{F}_2$matrices representing C_5.*

Rank	3	4	5
$\mathbb{F}_2$	5	16	11
$\mathbb{F}_3$	320	2592	4864

Table 1.2: Rank distribution of the cycle C_5 over $\mathbb{F}_2$ and $\mathbb{F}_3$.

In this thesis, we focus on rank distributions over $\mathbb{F}_2$. One reason for this is that $\mathbb{F}_2$ is the only finite field over which the list of numbers of $n \times n$ matrices of rank $0, 1, 2, \ldots, n$ is not a strictly increasing sequence, as discussed below. The same is true when restricting to symmetric matrices.

Another reason that the field of two elements is exceptional in this context is that, subject to a fixed ordering of the vertices, the off-diagonal entries of the matrix representing a graph are fully determined. This means that the set of all such matrices is an affine subspace of $M_n(\mathbb{F}_2)$ of dimension n.

Fisher and Alexander [6] provide the following formula for the number $M(n, k, q)$ of all $n \times n$ matrices of rank k over the field $\mathbb{F}_q$ (not restricted to symmetric matrices), where $0 \leqslant k \leqslant n$.

$$M(n,k,q) = \frac{\prod_{j=0}^{k-1} \left(q^n - q^j\right)^2}{\prod_{j=0}^{k-1} (q^k - q^j)}$$

Comparing $M(n,k,q)$ and $M(n,k+1,q)$, we have the following.

$$\begin{aligned} M(n,k+1,q) &= \frac{\prod_{j=0}^{k} \left(q^n - q^j\right)^2}{\prod_{j=0}^{k} (q^{k+1} - q^j)} = \frac{\left(q^n - q^k\right)^2 \prod_{j=0}^{k-1} \left(q^n - q^j\right)^2}{q^{k+1} \prod_{j=0}^{k} (q^k - q^{j-1})} = \frac{\left(q^n - q^k\right)^2}{q^{k+1}} \cdot \frac{\prod_{j=0}^{k-1} \left(q^n - q^j\right)^2}{\prod_{j=-1}^{k-1} (q^k - q^j)} \\ &= \frac{q^{2k}\left(q^{n-k} - 1\right)^2}{q^{k+1}\left(q^k - q^{-1}\right)} \cdot \frac{\prod_{j=0}^{k-1} \left(q^n - q^j\right)^2}{\prod_{j=0}^{k-1} (q^k - q^j)} = \frac{q^k\left(q^{n-k} - 1\right)^2}{q^{k+1} - 1} M(n,k,q) \end{aligned}$$

$M(n,k,q) > M(n,k+1,q)$ if and only if $\frac{q^k\left(q^{n-k}-1\right)^2}{q^{k+1}-1} < 1$. Rearranging gives $q^{2n-k} - 2q^n - q^{k+1} + q^k + 1 < 0$.

If $k \leqslant n-2$:

$$\begin{aligned} q^{2n-k} - 2q^n - q^{k+1} + q^k + 1 &= q^{2n-k} - 2q^n + (1-q)q^k + 1 \\ &\geqslant q^{n+2} - 2q^n + (1-q)q^{n-2} + 1 \\ &\geqslant q^{n+2} - 2q^n - q^{n-1} + q^{n-2} + 1 \\ &\geqslant q^{n-2}\left(q^4 - 2q^2 - q + 1\right) + 1 > 0 \end{aligned}$$

If $k = n-1$:

$$\begin{aligned} q^{2n-k} - 2q^n - q^{k+1} + q^k + 1 &= q^{n+1} - 2q^n - q^n + q^{n-1} + 1 \\ &= q^{n-1}(q^2 - 3q + 1) + 1 \end{aligned}$$

This expression is positive for $q \geqslant 3$. However, if $q = 2$, it is equal to $1 - 2^{n-1}$. For fixed $q \geqslant 3$ and fixed $n \geqslant 1$, we conclude that $M(n,k,q)$ strictly increases with k, for $0 \leqslant k \leqslant n$. For fixed $n \geqslant 2$, $M(n,k,2)$ increases with k for $0 \leqslant k \leqslant n-1$ but $M(n,n-1,2) > M(n,n,2)$. Therefore, the most frequently occurring rank in $M_n(\mathbb{F}_2)$ is not n but $n-1$.

Example 1.2.5. *Let A be any square matrix with order* $n = 5$ *over a finite field* $\mathbb{F}$. *It is clear that if* $\mathbb{F}$ *is a finite field, then the general linear group* $\mathrm{GL}_n(F)$ *of all invertible* $n \times n$ *matrices over* $\mathbb{F}$ *has only finitely many elements. If* $\mathbb{F} = \mathbb{F}_3$, *then the proportion of* 5×5 *matrices with full rank is* 0.561, *whereas the proportion of full rank* 5×5 *matrices over* $\mathbb{F}_2$ *is 0.298. Therefore the probability that an arbitrary matrix has full rank is much higher over* $\mathbb{F}_3$ *than* $\mathbb{F}_2$.

Example 1.2.6. *Let $n = 2$. The total number of 2×2 matrices with entries in $\mathbb{F}_2$ is*

$$|M_2(\mathbb{F}_2)| = 2^{2\times 2} = 16.$$

There are three possible ranks, with 1 matrix of rank 0, 9 of rank 1, and 6 of rank 2.

$$\text{Rank } 0: \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Rank } 1: \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{Rank } 2: \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Over $M_2(F_2)$, we see that matrices of rank $n-1$ occur more frequently than matrices of rank n. Table 1.3 shows how this compares to the rank distribution of 2×2 matrices over $\mathbb{F}_3$.

Rank	0	1	2	Total
$\mathbb{F}_2$	1	9	6	16
$\mathbb{F}_3$	1	32	48	81

Table 1.3: Number of 2×2 matrices of each rank over $\mathbb{F}_2$ and $\mathbb{F}_3$.

When we restrict attention to symmetric matrices, $\mathbb{F}_2$ is again an exception. MacWilliams [10, Theorem 2] provides the following formulae for the number $N(n, r, q)$ of symmetric $n \times n$ matrices over $\mathbb{F}_q$ of rank $r = 2s$ and rank $r = 2s + 1$, $s \in \mathbb{N}$.

$$N(n, 2s, q) = \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i} - 1} \cdot \prod_{i=0}^{2s-1} \left(q^{n-i} - 1\right), 2s \leqslant n \tag{1.1}$$

$$N(n, 2s+1, q) = \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i} - 1} \cdot \prod_{i=0}^{2s} \left(q^{n-i} - 1\right), 2s + 1 \leqslant n. \tag{1.2}$$

Again we compare these counts for consecutive ranks.

Suppose $r = 2s + 1 \leqslant n$:

$$\begin{aligned} N(n, 2s+1, q) &= \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i} - 1} \cdot \prod_{i=0}^{2s} \left(q^{n-i} - 1\right) \\ &= \left(q^{n-2s} - 1\right) \cdot \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i} - 1} \cdot \prod_{i=0}^{2s-1} \left(q^{n-i} - 1\right) = \left(q^{n-2s} - 1\right) \cdot N(n, 2s, q) \end{aligned}$$

Suppose $r = 2s \leqslant n$:

$$\begin{aligned} N(n,2s,q) &= \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i}-1} \cdot \prod_{i=0}^{2s-1} \left(q^{n-i}-1\right) \\ &= \frac{q^{2s}}{q^{2s}-1} \cdot \left(q^{n-(2s-1)}-1\right) \cdot \prod_{i=1}^{s-1} \frac{q^{2i}}{q^{2i}-1} \cdot \prod_{i=0}^{2s-2} \left(q^{n-i}-1\right) \\ &= \left(1 + \frac{1}{q^{2s}-1}\right) \cdot \left(q^{n+1-2s}-1\right) \cdot N(n,2s-1,q) \end{aligned}$$

Setting $q = 2$, we observe that

$$N(n,n,2) = N(n,n-1,2)$$

for odd n, and

$$N(n,n,2) = \left(1 + \frac{1}{2^n-1}\right) N(n,n-1,2)$$

for even n. Otherwise, if $r < n$ or $q > 2$, then $N(n,r,q)$ is a ($\geqslant 2$) multiple of $N(n,r-1,q)$. For this reason, we generally expect a connected graph of order n to be represented by at least as many matrices over $\mathbb{F}_q$ of rank r as of rank $r-1$. This suggests that exceptions to this pattern may be prevalent in the case $q = 2$ and $r = n$. Over finite fields of order greater than 2, we know of no connected graph whose rank distribution is not strictly increasing.

Example 1.2.7. *Suppose $n = 5$. We can see in Table 1.4 the number of symmetric matrices of each rank over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$. The number of matrices of rank $n-1$ and n matches over $\mathbb{F}_2$ since n is odd. In contrast, over $\mathbb{F}_3$, the number of rank r is more than double the number of rank $r-1$ for all $r \in \{1, \dots, n\}$. The numbers for each rank in the following table were obtained using Equations 1.1 and 1.2.*

Rank	$\mathbb{F}_2$	$\mathbb{F}_3$
0	1	1
1	31	242
2	620	21780
3	4340	1698840
4	13888	4586868
5	13888	9173736

Table 1.4: Number of symmetric 5×5 matrices of rank r over $\mathbb{F}_2$ and $\mathbb{F}_3$.

Example 1.2.8. *Suppose $n = 6$. The number of symmetric matrices of each rank over $\mathbb{F}_2$ and $\mathbb{F}_3$ is given in Table 1.5. Since n is even, the number of symmetric matrices of rank $n-1$ and n only differ slightly. Again, over $\mathbb{F}_3$, the number of rank n is at least double the number of rank $n-1$. The numbers for each rank in this example were obtained using Equations 1.1 and 1.2.*

The goal of this thesis is to identify classes of connected graphs which are represented by more matrices over $\mathbb{F}_2$ of rank $n-1$ than of rank n. In the opening

Rank	$\mathbb{F}_2$	$\mathbb{F}_3$
0	1	1
1	63	728
2	2,604	198,198
3	78,120	15,855,840
4	291,648	417,404,988
5	874,944	3,339,239,904
6	888,832	6,687,653,544

Table 1.5: Number of symmetric 6×6 matrices of rank r over $\mathbb{F}_2$ and $\mathbb{F}_3$.

chapters, we focus on connected graphs that have a path on all but one of their vertices as an induced subgraph. This is motivated by the special role of the path graph in the study of the minimum rank problem. Over any field $\mathbb{F}$, the path P_n is the unique graph of order n whose minimum rank over $\mathbb{F}$ is $n-1$ [5]. In later chapters, we turn our attention on connected graphs with an induced cycle on all but one of their vertices.

To demonstrate the variety in rank distribution possible across different classes of graphs, consider the examples of complete graphs, cycle graphs, and path graphs. If M is a matrix representing K_n over $\mathbb{F}_2$ with r zero entries on the diagonal, then the rank of M is $r+1$ if $r < n$. If $r = n$, then M has rank n if n is even, and rank $n-1$ if n is odd.

So for $r \leqslant n-2$, the number of matrices of rank r representing K_n is the number of ways to put $r-1$ zeros on the diagonal, $\binom{n}{r-1}$. If n is even, the number of rank $n-1$ is $\binom{n}{n-2}$ and of rank n is $\binom{n}{n-1}+1$. If n is odd, the number of rank $n-1$ is $\binom{n}{n-2}+1$ and of rank n is $\binom{n}{n-1}$.

Theorem 2.0.2 describes the rank distribution of the path graph P_n, and Theorem 4.0.1 describes the rank distribution of the cycle graph C_n. The table below summarises these results, along with the rank distribution of the complete graph K_n. We use R_k to denote the number of matrices over $\mathbb{F}_2$ of rank k representing Γ.

Γ	R_1	...	R_{n-3}	R_{n-2}	R_{n-1}	R_n
K_n	$\binom{n}{0}$	...	$\binom{n}{n-4}$	$\binom{n}{n-3}$	$\binom{n}{n-2}+\frac{1-(-1)^n}{2}$	$\binom{n}{n-1}+\frac{1+(-1)^n}{2}$
C_n		...		$\frac{1}{3}(2^{n-1}+(-1)^n)$	2^{n-1}	$\frac{1}{3}(2^n+(-1)^{n-1})$
P_n		...			$\frac{1}{3}(2^n+(-1)^{n-1})$	$\frac{1}{3}(2^{n+1}+(-1)^n)$

So the most frequently occurring rank(s) in the rank distribution of K_n is $\frac{n+2}{2}$ if n is even, and $\frac{n+1}{2}$ and $\frac{n+3}{2}$ if n is odd, while the most frequently occurring rank for C_n is $n-1$, and for P_n it is n.

The rank distribution of all symmetric matrices suggests that we can expect the peak to be at rank $n-1$ or rank n for nearly all graphs, with comparable frequency. This explains the focus on ranks $n-1$ and n in this thesis. The complete graph

K_n provides an example where the peak occurs at a different rank, an exceptional behaviour.

We remark that one may interpret an element of $S(\mathbb{F}_2, \Gamma)$ as the $\mathbb{F}_2$-adjacency matrix of the graph adapted from Γ by adding a loop at every vertex where the diagonal entry is 1. We refer to a graph constructed from Γ by adding loops at distinct vertices as a *looped extension*. Suppose that Γ' and Γ'' are looped extensions of Γ. Then Γ' and Γ'' are isomorphic to each other if and only if some automorphism of Γ maps the set of looped vertices of Γ' to the set of looped vertices of Γ''. For $1 < k < n-1$, the $\binom{n}{k}$ matrices that represent K_n and have rank k are those having $n-k+1$ non-zero entries on the diagonal. All of these matrices correspond to isomorphic looped extensions of K_n, with $n-k+1$ loops. In the row corresponding to the complete graph in the above table, each entry corresponds to a single isomorphism class of looped graphs (except that one of the last two entries corresponds to two classes). The complete graph is an extreme case, since up to isomorphism there is only one way to extend a complete graph by adding a fixed number of loops at distinct vertices. In the case of the path and cycle graphs, the situation is more complex. The automorphism group of the path graph has order 2. Most isomorphism types of looped extensions of P_n are counted twice by the numbers in the final row of the above table, except for those few whose loops are preserved by the non-trivial automorphism of the path P_n. The automorphism group of C_n is dihedral of order $2n$. Every isomorphism type of looped extension of C_n is counted up to $2n$ times by the numbers in the table above, depending on the number of orbits of the set of looped vertices under the action of the automorphism group of C_n. Most graphs whose $\mathbb{F}_2$-rank distributions are considered in this thesis have trivial automorphism group, which means that all distinct choices for the diagonal entries of a representing matrix correspond to mutually non-isomorphic looped extensions.

In this work, we restrict our attention to connected graphs because the rank distribution of any graph can be determined by the rank distributions of its connected components. The vertices of a disconnected graph can be ordered such that each matrix representing it is block-diagonal with blocks corresponding to its connected components, and so the rank of a matrix representing a disconnected graph is the sum of the ranks of the blocks corresponding to its components. Therefore, studying connected graphs captures all the essential complexity of the general case.

We have seen that the number of symmetric $n \times n$ matrices over $\mathbb{F}_q$ of rank r is at least double the number of rank $r-1$ for all $r \in \{1, \ldots, n\}$ with $(q, r) \neq (2, n)$, while the number of rank $n-1$ and of rank n are approximately equal over $\mathbb{F}_2$. Since every symmetric matrix represents some graph, it cannot be true for every positive integer n that $R_n(\Gamma) > R_{n-1}(\Gamma)$ for every graph Γ of order n. There exist graphs of order n represented by more matrices over $\mathbb{F}_2$ of rank $n-1$ than of rank n. Disconnected graphs alone do not account for this. There are more connected graphs of order n than disconnected graphs because the complement of a disconnected graph is connected, but the complement of a connected graph is not necessarily disconnected. The rank distribution of a disconnected graph is equal to the discrete convolution of the distributions of its connected components. Consequently, the more connected components a graph has, the more likely its most frequently occurring rank is to be strictly between its minimum and maximum rank. This is true over any finite field, and therefore does not account for the exceptional rank distribution of all symmetric matrices over $\mathbb{F}_2$.

The rest of the thesis is arranged as follows. In Chapter 2, we analyse the rank distribution for the path graph P_n. Chapter 3 presents an analysis of the rank distribution for the class $\mathcal{G}^P$ of graphs with an induced path on all but one vertex, first by describing formulae for the number of $\mathbb{F}_2$-matrices representing graphs of order n in $\mathcal{G}^P$ of rank $n-1$ and n in terms of the nullspace vectors of matrices representing P_n, and then by establishing recurrences for these formulae. We classify graphs in $\mathcal{G}^P$ for which the extra vertex has degree 1, then extend the classification step-by-step to degree 2, then 3, 4, 5, and prove that none exist for degree $\geqslant 6$. The full statement of these results appears in Theorem 3.3.2. In Chapter 4, we describe the rank distribution for the cycle graph C_n. Chapter 5 presents an analysis of the rank distribution for the class $\mathcal{G}^C$ of graphs with an induced cycle on all but one vertex, again expressing counting functions for the number of $\mathbb{F}_2$-matrices representing graphs of order n in $\mathcal{G}^C_d$ with rank $n-1$ and rank n in terms of nullspace vectors of matrices representing C_{n-1}. Among graphs in $\mathcal{G}^C$, we fully describe which have more matrix representations of rank $n-1$ than rank n for those with extra vertex of degree $\leqslant 3$. Theorem 5.4.8 proves that, for any positive even integer d, there exists a graph in $\mathcal{G}^C$ with extra vertex of degree d which is represented by more matrices of rank $n-1$ than rank n, highlighting the vast difference in behaviour between $\mathcal{G}^P$ and $\mathcal{G}^C$. Throughout this thesis, we write $\mathcal{G}^P_d$ (resp. $\mathcal{G}^C_d$) for the class of connected graphs that contain an induced path (resp. cycle) on all but one vertex and for which the extra vertex has degree d.

Chapter 2

Rank distribution for the path graph P_n over $\mathbb{F}_2$

For a positive integer n, we write P_n for the path graph on n vertices. We write $x_1, x_2, \ldots, x_n$ for the vertices of P_n, where x_1 and x_n are the two vertices of degree 1, and x_i is adjacent to x_{i+1} for $1 \leqslant i \leqslant n-1$. Over any field $\mathbb{F}$, every matrix that represents P_n with respect to this vertex ordering is tridiagonal with nonzero entries in the super-diagonal and sub-diagonal; therefore, it has rank at least $n-1$. Over any field $\mathbb{F}$, it is routine to check that if the nonzero off-diagonal entries of a matrix representing P_n are all 1, then the diagonal entries may be completed to obtain a matrix either of rank n or of rank $n-1$. The path P_n has a special role in the minimum rank problem for graphs; for $n \geqslant 1$ and for every field $\mathbb{F}$, P_n is the unique graph whose minimum rank over $\mathbb{F}$ is $n-1$ (see [5]).

We define an *indeterminate matrix* over $\mathbb{F}$ to be a matrix in which each entry is either an element of $\mathbb{F}$ or an indeterminate. A *completion* of an indeterminate matrix M over $\mathbb{F}$ is a matrix that results from an assignment of elements of $\mathbb{F}$ to the indeterminates of M. We say a symmetric indeterminate matrix M *represents* a graph Γ if, for all $i \neq j$, $M_{ij} = 0$ if and only if $x_i x_j$ is not an edge of Γ. Therefore any symmetric completion of M in which off-diagonal indeterminates are assigned non-zero values represents Γ.

We define $M(P_n)$ to be the indeterminate matrix over $\mathbb{F}_2$ with all entries in the first superdiagonal and the first subdiagonal equal to 1, indeterminates on the main diagonal, and zeros elsewhere.

$$M(P_n) = \begin{bmatrix} d_1 & 1 & 0 & \cdots & 0 \\ 1 & d_2 & 1 & \ddots & \vdots \\ 0 & 1 & d_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & d_n \end{bmatrix}$$

Our goal in this chapter is to determine $R_n(P_n)$ and $R_{n-1}(P_n)$, the number of $\mathbb{F}_2$-completions of $M(P_n)$ of ranks n and $n-1$, respectively. We denote the ith standard basis vector by e_i.

Lemma 2.0.1. *Let A be a completion of $M(P_n)$. Then A has rank n if and only if $e_1 \in \mathbb{F}_2^n$ is in the column space of A. Equivalently, A has rank n if and only if $e_n \in \mathbb{F}_2^n$ is in the column space of A.*

Proof. If A has rank n, then its column space is $\mathbb{F}_2^n$ which includes e_1. On the other hand, suppose that e_1 belongs to the column space of A. Since the first column of A has the form $ae_1 + e_2$ for $a \in \mathbb{F}_2$, it follows that e_2 is also in the column space of A. Applying this reasoning to each successive column of A, we observe that all the standard basis vectors of $\mathbb{F}_2^n$ belong to the column space of A, so A has rank n. The same argument applies to e_n, starting with column n. □

The following theorem shows that approximately one-third of $\mathbb{F}_2$-matrices representing P_n have rank $n-1$, with the remainder having rank n.

Theorem 2.0.2. $R_n(P_n) = \dfrac{1}{3}\left(2^{n+1} + (-1)^n\right), R_{n-1}(P_n) = \dfrac{1}{3}\left(2^n + (-1)^{n+1}\right)$

Proof. Let A' be a completion of $M(P_{n-1})$. Then A' has rank $n-1$ or $n-2$. Let A be the partial completion of $M(P_n)$ that has A' as its lower right $(n-1) \times (n-1)$ submatrix, and the indeterminate d_1 as its upper left entry.

First, suppose that A' has rank $n-2$. By Lemma 2.0.1, the first standard basis vector of $\mathbb{F}_2^{n-1}$ is not in the column space of A', nor is its transpose in the row space of A'. It follows that the matrix consisting of the last $n-1$ columns of A has rank $n-1$ and that both completions of A to an element of $M_n(\mathbb{F}_2)$ have rank n, since the first column is independent of the remaining columns, regardless of the value assigned to d_1. So A has rank n for both choices of d_1, which means that every rank $n-2$ completion of $M(P_{n-1})$ corresponds to two rank n completions of $M(P_n)$.

Now suppose that A' has rank $n-1$. Deleting the first row of A leaves an $(n-1) \times n$ matrix of rank $n-1$, whose right nullspace contains a unique non-zero vector $u \in \mathbb{F}_2^n$. The first entry of u is 1, since the columns of A' are linearly independent. The first entry of Au is $d_1 + u_2$, and it follows that one choice of a value of d_1 determines a completion of rank $n-1$ of $M(P_n)$ that has u in its right nullspace, and the other determines a completion of rank n. Therefore, each rank $n-1$ completion of $M(P_{n-1})$ corresponds to one rank n completion and one rank $n-1$ completion of $M(P_n)$.

We conclude that

$$R_n(P_n) = 2 \cdot R_{n-2}(P_{n-1}) + R_{n-1}(P_{n-1}), \qquad R_{n-1}(P_n) = R_{n-1}(P_{n-1}). \tag{2.1}$$

The result follows by induction on n, noting that $R_2(P_2) = 3$ and $R_1(P_2) = 1$.

□

Definition 2.0.3. *For an integer $n \geqslant 0$, we define*

$$F(n) = \frac{1}{3}(2^{n+1} + (-1)^n).$$

For $n \geqslant 1$, it follows from Theorem 2.0.2 that $F(n)$ is equal to the number of $\mathbb{F}_2$-matrices of rank n that represent P_n, and the number of rank n that represent P_{n+1}. We note the following properties of $F(n)$, which will be useful in Sections 3.2 and 3.3.

Lemma 2.0.4.

1. $F(n)$ *is the number of completions of the matrix consisting of the last* n *rows of* $M(P_{n+1})$, *whose rowspace avoids* $e_1^\top$.
2. $F(n)$ *is the number of completions of the matrix consisting of the first* n *rows of* $M(P_{n+1})$, *whose rowspace avoids* $e_{n+1}^\top$.

Proof. Deleting the first row from a completion of $M(P_{n+1})$ of rank n leaves an $n \times (n+1)$ matrix of rank n, whose rowspace avoids $e_1^\top$ by Lemma 2.0.1. On the other hand, let A be a completion of the last n rows of $M(P_{n+1})$ whose rowspace avoids $e_1^\top$. The rowspace of A intersects $\langle e_1^\top, e_2^\top \rangle$ in a 1-dimensional subspace that contains exactly one of $e_2^\top$ and $e_1^\top + e_2^\top$. Hence, the insertion of an additional row at the top can extend A in exactly one way to a completion of $M(P_{n+1})$ of rank n. These observations establish a bijective correspondence that proves the first statement. The second is proved in a similar way. □

The following Lemma presents several expressions involving $F(n)$, which will be used in subsequent proofs.

Lemma 2.0.5. *For all positive integers* p *and* q, *the function* F *satisfies the following recurrence relations.*

1. $2F(p-1) = F(p) + (-1)^{p+1}$.
2. $4F(q-1)F(p-1) = F(q)F(p) + (-1)^{p+1}F(q) + (-1)^{q+1}F(p) + (-1)^{p+q}$.
3. $4F(p-2) = F(p) + (-1)^p$

Proof. 1. We express $2F(p-1)$ as follows.

$$2F(p-1) = 2\frac{1}{3}\left(2^p + (-1)^{p-1}\right) = \frac{2}{3}\left(2^p + (-1)^{p-1}\right)$$

Distributing the factor $\frac{2}{3}$, and noting $(-1)^{p-1} = (-1)^{p+1}$, we obtain

$$\begin{aligned} 2F(p-1) &= \frac{2}{3}2^p + \frac{2}{3}(-1)^{p-1} \\ &= \frac{1}{3}2^{p+1} + \frac{2}{3}(-1)^{p+1}. \end{aligned}$$

We now split the second term as follows.

$$2F(p-1) = \left(\frac{1}{3}2^{p+1} - \frac{1}{3}(-1)^{p+1}\right) + (-1)^{p+1}$$

Recognizing the first group of terms as $F(p)$, we conclude

$$2F(p-1) = F(p) + (-1)^{p+1}.$$

2. We begin with identity

$$2F(p-1) = F(p) + (-1)^{p+1}.$$

Therefore

$$\begin{aligned} 4F(p-1)F(q-1) &= \left(F(p) + (-1)^{p+1}\right)\left(F(q) + (-1)^{q+1}\right) \\ &= F(p)F(q) + (-1)^{q+1}F(p) + (-1)^{p+1}F(q) + (-1)^{p+q}. \end{aligned}$$

3. We express $4F(p-2)$ as follows.

$$\begin{aligned} 4F(p-2) &= 4\left(\tfrac{1}{3}2^{p-1} + \tfrac{1}{3}(-1)^{p-2}\right) \\ &= \tfrac{1}{3}2^{p+1} + \tfrac{4}{3}(-1)^p \\ &= \tfrac{1}{3}\left(2^{p+1} + (-1)^p\right) + (-1)^p \\ &= F(p) + (-1)^p \end{aligned}$$

This completes the proof.

□

Chapter 3

Graphs with a long induced path

This chapter is concerned with the class $\mathcal{G}^P$ of connected graphs on n vertices containing P_{n-1} as an induced subgraph on all but one vertex (called the extra vertex) . The motivation for studying this class of graphs is provided by Theorem 2.0.2, which fully describes the $\mathbb{F}_2$-rank distribution of P_n. For a graph Γ of order n in $\mathcal{G}^P$, we investigate the relationship between the rank of an $\mathbb{F}_2$-matrix representing Γ and that of its submatrix corresponding to an induced subgraph isomorphic to P_{n-1}.

For any graph Γ of order n in $\mathcal{G}^P$, we list the vertices of Γ as $x, x_1, \ldots, x_{n-1}$, where the subgraph induced on $\{x_1, \ldots, x_{n-1}\}$ is a path with edges $x_i x_{i+1}$ for $1 \leqslant i \leqslant n-2$ (see Figure 5.2).

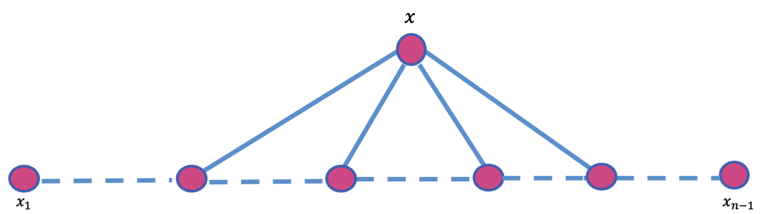


Figure 3.1: A graph Γ with a long induced path

We write $M(\Gamma)$ for the indeterminate matrix that generically represents Γ with respect to this vertex ordering. Then

$$M(\Gamma) = \left(\begin{array}{c|c} d_0 & v^\top \\ \hline v & M\left(P_{n-1}\right) \end{array} \right),$$

where the upper left entry d_0 is an indeterminate and the vector $v \in \mathbb{F}_2^{n-1}$ records the incidences at the vertex x, meaning $v_i = 1$ if and only if x_i is adjacent to x. Since every $\mathbb{F}_2$-completion of $M\left(P_{n-1}\right)$ has rank at least $n-2$, every $\mathbb{F}_2$-matrix representing Γ has one of three possible ranks: $n-2, n-1$ or n.

The following theorem details how the rank of a completion of $M(P_{n-1})$ determines the ranks of its two extensions to completions of $M(\Gamma)$.

Theorem 3.0.1. *Let A be a completion of $M(P_{n-1})$ and let A_0 and A_1 be the completions of $M(\Gamma)$ respectively given by*

$$A_0 = \left(\begin{array}{c|c} 0 & v^\top \\ \hline v & A \end{array}\right), \quad A_1 = \left(\begin{array}{c|c} 1 & v^\top \\ \hline v & A \end{array}\right).$$

Then

1. *If* $\mathrm{rank}(A) = n-1$, *then one of* A_0 *and* A_1 *has rank* $n-1$ *and the other has rank* n.
2. *If* $\mathrm{rank}(A) = n-2$ *and* $v^\top \notin \mathrm{rowspace}(A)$, *then both* A_0 *and* A_1 *have rank* n.
3. *If* $\mathrm{rank}(A) = n-2$ *and* $v^\top \in \mathrm{rowspace}(A)$, *then one of* A_0 *and* A_1 *has rank* $n-2$ *and the other has rank* $n-1$.

Proof. Let M denote the indeterminate matrix obtained from $M(\Gamma)$ by completing $M(P_{n-1})$ to A, and retaining the indeterminate d_0 in the $(1,1)$ position.

1. Suppose that $\mathrm{rank}(A) = n-1$. Then A_0 and A_1 both have rank at least $n-1$. Let A' denote the $(n-1)\times n$ submatrix of M consisting of rows 2 through n, which are linearly independent in $\mathbb{F}_2^n$. The rows of A form a basis of $\mathbb{F}_2^{n-1}$, and there is a unique $w \in \mathbb{F}_2^{n-1}$ for which $w^\top A = v^\top$. Then $w^\top A'$ is either equal to the first row of A_0 or of A_1, and exactly one of A_0 and A_1 has rank $n-1$. The other has rank n, since its first row is not a linear combination of subsequent rows.
2. Now suppose that $\mathrm{rank}(A) = n-2$ and that $v^\top$ is not in the rowspace of A. Then (since A is symmetric) v is not a linear combination of the columns of A. Therefore $(v|A)$ is an $(n-1)\times n$ matrix of rank $n-1$. Since $v^\top$ is not in the rowspace of A, it follows that $(d_0|v^\top)$ is not in the rowspace of $(v|A)$ for either choice of d_0. It follows that extending A to either A_0 or A_1 increases the rank from $n-2$ to n.
3. Since $v^\top$ is a linear combination of the rows of A, either $(0|v^\top)$ or $(1|v^\top)$ is a linear combination of the rows of the $(n-1)\times n$ matrix $(v|A)$, which has rank $n-2$. Hence at least one of A_0 and A_1 has rank $n-2$. Both have rank $n-2$ if and only if the transpose of $e_1 \in \mathbb{F}_2^n$ belongs to the rowspace of $(v|A)$, which means $u^\top(v|A) = e_1^\top$ for some $u \in \mathbb{F}_2^n$. This is impossible, since if $u^\top A = 0$ then $u^\top v = 0$ also, as v is in the column space of A.

□

When the matrix A of Theorem 3.0.1 has rank $n-2$, there exists a unique nonzero right vector u in the right nullspace of A. In this case, it is useful to determine whether the vector v is orthogonal to u or not. If $u^\top v = 0$, then $M(\Gamma)$ has one completion each of rank $n-1$ and $n-2$. If $u^\top v = 1$, then both choices for d_0

result in a rank n completion of $M(\Gamma)$. As a result, we restrict our attention to completions of $M(\Gamma)$ for which the lower-right $(n-1)\times(n-1)$ submatrix has rank $n-2$.

For a graph Γ in the class $\mathcal{G}^P$, we write $A(\Gamma)$ and $B(\Gamma)$ respectively for the numbers of matrices of rank n and $n-1$ that represent Γ over $\mathbb{F}_2$ with respect to the vertex ordering $\{x, x_1, \dots, x_{n-1}\}$, and for which the lower right $(n-1)\times(n-1)$ submatrix corresponding to the path on $x_1, \dots, x_{n-1}$ has rank $n-2$. From Theorem 3.0.1 it follows that

$$R_n(\Gamma) - R_{n-1}(\Gamma) = A(\Gamma) - B(\Gamma).$$

We write $\alpha(\Gamma) = A(\Gamma) - B(\Gamma)$, and proceed to identify those $\Gamma \in \mathcal{G}^P$ for which $\alpha(\Gamma)$ is negative.

The first case (1) of Theorem 3.0.1 establishes that completions of $M(\Gamma)$ whose lower right $(n-1)\times(n-1)$ submatrix has full rank contribute equally to $R_{n-1}(\Gamma)$ and $R_n(\Gamma)$, and these contributions cancel in $\alpha(\Gamma)$. The simplifying effect of this cancellation is crucial to the methods that are applied throughout this thesis, and it is particular to the field $\mathbb{F}_2$. Over the finite field $\mathbb{F}_q$, the analogous statement to item 1 of Theorem 3.0.1 is that one choice of the indeterminate in the $(1,1)$-position corresponds to a matrix of rank $n-1$, and the remaining $q-1$ choices correspond to matrices of rank n. For $q > 2$, the contribution over $\mathbb{F}_q$ to $R_n(\Gamma)$ arising from full rank completions of $M(P_{n-1})$ exceeds the corresponding contribution to $R_{n-1}(\Gamma)$ by a factor of $q-1$.

3.1 Vectors in the nullspace of completions of $M(P_n)$

We now consider which column vectors over $\mathbb{F}_2$ may occur as the unique nonzero element of the right nullspace of a matrix representing P_n in $M_n(\mathbb{F}_2)$.

Lemma 3.1.1. *Suppose that $Au = 0$, for a matrix $A \in M_n(\mathbb{F}_2)$ that represents P_n, and a nonzero column vector u. Then the first and last entries of u are both 1, and u has no pair of consecutive zero entries.*

Proof. We write $d_1, \dots, d_n$ for the diagonal entries of A, and $u_1, \dots, u_n$ for the entries of u. Since $Au = 0$, we have the following:

- $d_1u_1 + u_2 = 0$
- $u_{i-1} + d_iu_i + u_{i+1} = 0$, for $2 \leqslant i \leqslant n-1$
- $u_{n-1} + d_nu_n = 0$

If $u_1 = 0$, then from the first of the above equations, it follows that $u_2 = 0$. Applying the second equation to the successive triples (u_{i-1}, u_i, u_{i+1}) from $i = 2$, it follows that $u_i = 0$ for all i. A similar argument applies if $u_n = 0$, working through the triples in the opposite order. Thus, the zero vector is the only vector with the first or last entry equal to zero in the nullspace of any matrix that represents P_n.

Suppose now that $u_i = u_{i+1} = 0$ for some i with $2 \leqslant i \leqslant n-2$. Then u_{i-1} and u_{i+2} are also equal to zero, since $u_{i-1} + d_i u_i + u_{i+1} = 0$ and $u_i + d_{i+1}u_{i+1} + u_{i+2} = 0$. Repeating this argument, it follows that $u = 0$. □

For a positive integer n, we write U_n for the set of vectors in $\mathbb{F}_2^n$ that have no consecutive zero entries and have first and last entries equal to 1. Lemma 3.1.1 shows that every non-zero vector that is in the nullspace of a completion of $M(P_n)$ belongs to U_n. In the next lemma, we show that U_n is exactly the set of non-zero vectors that occur in the nullspace of some rank $n-1$ completion of $M(P_n)$, and determine the number of such completions with a particular 1-dimensional nullspace.

Lemma 3.1.2. *Let $u \in U_n$ and let $z(u)$ be the number of zero entries in u. Then the number of matrices A that represent P_n and satisfy $Au = 0$ is $2^{z(u)}$.*

Proof. We write $u_1, \dots, u_n$ for the entries of u, and note that $u_1 = u_n = 1$. Let A be a completion of $M(P_n)$, and write $d_1, \dots, d_n$ for the diagonal entries of A. Then $Au = 0$ if and only if the following conditions are satisfied.

- $d_1 + u_2 = 0$
- For $2 \leqslant i \leqslant n-1$, $u_{i-1} + d_i u_i + u_{i+1} = 0$
- $u_{n-1} + d_n = 0$

The first and last equations above are satisfied if (and only if) $d_1 = u_2$ and $d_n = u_{n-1}$. For $2 \leqslant i \leqslant n-1$, we consider the cases $u_i = 1$ and $u_i = 0$ separately. If $u_i = 1$, then $u_{i-1} + d_i u_i + u_{i+1} = 0$ is satisfied only by $d_i = u_{i-1} + u_{i+1}$. If $u_i = 0$, then $u_{i-1} + u_{i+1} = 1 + 1 = 0$, and the equation is satisfied by both $d_i = 0$ and $d_i = 1$. It follows that every element of U_n belongs to the nullspace of *some* completion of $M(P_n)$ of rank $n-1$, and that for a particular $u \in U_n$ with $z(u)$ zero entries, the number of choices for the diagonal entries of such a completion is $2^{z(u)}$. □

Example 3.1.3. *Consider a symmetric matrix A that represents P_5 over $\mathbb{F}_2$ with nullspace vector $u = (1, 0, 1, 1, 1)^\top$.*

$$Au = \begin{bmatrix} d_1 & 1 & 0 & 0 & 0 \\ 1 & d_2 & 1 & 0 & 0 \\ 0 & 1 & d_3 & 1 & 0 \\ 0 & 0 & 1 & d_4 & 1 \\ 0 & 0 & 0 & 1 & d_5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{matrix} d_1 = 0 \\ \\ d_3 = 1 \\ d_4 = 0 \\ d_5 = 1 \end{matrix}$$

Four of the diagonal entries are determined, whereas d_2 *remains free to choose. The number of such matrices is therefore* $2^{z(u)} = 2^1 = 2$.

The statements and proofs of Lemmas 3.1.1 and 3.1.2 have a particularly simple form over the field $\mathbb{F}_2$, but the same methods establish analogous results over $\mathbb{F}_q$ for any prime power q. A non-zero vector in $\mathbb{F}_q^n$ is in the 1-dimensional nullspace of a matrix representing P_n over $\mathbb{F}_q$ if and only if it has non-zero first and last entries and has no consecutive pair of zero entries. For a vector u with these properties, the number of matrices that represent P_n over $\mathbb{F}_q$ and have nullspace spanned by u is $q^{z(u)}(q-1)^{n-1-z(u)}$. In the case $q = 2$, this expression is just $2^{z(u)}$.

The following lemma expresses $F(n) = \frac{1}{3}(2^{n+1} + (-1)^n)$ as a sum over all nullspace vectors of $M(P_{n+1})$.

Lemma 3.1.4. *Let* U_n *be the set of vectors in* $\mathbb{F}_2^n$ *whose first and last entries are equal to 1 and with no consecutive zeros. Then, for every positive integer* n*,*

$$F(n) = \sum_{u \in U_{n+1}} 2^{z(u)}$$

Proof. For each vector $u \in U_{n+1}$, Lemma 3.1.2 implies the number of matrices A that represent P_{n+1} and satisfy $Au = 0$ is $2^{z(u)}$, where $z(u)$ denotes the number of zero entries in u. Since $F(n)$ counts all matrices of rank n representing P_{n+1}, as noted immediately after Definition 2.0.3, it follows that the total number of such completions is

$$F(n) = \sum_{u \in U_{n+1}} 2^{z(u)}.$$

□

For any positive integer n, we write S_n for the set of vectors in $\mathbb{F}_2^n$ that have no zero entries in consecutive positions (with no conditions on the first and last entry). We note that S_{n-1} is in bijective correspondence with U_{n+1}, via a correspondence that deletes the first and last entry (both 1) from an element of U_{n+1}, or (in the other direction) appends a 1 as the first and last entries of an element of S_{n-1}. This correspondence preserves the number of zero entries, so the following is an immediate consequence of Lemma 3.1.4.

Corollary 3.1.5. *For every positive integer* n*,*

$$F(n) = \sum_{u \in S_{n-1}} 2^{z(u)}.$$

Remark. Define $U = U_n \cup \{0\}$. Since the rank of any completion of $M(P_n)$ is at least $n-1$, no completion has a nullspace of dimension greater than 1. Additionally, all such nullspaces are subsets of U. The set U is not a subspace of $\mathbb{F}_2^n$ because adding distinct non-zero vectors in U produces a non-zero vector whose first and last entries are 0, which therefore does not belong to U. No 2-dimensional subspace of $\mathbb{F}_2^n$ lies entirely within U for the same reason. This observation allows us to understand why no completion of $M(P_n)$ has rank less than $n-1$ solely from the description of the vectors in the nullspaces of its completions.

Lemma 3.1.2 allows us to characterise $A(\Gamma)$ and $B(\Gamma)$ in the following way. For column vectors in $\mathbb{F}_2^n$, we write $\perp$ for the relation of orthogonality with respect to the standard scalar product.

Theorem 3.1.6. *Let $\Gamma \in \mathcal{G}^P$ have order n, and let v be the vector in $\mathbb{F}_2^{n-1}$ consisting of the last $n-1$ entries of the first column of $M(\Gamma)$. Let $z(u)$ be the number of zero entries in u. Then*

1. $A(\Gamma) = \sum_{\substack{u \in U_{n-1} \\ u \not\perp v}} 2^{z(u)+1}$
2. $B(\Gamma) = \sum_{\substack{u \in U_{n-1} \\ u \perp v}} 2^{z(u)}$

Proof. Let $u \in U_{n-1}$, and let $T(u)$ be the set of completions of $M(\Gamma)$ whose lower right $(n-1) \times (n-1)$ submatrix is a completion of $M(P_{n-1})$ with nullspace $\langle u \rangle$. By Lemma 3.1.2, u is in the nullspace of $2^{z(u)}$ rank $n-2$ completions of $M(P_{n-1})$, each of which extends to two elements of $T(u)$ through the choice of a value for d_0. Hence $|T(u)| = 2^{z(u)+1}$. Note that $u \perp v \iff v^\top u = 0$, meaning $u \perp v$ if and only if $v^\top$ is in the rowspace of every rank $n-2$ completion of $M(P_{n-1})$ whose nullspace contains u.

If $u \not\perp v$, then it follows from Theorem 3.0.1 that every element of $T(u)$ has rank n. If $u \perp v$, then it follows from Theorem 3.0.1 that $2^{z(u)}$ elements of $T(u)$ have rank $n-1$ and $2^{z(u)}$ have rank $n-2$. Together, these imply the result. □

3.2 Recurrences for $A(\Gamma)$ and $B(\Gamma)$

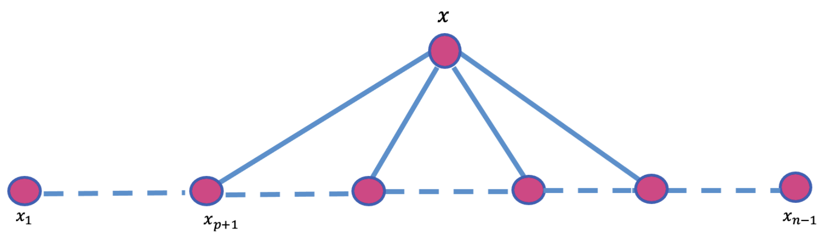


Figure 3.2: Γ with induced path

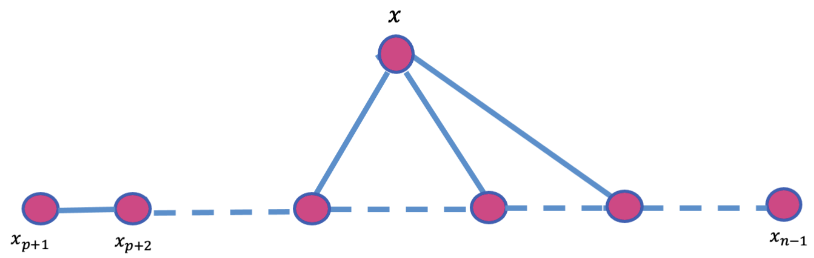


Figure 3.3: Γ_1

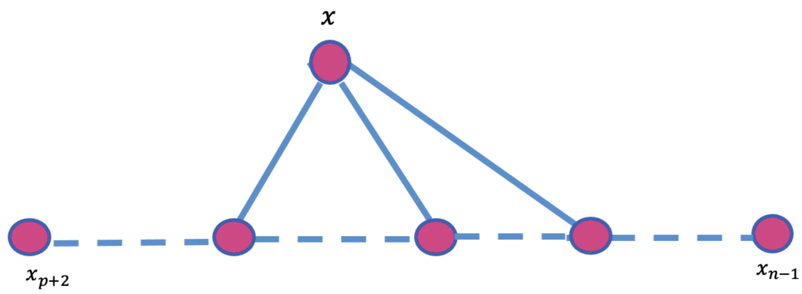


Figure 3.4: Γ_2

Let $\Gamma \in \mathcal{G}^P$, with vertices $x, x_1, \ldots, x_{n-1}$. Let p be the minimal non-negative integer with x_{p+1} adjacent to x, so that p is the number of edges in the path $x_1 \ldots x_{p+1}$. Let Γ_1 be the graph obtained from Γ by deleting $x_1, \ldots, x_p$ and their incident edges, and deleting the edge xx_{p+1} (see Figure 3.3). Let Γ_2 be the graph obtained from Γ_1 by deleting the vertex x_{p+1} and its incident edge (see Figure 3.4). In this section we establish expressions for $A(\Gamma)$ and $B(\Gamma)$ in terms of the corresponding quantities for Γ_1 and Γ_2.

The degree of the vertex x in both Γ_1 and Γ_2 is $\deg_\Gamma(x) - 1$, allowing for recursive analysis of $\alpha(\Gamma)$ in terms of the degree of the vertex x for graphs in $\mathcal{G}^P$.

For a positive integer $t \geqslant 2$, we define $S(t)$ to be the $(t-1) \times t$ indeterminate matrix obtained from $M(P_t)$ by deleting the first row. Every $\mathbb{F}_2$-completion of

$S(t)$ has rank $t-1$ and has a 1-dimensional right nullspace in $\mathbb{F}_2^t$. For a vector $v \in \mathbb{F}_2$, we define $\Gamma(v)$ to be the graph on $t+1$ vertices whose indeterminate matrix is

$$\begin{bmatrix} * & v^\top \\ v & M(P_t) \end{bmatrix}.$$

We denote this matrix by $M(v)$.

Theorem 3.2.1. *Let $\Gamma \in \mathcal{G}^P$. Then for Γ_1, Γ_2, and p defined as above:*

1. $A(\Gamma) = 2F(p)B(\Gamma_1) + 2F(p-1)A(\Gamma_2)$.
2. $B(\Gamma) = \frac{1}{2}F(p)A(\Gamma_1) + 2F(p-1)B(\Gamma_2)$.

The proof of Theorem 3.2.1 is presented in a series of steps. We recall that $A(\Gamma)$ and $B(\Gamma)$ are respectively the numbers of completions of $M(\Gamma)$ of rank n and rank $n-1$, in which the lower right $(n-1)\times(n-1)$ submatrix is a completion of $M(P_{n-1})$ of rank $n-2$. We write v for the vector in $\mathbb{F}_2^{n-1}$ consisting of the last $n-1$ entries of the first column of $M(\Gamma)$, which records the neighbours of x in Γ. We note that the first p entries of v are zeros, and the first nonzero entry of v is in position $p+1$.

We write $C(v)$ for the set of completions of rank $n-2$ of $M(P_{n-1})$ whose columnspace contains v, and $\overline{C}(v)$ for the set of rank $n-2$ completions of $M(P_{n-1})$ whose columnspace excludes v. Suppose that M' is a completion of $M(\Gamma)$ that contributes either to $A(\Gamma)$ or $B(\Gamma)$, and let M be the corresponding completion of $M(P_{n-1})$, which has rank $n-2$. Theorem 3.0.1 implies that M' contributes to $A(\Gamma)$ if $M \in \overline{C}(v)$ and to $B(\Gamma)$ if $M \in C(v)$. Every $M \in \overline{C}(v)$ extends in two ways to a rank n completion of $M(\Gamma)$, since both choices for the upper left entry result in matrices of rank n. However, every $M \in C(v)$ extends in only one way to a rank $n-1$ completion of $M(\Gamma)$, since the two choices for the upper left entry result in one matrix of rank $n-1$ and one of rank $n-2$. Hence

$$A(\Gamma) = 2|\overline{C}(v)|, \quad B(\Gamma) = |C(v)|. \tag{3.1}$$

To prove Theorem 3.2.1, we need to express $|C(v)|$ and $|\overline{C}(v)|$ in terms of p and the graphs Γ_1 and Γ_2. Each element M of $C(v)$ or $\overline{C}(v)$ has a unique nonzero vector u_M in its right nullspace. The entry $u_M[p+1]$ in position $p+1$ of u_M is either 1 or 0. We define

$$\begin{aligned} C_1(v) &= \{M \in C(v) : u_M[p+1] = 1\}, & C_0(v) &= \{M \in C(v) : u_M[p+1] = 0\} \\ \overline{C}_1(v) &= \{M \in \overline{C}(v) : u_M[p+1] = 1\}, & \overline{C}_0(v) &= \{M \in \overline{C}(v) : u_M[p+1] = 0\}. \end{aligned}$$

The proof of Theorem 3.2.1 depends on enumerating the elements of the four sets above in terms of Γ_1 and Γ_2.

Lemma 3.2.2. *For* $t \geqslant 2$ *and* $e_1, v \in \mathbb{F}_2^t$,

1. $B(\Gamma(v))$ *is the number of completions of* $S(t)$ *whose rowspace includes* $v^\top$ *and not* $e_1^\top$.
2. $\frac{1}{2}A(\Gamma(v))$ *is the number of completions of* $S(t)$ *whose rowspace includes neither* $v^\top$ *nor* $e_1^\top$.

Proof. The matrix $M(v)$ has $M(P_t)$ as its lower right $t \times t$ submatrix and $S(t)$ as its lower right $(t-1) \times t$ submatrix. Let S be a completion of $S(t)$. If the rowspace of S includes $e_1^\top$, then it follows from Lemma 2.0.1 that both extensions of S to completions of $M(P_t)$ have rank t. Since $A(\Gamma(v))$ and $B(\Gamma(v))$ count completions with lower-right $t \times t$ submatrices of rank $t-1$, the extensions of these matrices to completions of $M(v)$ do not contribute to either $A(\Gamma(v))$ or $B(\Gamma(v))$.

If the rowspace of S does not include $e_1^\top$, then S has a unique extension to a completion S' of rank $t-1$ of $M(P_t)$, whose rowspace is equal to that of S. The two extensions of S' to completions of $\Gamma(v)$, determined by assigning a value from $\mathbb{F}_2$ to the upper left entry, potentially contribute to $A(\Gamma(v))$ or $B(\Gamma(v))$. If v does not belong to the rowspace of S' (or equivalently S), then Theorem 3.0.1 implies that both of these extensions have rank $t+1$, and both contribute to $A(\Gamma(v))$. On the other hand, if v belongs to the rowspace of S', then Theorem 3.0.1 implies that one extension of S' to a completion of S has rank $t-1$ and the other has rank t, the latter of which contributes to $B(\Gamma(v))$.

By Lemma 2.0.1, every completion of $M(v)$ that is counted by either $A(\Gamma(v))$ or $B(\Gamma(v))$ has a lower right $(t-1) \times t$ submatrix whose rowspace excludes $e_1^\top$. Therefore among the completions of S whose rowspace excludes $e_1^\top$, the number whose rowspace includes $v^\top$ is $B(\Gamma(v))$, and the number whose rowspace excludes $v^\top$ is $\frac{1}{2}A(\Gamma(v))$. □

Lemma 3.2.3. $|C_0(v)| = 2F(p-1)B(\Gamma_2)$ *and* $|\overline{C}_0(v)| = F(p-1)A(\Gamma_2)$.

Proof. Let M be a completion of $M(P_{n-1})$, and let L and R respectively denote its upper left $p \times p$ submatrix and its lower right $(n-p-2) \times (n-p-2)$ submatrices, which are respectively completions of $M(P_p)$ and $M(P_{n-p-2})$. Suppose that $M \in C_0(v) \cup \overline{C}_0(v)$. The right nullspace of M contains a unique non-zero element u, with $u[p+1] = 0$ and $u[p] = u[p+2] = 1$. Then $Lu_1 = 0$ and $Ru_2 = 0$, where u_1 and u_2 are respectively the elements of $\mathbb{F}_2^p$ and $\mathbb{F}_2^{n-p-2}$ consisting of the first p and the last $n-p-2$ components of u. We define v_1 and v_2 similarly. Since neither u_1 nor u_2 is the zero vector, it follows that L and R are both rank deficient. The vector $v^\top$ belongs to the rowspace of M if and only if $v_2^\top u_2 = 0$, which occurs if and only if $v_2^\top$ is in the rowspace of R.

On the other hand, let L be any completion of rank $p-1$ of $M(P_p)$ and let R be any completion of rank $n-p-3$ of $M(P_{n-p-2})$. Let u_1 and u_2 be the non-zero elements of the right nullspaces of L and R respectively, and note that the last entry of u_1 and the first entry of u_2 are both 1. If the upper left $p \times p$ submatrix of $M(P_{n-1})$ is completed to L and the lower right $(n-p-2)\times(n-p-2)$ region is completed to R, then both assignments of a value to the indeterminate in row $p+1$ result in a matrix of rank $n-2$, whose right nullspace contains the vector $u = [u_1\ 0\ u_2]$. Hence every choice for L and R contributes twice either to $|C_0(v)|$ or to $|\overline{C}_0(v)|$, according to whether v_2 belongs to the rowspace of R or not. The number of choices for L is $R_{p-1}(P_p) = F(p-1)$. Since $\Gamma_2 = \Gamma(v_2)$, the number of choices for R with v_2 in its rowspace is $B(\Gamma_2)$, and the number of choices for R with rowspace excluding v_2 is $\frac{1}{2}A(\Gamma_2)$. Hence $|C_0(v)| = 2\times F(p-1)B(\Gamma_2)$ and $|\overline{C}_0(v)| = 2\times F(p-1)\times\frac{1}{2}A(\Gamma_2) = F(p-1)A(\Gamma_2)$. □

Lemma 3.2.4. $|C_1(v)| = \frac{1}{2}F(p)A(\Gamma_1)$ *and* $|\overline{C}_1(v)| = F(p)B(\Gamma_1)$.

Proof. Let $M \in C_1(v)\cup\overline{C}_1(v)$, so M is a completion of $M(P_{n-1})$ of rank $n-2$, whose right nullspace includes a single non-zero vector u with 1 in position $p+1$. Then $e_{p+1}^\top$ is not in the rowspace of M, since $e_{p+1}^\top u = 1$. The vector $v^\top$ belongs to the rowspace of M if and only if $v^\top u = 0$, which occurs if and only if $v_2^\top u_2 = 0$, where v_2 and u_2 are the vectors in $\mathbb{F}_2^{n-1-p}$ respectively consisting of the last $n-1-p$ entries of v and of u. Let R be the submatrix of M in rows $p+2$ through $n-1$ and columns $p+1$ through $n-1$. Then R has rank $n-p-2$ and its rowspace comprises exactly those vectors $w^\top$ with $w^\top u_2 = 0$. It follows that $v^\top$ belongs to the rowspace of M if and only if $v_2^\top$ belongs to the rowspace of R. Let v_2' be the element $v_2 + e_1$ of $\mathbb{F}_2^{n-p-1}$, which differs from v_2 only in its first entry, which is 0. Then $(v_2')^\top$ is in the rowspace of R if and only if $v_2^\top$ is not. Thus $M \in C_1(v)$ if and only if $(v_2')^\top$ is not in the rowspace of R. Alternatively $M \in \overline{C}_1(v)$, which occurs if and only if $(v_2')^\top$ is in the rowspace of R. Since $\Gamma_1 = \Gamma(v_2')$, if follows from Lemma 3.2.2 that the number of possibilities for R in an element of $C_1(v)$ or $\overline{C}_1(v)$ are respectively bounded above by $B(\Gamma_1)$ and $\frac{1}{2}A(\Gamma_1)$. If L is the upper left $p\times(p+1)$ submatrix of M, then the rowspace of L excludes $e_{p+1}^\top$ and L occurs as the first p rows of a unique completion of $M(P_{p+1})$ of rank p. It follows that the number of possibilities for L is at most $R_p(P_{p+1}) = F(p)$.

On the other hand, let L' be a completion of the first p rows of $M(P_{p+1})$ whose rowspace does not contain e_{p+1}, and let R' be a completion of $S(n-p-1)$ whose rowspace does not contain $e_1^\top$. Let the non-zero vectors in the right nullspaces of L' and R' be u_1 and u_2 respectively. Noting that the last entry of u_1 and the first entry of u_2 are both 1, let u be the vector in $\mathbb{F}_2^{n-1}$ that coincides with u_1 in its first $p+1$ entries and with u_2 in its last $n-1-p$ entries. Completing the upper left region of $M(P_{n-1})$ to L', and the lower right region to R' leaves a single choice for the indeterminate in row $p+1$, to ensures the resulting matrix M' satisfies $M'u = 0$ and hence belongs to $C_1(v)\cup\overline{C}_1(v)$. Thus the pair (L', R') determines a unique element M' of $C_1(v)\cup\overline{C}_1(v)$, which belongs to $C_1(v)$ if $(v_2')^\top$ is not in the rowspace of R' and to $\overline{C}_1(v)$ otherwise. Since the number of possibilities for L' is $F(p)$, the conclusion follows from Lemma 3.2.2. □

We complete the proof of Theorem 3.2.1 by noting that

$$\begin{aligned} A(\Gamma) &= 2|\overline{C}(v)| = 2|\overline{C}_1(v)| + 2|\overline{C}_0(v)| = 2F(p)B(\Gamma_1) + 2F(p-1)A(\Gamma_2), \\ B(\Gamma) &= |C(v)| = |C_1(v)| + |C_0(v)| = \frac{1}{2}F(p)A(\Gamma_1) + 2F(p-1)B(\Gamma_2). \end{aligned}$$

3.3 Characterising all graphs in $\mathcal{G}^P$ with negative α

Recall $\alpha(\Gamma) = A(\Gamma) - B(\Gamma)$ counts the difference in the number of matrices of rank n and $n-1$ representing $\Gamma \in \mathcal{G}^P$ over $\mathbb{F}_2$. Using Theorem 3.2.1, we derive the following expression for $\alpha(\Gamma)$.

$$\begin{aligned}\alpha(\Gamma) &= 2F(p)B(\Gamma_1) + 2F(p-1)A(\Gamma_2) - \frac{1}{2}F(p)A(\Gamma_1) - 2F(p-1)B(\Gamma_2) \\ &= \frac{1}{2}F(p)\big(4B(\Gamma_1) - A(\Gamma_1)\big) + 2F(p-1)\big(A(\Gamma_2) - B(\Gamma_2)\big)\end{aligned}$$

To simplify the recurrence, we replace $A(\Gamma_2) - B(\Gamma_2)$ with $\alpha(\Gamma_2)$ and define $\beta(\Gamma) = 4B(\Gamma) - A(\Gamma)$. The recurrence relation then becomes $\alpha(\Gamma) = \frac{1}{2}F(p)\beta(\Gamma_1) + 2F(p-1)\alpha(\Gamma_2)$.

By Theorem 3.2.1, we derive the following recurrence formula for $\beta(\Gamma)$.

$$\begin{aligned}\beta(\Gamma) &= 4B(\Gamma) - A(\Gamma) \\ &= 4\left(\frac{1}{2}F(p)A(\Gamma_1) + 2F(p-1)B(\Gamma_2)\right) - (2F(p)B(\Gamma_1) + 2F(p-1)A(\Gamma_2)) \\ &= 2F(p)\left(A(\Gamma_1) - B(\Gamma_1)\right) + 2F(p-1)\left(4B(\Gamma_2) - A(\Gamma_2)\right) \\ &= 2F(p)\alpha(\Gamma_1) + 2F(p-1)\beta(\Gamma_2)\end{aligned}$$

We summarise these recurrences in the following lemma.

Lemma 3.3.1.

1. $\alpha(\Gamma) = \frac{1}{2}F(p)\beta(\Gamma_1) + 2F(p-1)\alpha(\Gamma_2)$
2. $\beta(\Gamma) = 2F(p)\alpha(\Gamma_1) + 2F(p-1)\beta(\Gamma_2)$

In this section, we use these recurrences to determine all $\Gamma \in \mathcal{G}^P$ with $\alpha(\Gamma) < 0$. For an integer $d \geqslant 1$, we write $\mathcal{G}^P_d$ for the class of graphs in $\mathcal{G}^P$ in which the vertex that does not belong to the induced path has degree d. If $\Gamma \in \mathcal{G}^P_d$ for $d > 1$, the Γ_1 and Γ_2 belong to $\mathcal{G}^P_{d-1}$. The recurrence relations above express the values of α and β for a graph in $\mathcal{G}^P_d$ in terms of corresponding values for graphs in $\mathcal{G}^P_{d-1}$.

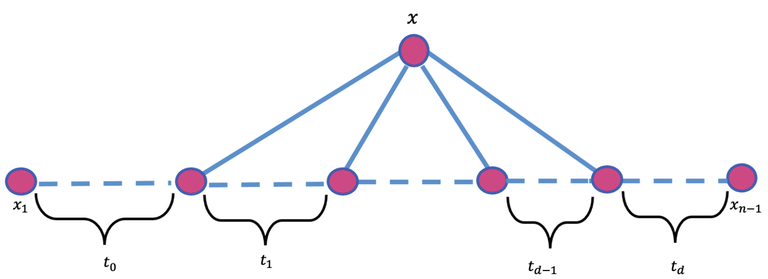


Figure 3.5: $\Gamma(t_0, t_1, \cdots, t_d)$

We write $\Gamma(t_0, t_1, \ldots, t_d)$ for the graph in $\mathcal{G}_d^P$ with the following properties, where x is the vertex not in the induced path $P = x_1x_2\ldots x_{n-1}$, and the i^{th} neighbour of x is the neighbour of i^{th} least index in P. Each t_i represents the number of edges in the path P lying between two neighbours of x. More precisely:

- $t_0 \geqslant 0$ is the number of edges in P between x_1 and the first neighbour of x.
- For each $i \in \{1, \ldots, d-1\}$, the value $t_i \geqslant 1$ is the number of edges of P between the i^{th} and $(i+1)^{\text{th}}$ neighbours of x.
- $t_d \geqslant 0$ is the number of edges of P between the last neighbour of x and the endpoint x_{n-1}.

For $\Gamma = \Gamma(t_0, t_1, \ldots, t_d)$, we write $\alpha(t_0, t_1, \ldots, t_d) = \alpha(\Gamma)$, and similar for β, A, and B.

The function β plays an important role in determining whether $\alpha(\Gamma)$ is negative for given $\Gamma \in \mathcal{G}^P$, since Lemma 3.3.1 implies that $\alpha(\Gamma)$ can only be negative if either $\beta(\Gamma_1) < 0$ or $\alpha(\Gamma_2) < 0$. This allows us to determine which $\Gamma \in \mathcal{G}_d^P$ have $\alpha(\Gamma) < 0$ solely in terms of graphs in $\mathcal{G}_{d-1}^P$. We begin by classifying all $\Gamma \in \mathcal{G}_1^P$ with $\alpha(\Gamma) < 0$ and all with $\beta(\Gamma) < 0$. We then use the results from $\mathcal{G}_1^P$ to determine all $\Gamma \in \mathcal{G}_2^P$ with $\alpha(\Gamma) < 0$ and all with $\beta(\Gamma) < 0$, continuing similarly for $\mathcal{G}_3^P$, $\mathcal{G}_4^P$, and $\mathcal{G}_5^P$. Finally, we show that no $\Gamma \in \mathcal{G}_d^P$ has $\alpha(\Gamma) < 0$ or $\beta(\Gamma) < 0$ for $d \geqslant 6$. These results are then summarised in Theorem 3.3.2, which describes fully all graphs in $\mathcal{G}^P$ represented by more matrices of rank $n-1$ than rank n.

Theorem 3.3.2. *The following are all graphs in* $\mathcal{G}^P$ *represented by more matrices of rank* $n-1$ *than rank* n *(see Figure 3.11).*

- $\Gamma(0, s, t)$ *with* $\min(s, t)$ *even.*
- $\Gamma(s, 2, t)$ *with* $\min(s, t)$ *even.*
- $\Gamma(0, s, 1, t)$ *with* $\min(s, t)$ *odd.*
- $\Gamma(s, 1, 1, t)$ *with* $\min(s, t)$ *even.*
- $\Gamma(0, s, 2, t, 0)$ *with* $\min(s, t)$ *even.*
- $\Gamma(0, s, 1, 1, t, 0)$ *with* $\min(s, t)$ *even.*

The following outlines some methods used throughout this section.

- $\Gamma(t_0, t_1, \ldots, t_{d-1}, t_d)$ is isomorphic to $\Gamma(t_d, t_{d-1}, \ldots, t_1, t_0)$ by symmetry. This means that $\alpha(t_0, t_1, \ldots, t_{d-1}, t_d) = \alpha(t_d, t_{d-1}, \ldots, t_1, t_0)$, and similar for β. We use Lemma 3.3.1 on both forms of α (or β) to find pairs of conditions which must be satisfied simultaneously, since one form is negative if and only if the other form is negative. These pairs often contradict one another, reducing the number of cases that need to be checked.

- While $t_0, t_d \geqslant 0$ for any graph $\Gamma = \Gamma(t_0, t_1, \ldots, t_{d-1}, t_d)$, all internal values $t_1, \ldots, t_{d-1} \geqslant 1$. This is because the number of edges between the first (or last) vertex in the induced path P of Γ and the first (or last) neighbour of the extra vertex x in P may be 0, but the number of edges between any pair of neighbours of x in P must be at least 1. Many of the conditions for α or β to be negative require at least one of t_0 or t_d to be 0. When determining when α and β are negative in $\mathcal{G}_d^P$ using the results from $\mathcal{G}_{d-1}^P$ and Lemma 3.3.1, this often requires an internal value to be 0, which is impossible. This again allows us to reduce the number of cases that need to be checked.

- If we have shown that α or β is equal to an expression involving terms with a power of -1 as a coefficient, and we want to prove α or β is positive for all values in this case, we may use the fact that it is greater than or equal to the expression resulting from letting all powers of -1 be negative simultaneously.

- For positive t_0 and t_d, we note that $\alpha(t_0, t_1, \ldots, t_d) = \alpha(0, t_0, \ldots, t_d, 0)$ and $\beta(t_0, t_1, \ldots, t_d) = \beta(0, t_0, \ldots, t_d, 0)$. This can be deduced directly from Lemma 3.3.1 using a symmetry argument, or from the arguments of Section 3.1. It is a consequence of the fact that the vector $(e_1 + e_{n-1})^\top u = 0$ for all $u \in U_{n-1}$, since every $u \in U_{n-1}$ has 1 as its first and last entry. For a graph Γ of order n in $\mathcal{G}^P$, changing the adjacency status of the vertex x with both x_1 and x_{n-1} has no effect on the values of A, B, α or β. In particular, if $\Gamma \in \mathcal{G}_d^P$ has two vertices x_1 and x_{n-1} of degree 1, then the graph $\Gamma' \in \mathcal{G}_{d+2}^P$ obtained from Γ by adding the edges xx_1 and xx_{n-1} satisfies $\alpha(\Gamma') = \alpha(\Gamma)$ and $\beta(\Gamma') = \beta(\Gamma)$.

- Expressions for α and β are found using identities in Lemma 2.0.5. Calculations were done by hand and checked against a SageMath [11] program which is included as an appendix.

3.3.1 Degree 1

In this subsection, we study graphs of the form $\Gamma(s,t) \in \mathcal{G}_1^P$ for positive integers s and t(see Figure 3.6).

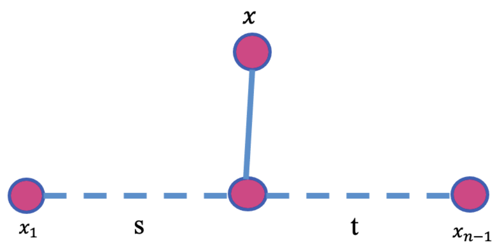


Figure 3.6: $\Gamma(s,t)$

Recall $F(n) = \frac{1}{3}(2^{n+1} + (-1)^n)$ and note $F(0) = F(1) = 1$, $F(2) = 3$, $F(3) = 5$, $F(4) = 11$, $F(5) = 21$.

Lemma 3.3.3. *Let* s *and* t *be positive integers.*

$$\alpha(s,t) = 2F(s)F(t) - 2F(s-1)F(t-1)$$
$$\beta(s,t) = 8F(s-1)F(t-1) - 2F(s)F(t)$$

Proof. The number of rank-deficient completions of $M(P_{s+t+1})$ whose right nullspace contains a vector with 1 in position $s+1$ is $\frac{1}{2}A(s,t)$. To form such a completion of $M(P_{s+t+1})$, we may complete the upper left $s \times (s+1)$ region to a matrix L whose rowspace avoids $e_{s+1}^\top$, and complete the lower right $t \times (t+1)$ region to a matrix R whose rowspace avoids the vector $e_1^\top$ in $\mathbb{F}_2^{t+1}$. If u_L and u_R are the unique non-zero vectors in the right nullspaces of L and R respectively, then the last entry of u_L and the first entry of u_R are both 1. There is one way to complete row $s+1$ so that it is orthogonal to the vector $u \in \mathbb{F}_2^{s+t+1}$ whose first $s+1$ and last $t+1$ entries respectively coincide with u_L and u_R. By Lemma 2.0.4, the number of choices for L and R are respectively $F(s)$ and $F(t)$, hence $A(s,t) = 2F(s)F(t)$.

The number of rank-deficient completions of $M(P_{s+t+1})$ whose right nullspace contains a non-zero vector with 0 in position $s+1$ is $B(s,t)$. To form such a completion, we complete the upper left $s \times s$ region to a completion of $M(P_s)$ of rank $s-1$, complete the lower right $t \times t$ region to a completion of $M(P_t)$ of rank $t-1$, and assign either value to the indeterminate in row $s+1$. The number of choices for the upper left and lower right matrices are respectively $F(s-1)$ and $F(t-1)$, hence $B(s,t) = 2F(s-1)F(t-1)$.

Since $\alpha = A - B$ and $\beta = 4B - A$, this implies the result. □

Theorem 3.3.4. $\alpha(s,t)$ *is never negative and is zero if and only if* $s = t = 1$.

Proof. From Lemma 3.3.3, $\alpha(s,t) = 2F(s)F(t) - 2F(s-1)F(t-1)$. Since $F(s) \geqslant F(s-1)$ and $F(t) \geqslant F(t-1)$, it follows that $\alpha(\Gamma)$ is always non-negative. Moreover,

$\alpha(\Gamma)$ is equal to zero if and only if $s = t = 1$, since $F(n) = F(n-1)$ only for $n = 1$. □

We now identify when $\beta(s,t)$ is negative, which is used later to determine when $\alpha(r,s,t) < 0$.

Theorem 3.3.5. $\beta(s,t) < 0$ *if and only if* $\min(s,t)$ *is even.*

Proof. From Lemma 3.3.3, $\beta(s,t) = 8F(s-1)F(t-1) - 2F(s)F(t)$. By Lemma 2.0.5, we derive:

$$\beta(s,t) = 2\left[(-1)^{s+1}F(t) + (-1)^{t+1}F(s) + (-1)^{s+t}\right]$$

Therefore $\beta(s,t)$ can be negative, and this happens exactly when $\min(s,t)$ is even. □

3.3.2 Degree 2

In this subsection, we study graphs of the form $\Gamma(r,s,t) \in \mathcal{G}_2^P$ for positive integers r, s, and t (see Figure 3.7).

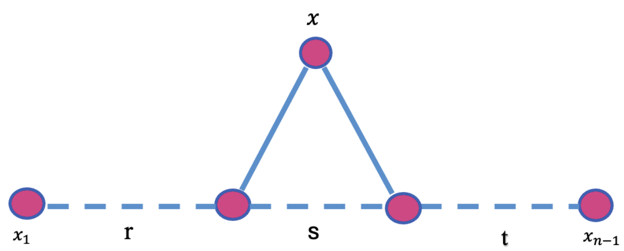


Figure 3.7: $\Gamma(r,s,t)$

Theorem 3.3.6. $\alpha(r,s,t)$ *is negative if and only if either:*

Case 1: $r = 0$ *and* $\min(s,t)$ *is even, or* $t = 0$ *and* $\min(s,r)$ *is even*

Case 2: $s = 2$ *and* $\min(r,t)$ *is even*

Proof. By Lemma 3.3.1, and since $\alpha(r,s,t) = \alpha(t,s,r)$:

$$\begin{aligned} \alpha(r,s,t) &= \frac{1}{2}F(r)\beta(s,t) + 2F(r-1)\alpha(s-1,t) \\ &= \frac{1}{2}F(t)\beta(s,r) + 2F(t-1)\alpha(s-1,r) \end{aligned}$$

By Theorem 3.3.4, the α terms are not negative for any values of r, s, t. Therefore $\alpha(r,s,t)$ can only be negative if both β terms are negative. By Theorem 3.3.5, this happens if and only if $\min(s,t)$ and $\min(s,r)$ are both even.

Case 1: Suppose $r = 0$. Then $\alpha(r, s, t) = \frac{1}{2}F(0)\beta(s, t) + 2F(0 - 1)\alpha(s - 1, t) = \frac{1}{2}\beta(s, t)$, which is negative if and only if $\min(s, t)$ is even. Similar for $t = 0$.

Case 2: Suppose $r, t > 0$. Then $r, s, t \geqslant 2$ since $\min(s, t)$ and $\min(s, r)$ are both even.

Suppose first that $s \geqslant 3$. Then $F(s) \geqslant 5$, $F(r) \geqslant 3$ and $F(t) \geqslant 3$. Then $\alpha(r, s, t)$ can be expressed as follows, using the identities in Lemma 2.0.5.

$$\alpha(r,s,t) = \frac{1}{4}\Big[3F(r)F(s)F(t) + 3(-1)^{t+1}F(r)F(s) + 3(-1)^{r+1}F(s)F(t) + 9(-1)^{s+1}F(r)F(t) + 5(-1)^{s+t}F(r) + (-1)^{r+t+1}F(s) + 5(-1)^{r+s}F(t) + (-1)^{s+r+t+1}\Big]$$

If s is odd, then r and t are both even, giving the following.

$$\begin{aligned}\alpha(r,s,t) &= \frac{1}{4}\Big[3F(r)F(s)F(t) - 3F(r)F(s) - 3F(s)F(t) - F(s) + 9F(r)F(t) - 5F(r) - 5F(t) + 1\Big] \\ &= \frac{1}{4}\Big[\underbrace{F(r)F(s)F(t) - 3F(r)F(s)}_{\geqslant 0 \text{ since } F(t)\geqslant 3} + \underbrace{F(r)F(s)F(t) - 3F(s)F(t)}_{\geqslant 0 \text{ since } F(r)\geqslant 3} + \underbrace{F(r)F(s)F(t) - F(s)}_{\geqslant 8F(s) \text{ since } F(r),F(t)\geqslant 3} \\ &\qquad + \underbrace{2F(r)F(t) - 5F(r)}_{\geqslant F(r) \text{ since } F(t)\geqslant 3} + \underbrace{2F(r)F(t) - 5F(t)}_{\geqslant F(t) \text{ since } F(r)\geqslant 3} + 5F(r)F(t) + 1\Big] \\ &\geqslant \frac{1}{4}\big[8F(s) + F(r) + F(t) + 5F(r)F(t) + 1\big] > 0\end{aligned}$$

If s is even and r is odd, then $F(s) \geqslant 11$, giving the following.

$$\begin{aligned}\alpha(r,s,t) &\geqslant \frac{1}{4}\Big[3F(r)F(s)F(t) - 3F(r)F(s) + 3F(s)F(t) - 9F(r)F(t) - 5F(r) - F(s) - 5F(t) - 1\Big] \\ &= \frac{1}{4}\Big[\underbrace{F(r)F(s)F(t) - 3F(r)F(s)}_{\geqslant 0 \text{ since } F(t)\geqslant 3} + \underbrace{F(r)F(s)F(t) - 9F(r)F(t)}_{\geqslant 2F(r)F(t) \text{ since } F(s)\geqslant 11} + \underbrace{F(s)F(t) - F(s)}_{\geqslant 2F(s) \text{ since } F(t)\geqslant 3} \\ &\qquad + \underbrace{F(r)F(s)F(t) - 5F(r) - 5F(t) - 1}_{\geqslant 0 \text{ since } F(s)\geqslant 11} + 2F(s)F(t)\Big] \\ &\geqslant \frac{1}{4}\big[2F(r)F(t) + 2F(s) + 2F(s)F(t)\big] > 0\end{aligned}$$

By symmetry, $\alpha(r, s, t)$ is positive if s is even and t is odd. For s, r, t even, we have the following.

$$\begin{aligned}\alpha(r,s,t) &= \frac{1}{4}\Big[3F(r)F(s)F(t) - 3F(r)F(s) - 3F(s)F(t) - 9F(r)F(t) + 5F(r) - F(s) + 5F(t) - 1\Big] \\ &= \frac{1}{4}\Big[\underbrace{F(r)F(s)F(t) - 3F(r)F(s)}_{\geqslant 0 \text{ since } F(t)\geqslant 3} + \underbrace{F(r)F(s)F(t) - 3F(s)F(t)}_{\geqslant 0 \text{ since } F(r)\geqslant 3} + \underbrace{F(r) - 1}_{\geqslant 2 \text{ since } F(r)\geqslant 3} \\ &\qquad + \underbrace{\frac{9}{11}F(r)F(s)F(t) - 9F(r)F(t)}_{\geqslant 0 \text{ since } F(s)\geqslant 11} + \underbrace{\frac{2}{11}F(r)F(s)F(t) - F(s)}_{\geqslant \frac{7}{11}F(s) \text{ since } F(r),F(t)\geqslant 3} + 4F(r) + 5F(t)\Big] \\ &\geqslant \frac{1}{4}\Big[2 + \frac{7}{11}F(s) + 4F(r) + 5F(t)\Big] > 0\end{aligned}$$

So $\alpha(r,s,t) > 0$ if $r,t > 0$ and $s \geqslant 3$. If $s = 2$, the formula for $\alpha(r,s,t)$ simplifies as follows.

$$\alpha(r,2,t) = (-1)^{r+1}F(t) + (-1)^{t+1}F(r) + (-1)^{r+t+1}$$

This is negative exactly when $\min(r,t)$ is even.

□

Theorem 3.3.7. $\beta(r,s,t)$ *is negative if and only if* $s = 1$ *and* $\min(r,t)$ *are odd.*

Proof. By Lemma 3.3.3 and since $\beta(r,s,t) = \beta(t,s,r)$:

$$\begin{aligned}\beta(r,s,t) &= 2F(r)\alpha(s,t) + 2F(r-1)\beta(s-1,t) \\ &= 2F(t)\alpha(s,r) + 2F(t-1)\beta(s-1,r)\end{aligned}$$

By Theorem 3.3.4, $\alpha(s,t)$ and $\alpha(s,r)$ are never negative. Then $\beta(r,s,t)$ can be negative only if both $\beta(s-1,t)$ and $\beta(s-1,r)$ are negative. By Theorem 3.3.5, this implies $\min(s-1,t)$ and $\min(s-1,r)$ are both even.

If $r = 0$, then $\beta(0,s,t) = 2\alpha(s,t)$, which is always positive. Similarly, $\beta(r,s,t) > 0$ if $t = 0$.

If $r,t \geqslant 1$, we simplify $\beta(r,s,t)$ using $2F(n-1) = F(n) + (-1)^{n+1}$ from Lemma 2.0.5 as follows.

$$\begin{aligned}\beta(r,s,t) &= 3F(r)F(s)F(t) + 3(-1)^s F(r)F(t) + (-1)^{r+t}F(s) \\ &\quad - 2(-1)^{s+t}F(r) - 2(-1)^{s+r}F(t) + (-1)^{s+t+r}\end{aligned}$$

First suppose $s \geqslant 2$. Then $s \geqslant 3$ and $t \geqslant 2$, since $\min(s-1,t)$ is even. This implies $F(t) \geqslant 3$ and $F(s) \geqslant 5$. As a result, $2F(r)F(s)F(t) + 3(-1)^s F(r)F(t) \geqslant 7F(r)F(t)$ giving the following.

$$\beta(r,s,t) \geqslant F(r)F(s)F(t) + 7F(r)F(t) + (-1)^{r+t}F(s) + (-1)^{s+t+r} - 2(-1)^{s+t}F(r) - 2(-1)^{r+s}F(t)$$

We note that $7F(r)F(t) > 2F(r) + 2F(t) + 1$ and $F(r)F(s)F(t) > F(s)$. Hence $\beta(r,s,t) > 0$ for $s \geqslant 2$.

It only remains to consider $s = 1$. We simplify $\beta(r, s, t)$ using Lemma 2.0.5 as follows.

$$\begin{aligned}\beta(r,1,t) &= 3F(r)F(t) - 3F(r)F(t) + 2(-1)^t F(r) + (-1)^{r+t} + 2(-1)^r F(t) + (-1)^{r+t+1} \\ &= 2\left[(-1)^t F(r) + (-1)^r F(t)\right]\end{aligned}$$

Thus $\beta(r, s, t)$ is negative if and only if $s = 1$ and $\min(r, t)$ is odd.

□

3.3.3 Degree 3

In this subsection, we study graphs of the form $\Gamma(q, r, s, t) \in \mathcal{G}_3^P$ for positive integers q, r, s, and t (see Figure 3.8).

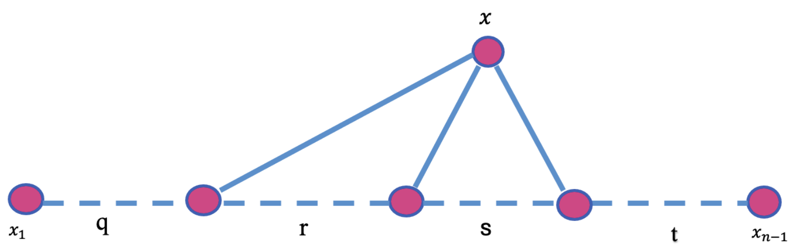


Figure 3.8: $\Gamma(q, r, s, t)$

Theorem 3.3.8. $\alpha(q, r, s, t)$ *is negative if and only if either:*

Case 1: $(q, s) = (0, 1)$ *and* $\min(r, t)$ *is odd, or* $(t, r) = (0, 1)$ *and* $\min(q, s)$ *is odd.*

Case 2: $s = r = 1$ *and* $\min(q, t)$ *is even.*

Proof. Since $\alpha(q, r, s, t) = \alpha(t, s, r, q)$, we have

$$\begin{aligned}\alpha(q,r,s,t) &= \frac{1}{2}F(q)\beta(r,s,t) + 2F(q-1)\alpha(r-1,s,t) \\ &= \frac{1}{2}F(t)\beta(s,r,q) + 2F(t-1)\alpha(s-1,r,q)\end{aligned}$$

Case 1: If $q = 0$, then Lemma 3.3.1 implies $\alpha(0, r, s, t) = \frac{1}{2}\beta(r, s, t)$. By Theorem 3.3.7, this is negative if and only if $s = 1$ and $\min(r, t)$ is odd. By symmetry, $t = 0$ gives the other result in Case 1.

Case 2: If $q, t \geqslant 1$, Theorems 3.3.6 and 3.3.7 and the above equations for $\alpha(q, r, s, t)$ imply that it can only be negative if both of the following hold.

- $\underbrace{s = 1 \text{ and } \min(r,t) \text{ is odd},}_{\beta(r,s,t)<0}$ or $\underbrace{r-1 = 0 \text{ and } \min(s,t) \text{ is even},}_{\alpha(r-1,s,t)<0}$ or $\underbrace{s = 2 \text{ and } \min(r-1,t) \text{ is even}}_{\alpha(r-1,s,t)<0}$.
- $\underbrace{r = 1 \text{ and } \min(s,q) \text{ is odd},}_{\beta(s,r,q)<0}$ or $\underbrace{s-1 = 0 \text{ and } \min(r,q) \text{ is even},}_{\alpha(s-1,r,q)<0}$ or $\underbrace{r = 2 \text{ and } \min(s-1,q) \text{ is even}}_{\alpha(s-1,r,q)<0}$.

Therefore we need only consider the cases in which either $r \in \{1,2\}$ or $s \in \{1,2\}$.

$\underline{r = 2}$: We simplify $\alpha(q,r,s,t)$ using $2F(n-1) = F(n) + (-1)^{n+1}$ from Lemma 2.0.5 as follows.

$$\alpha(q,2,s,t) = \frac{1}{2}\Big[12F(q)F(s)F(t) + 3(-1)^{q+1}F(s)F(t) + 2(-1)^{s+t+1}F(q) + (-1)^{q+t}F(s) + 7(-1)^{q+s}F(t) + 3(-1)^{q+s+t+1}\Big]$$

If $q = 1$, then $\alpha(q,2,s,t)$ simplifies as follows.

$$\begin{aligned}\alpha(1,2,s,t) &= \frac{1}{2}\Big[15F(s)F(t) + (-1)^{t+1}F(s) + 7(-1)^{s+1}F(t) + (-1)^{s+t}\Big] \\ &\geqslant \frac{1}{2}\Big[15F(s)F(t) - F(s) - 7F(t) - 1\Big] \\ &\geqslant \frac{1}{2}\Big[15F(s)F(t) - F(s)F(t) - 7F(s)F(t) - F(s)F(t)\Big] \geqslant \frac{1}{2}\Big[6F(s)F(t)\Big] > 0\end{aligned}$$

If $q \geqslant 2$, then $F(q) \geqslant 3$, and $\alpha(q,2,s,t)$ simplifies as follows.

$$\begin{aligned}\alpha(q,2,s,t) &= \frac{1}{2}\Big[2F(q)F(s)F(t) + 2(-1)^{s+t+1}F(q) \\ &\quad + 10F(q)F(s)F(t) + 3(-1)^{q+1}F(s)F(t) + (-1)^{q+t}F(s) + 7(-1)^{q+s}F(t) \\ &\quad + 3(-1)^{q+s+t+1}\Big] \\ &\geqslant \frac{1}{2}\Big[\underbrace{2F(q)F(s)F(t) - 2F(q)}_{\geqslant 0} + \underbrace{30F(s)F(t) - 3F(s)F(t) - F(s) - 7F(t) - 3}_{\geqslant 16F(s)F(t)}\Big] > 0\end{aligned}$$

Therefore $\alpha(q,r,s,t) > 0$ if $r = 2$. Similarly, by symmetry, $\alpha(q,r,s,t) > 0$ if $s = 2$.

$\underline{r = 1}$: We simplify $\alpha(q,r,s,t)$ using $2F(n-1) = F(n) + (-1)^{n+1}$ from Lemma 2.0.5 as follows.

$$\alpha(q,1,s,t) = \frac{1}{2}\Big[3F(q)F(s)F(t) + 3(-1)^{s}F(q)F(t) + 3(-1)^{t+1}F(q)F(s) + (-1)^{s+t+1}F(q) + 2(-1)^{q+t}F(s) + 2(-1)^{q+s}F(t) + 2(-1)^{q+s+t+1}\Big]$$

We have dealt with $s = 2$ previously, so we now separately consider $s = 1$ and $s \geqslant 3$.

Suppose $s \geqslant 3$. Since $r = 1$, it follows from the bullet points above that for $\alpha(q,r,s,t) < 0$, $\min(s,t)$ is even and $\min(s,q)$ is odd. This means that $t \geqslant 2$. Separating the $3F(q)F(s)F(t)$ term in the expression for $\alpha(q,1,s,t)$ to dominate the potentially negative terms gives the following.

$$\begin{aligned}
2\alpha(q,1,s,t) =& \Big[F(q)F(s)F(t) + 3(-1)^{s}F(q)F(t) + 2(-1)^{q+s}F(t)\Big] \\
&+ \Big[F(q)F(s)F(t) + 3(-1)^{t+1}F(q)F(s)\Big] + \Big[\frac{2}{3}F(q)F(s)F(t) + 2(-1)^{q+t}F(s)\Big] \\
&+ \Big[\frac{1}{3}F(q)F(s)F(t) + (-1)^{s+t+1}F(q) + 2(-1)^{q+s+t+1}\Big] \\
\geqslant& \Big[\underbrace{5F(q)F(t) - 3F(q)F(t) - 2F(t)}_{\geqslant 0}\Big] + \Big[\underbrace{3F(q)F(s) - 3F(q)F(s)}_{\geqslant 0}\Big] \\
&+ \Big[\underbrace{2F(q)F(s) - 2F(s)}_{\geqslant 0}\Big] + \Big[\underbrace{5F(q) - F(q) - 2}_{\geqslant 2F(q)}\Big] > 0
\end{aligned}$$

Similarly, by symmetry, $\alpha(q,r,s,t) > 0$ if $s = 1$ and $r \geqslant 3$.

Now suppose $s = r = 1$.

$$\begin{aligned}
\alpha(q,1,1,t) &= \frac{1}{2}\Big[3F(q)F(t) - 3F(q)F(t) + 3(-1)^{t+1}F(q) \\
&\qquad + (-1)^{t}F(q) + 2(-1)^{q+t} + 2(-1)^{q+1}F(t) + 2(-1)^{q+t}\Big] \\
&= (-1)^{t+1}F(q) + (-1)^{q+1}F(t) + 2(-1)^{q+t}
\end{aligned}$$

This is negative if and only if $\min(q,t)$ is even.

□

Theorem 3.3.9. $\beta(q,r,s,t)$ *is negative if and only if either:*

Case 1: $(q,s) = (0,2)$ *and* $\min(r,t)$ *is even, or* $(t,r) = (0,2)$ *and* $\min(q,s)$ *is even.*

Case 2: $q = t = 0$, *and* $\min(r,s)$ *is even.*

Proof. Since $\beta(q,r,s,t) = \beta(t,s,r,q)$, we have

$$\begin{aligned}
\beta(q,r,s,t) &= 2F(q)\alpha(r,s,t) + 2F(q-1)\beta(r-1,s,t) \\
&= 2F(t)\alpha(s,r,q) + 2F(t-1)\beta(s-1,r,q)
\end{aligned}$$

If $q = 0$, then $\beta(0, r, s, t) = 2\alpha(r, s, t)$ which is negative if $s = 2$ and $\min(r, t)$ is even (Case 1), $r = 0$ and $\min(s, t)$ is even (not possible since $r \geqslant 1$), or $t = 0$ and $\min(s, r)$ is even (Case 2). Similarly, if $t = 0$, then $\beta(q, r, s, 0)$ is negative if $r = 2$ and $\min(s, q)$ is even (finishing Case 1), $s = 0$ and $\min(r, q)$ is even (not possible since $s \geqslant 1$), or $q = 0$ and $\min(s, r)$ is even (Case 2).

If $q, t \geqslant 1$, and since $r, s \geqslant 1$, Theorems 3.3.6 and 3.3.7 and the above expressions for $\beta(q, r, s, t)$ imply that it can only be negative if both of the following hold.

- $\underbrace{s = 2 \text{ and } \min(r, t) \text{ is even}}_{\alpha(r,s,t)<0}$, or $\underbrace{s = 1 \text{ and } \min(r-1, t) \text{ is odd}}_{\beta(r-1,s,t)<0}$.
- $\underbrace{r = 2 \text{ and } \min(s, q) \text{ is even}}_{\alpha(s,r,q)<0}$, or $\underbrace{r = 1 \text{ and } \min(s-1, q) \text{ is odd}}_{\beta(s-1,r,q)<0}$.

We therefore may restrict our attention to $r, s \in \{1, 2\}$.

<u>$r = s = 1$:</u> We simplify $\beta(q, r, s, t)$ as follows.

$$\begin{aligned}\beta(q, 1, 1, t) &= 2\big[6F(q)F(t) + (-1)^{q+1}F(t) + (-1)^{t+1}F(q) + (-1)^{q+t}\big] \\ &\geqslant 2\big[6F(q)F(t) - F(t) - F(q) - 1\big] \geqslant 2 \cdot 3F(q)F(t) > 0\end{aligned}$$

<u>$r = 1$, $s = 2$:</u> We simplify $\beta(q, r, s, t)$ as follows.

$$\begin{aligned}\beta(q, 1, 2, t) &= 2\big[6F(q)F(t) + 5(-1)^{q+1}F(t) + (-1)^{t}F(q) + (-1)^{q+t+1}\big] \\ &= 2\big[\underbrace{5F(q)F(t) + 5(-1)^{q+1}F(t) + (-1)^{q+t+1}}_{>0} + \underbrace{F(q)F(t) + (-1)^{t}F(q)}_{\geqslant 0}\big]\end{aligned}$$

<u>$r = s = 2$:</u> We simplify $\beta(q, r, s, t)$ as follows.

$$\beta(q, 2, 2, t) = 2\big[6F(q)F(t) + 7(-1)^{q+1}F(t) + 7(-1)^{t+1}F(q) + 3(-1)^{q+t}\big]$$

If $q = 1$, then $\beta(q, 2, 2, t)$ simplifies as follows.

$$\beta(1, 2, 2, t) = 2\big[13F(t) + 10(-1)^{t+1}\big] \geqslant 2 \cdot 3F(t) > 0$$

By symmetry, $\beta(q, 2, 2, 1) > 0$.

If $q, t \geqslant 2$, then $F(q), F(t) \geqslant 3$, and we rearrange $\beta(q, 2, 2, t)$ as follows.

$$\begin{aligned}\beta(q, 2, 2, t) &= 2\big[3F(q)F(t) + 7(-1)^{q+1}F(t) + 3F(q)F(t) + 7(-1)^{t+1}F(q) + 3(-1)^{q+t}\big] \\ &\geqslant 2\big[9F(t) - 7F(t) + 9F(q) - 7F(q) - 3\big] = 2\big[2F(t) + 2F(q) - 3\big] > 0\end{aligned}$$

Therefore $\beta(q, r, s, t) > 0$ for all $q, t \geqslant 1$. $\square$

3.3.4 Degree 4

In this subsection, we study graphs of the form $\Gamma(p, q, r, s, t) \in \mathcal{G}_4^P$ for positive integers p, q, r, s, and t (see Figure 3.9).

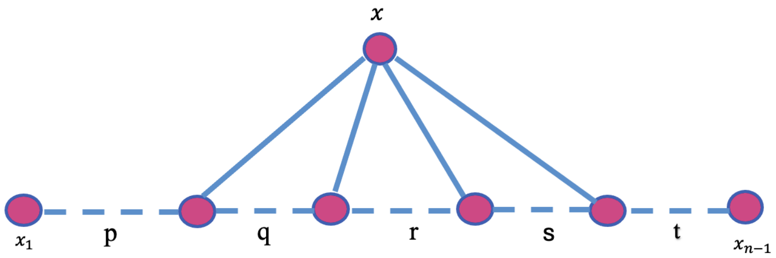


Figure 3.9: $\Gamma(p, q, r, s, t)$

Theorem 3.3.10. $\alpha(p, q, r, s, t) < 0$ *if and only if* $p = t = 0, r = 2$ *and* $\min(q, s)$ *is even.*

Proof. By Lemma 3.3.1, and since $\alpha(p, q, r, s, t) = \alpha(t, s, r, q, p)$:

$$\begin{aligned}\alpha(p, q, r, s, t) =& \frac{1}{2}F(p)\beta(q, r, s, t) + 2F(p-1)\alpha(q-1, r, s, t)\\ =& \frac{1}{2}F(t)\beta(s, r, q, p) + 2F(t-1)\alpha(s-1, r, q, p)\end{aligned}$$

First suppose $p = 0$. Then $\alpha(0, q, r, s, t) = \frac{1}{2}\beta(q, r, s, t)$, so $\alpha(0, q, r, s, t)$ can only be negative if $\beta(q, r, s, t)$ is negative. By Theorem 3.3.9, and since $q, r, s \geqslant 1$, this means $p = t = 0$, $r = 2$, and $\min(q, s)$ is even. Similarly, this is the only case for which $\alpha(p, q, r, s, 0)$ is negative.

Suppose $p, t > 0$. Theorems 3.3.8 and 3.3.9 and the above expressions for $\alpha(p, q, r, s, t)$ imply that it can only be negative if both of the following hold.

- $\underbrace{q = s = 1 \text{ and } \min(r, t) \text{ is odd}}_{\alpha(q-1, r, s, t) < 0}$, or $\underbrace{r = s = 1 \text{ and } \min(q-1, t) \text{ is even}}_{\alpha(q-1, r, s, t) < 0}$.
- $\underbrace{q = s = 1 \text{ and } \min(r, p) \text{ is odd}}_{\alpha(s-1, r, q, p) < 0}$, or $\underbrace{r = q = 1 \text{ and } \min(s-1, p) \text{ is even}}_{\alpha(s-1, r, q, p) < 0}$.

From the above cases, $\alpha(p, q, r, s, t)$ can only be negative if $q = s = 1$.

$$\alpha(p, 1, r, 1, t) = 3F(p)F(r)F(t) - 3(-1)^r F(p)F(t) - (-1)^{r+t}F(p) - (-1)^{p+r}F(t) - (-1)^{p+t}F(r)$$

If $r = 1$, we obtain the following.

$$\begin{aligned}\alpha(p,1,1,1,t) &= 6F(p)F(t) + (-1)^t F(p) + (-1)^p F(t) - (-1)^{p+t} \\ &\geqslant 6F(p)F(t) - F(p) - F(t) - 1 \geqslant 3F(p)F(t) > 0\end{aligned}$$

If $r > 1$, we simplify $\alpha(p,q,r,s,t)$ as follows.

$$\begin{aligned}\alpha(p,1,r,1,t) =& F(p)F(r)F(t) - 3(-1)^r F(p)F(t) \\ &+ F(p)F(r)F(t) - (-1)^{r+t}F(p) - (-1)^{p+r}F(t) \\ &+ F(p)F(r)F(t) - (-1)^{p+t}F(r) \\ \geqslant& \underbrace{3F(p)F(t) - 3F(p)F(t)}_{=0} + \underbrace{3F(p)F(t) - F(p) - F(t)}_{\geqslant F(p)F(t)} + \underbrace{F(p)F(r)F(t) - F(r)}_{\geqslant 0}\end{aligned}$$

Therefore $\alpha(p,q,r,s,t)$ is negative if and only if $p = t = 0$, $r = 2$, and $\min(q,s)$ is even.

□

Theorem 3.3.11. $\beta(p,q,r,s,t)$ *is negative if and only if either:*

Case 1: $p = 0$, $r = s = 1$ *and* $\min(q,t)$ *is even, or* $t = 0$, $q = r = 1$ *and* $\min(p,s)$ *is even.*

Case 2: $p = t = 0$, $r = 1$ *and* $\min(q,s)$ *is odd.*

Proof. Since $\beta(p,q,r,s,t) = \beta(t,s,r,q,p)$, we have

$$\begin{aligned}\beta(p,q,r,s,t) &= 2F(p)\alpha(q,r,s,t) + 2F(p-1)\beta(q-1,r,s,t) \\ &= 2F(t)\alpha(s,r,q,p) + 2F(t-1)\beta(s-1,r,q,p)\end{aligned}$$

If $p = t = 0$, since $\alpha(0,q,r,s,0) = \alpha(q,r,s)$, this is negative if and only if $r = 1$ and $\min(q,s)$ is odd (Case 2). If $p = 0$ and $t > 0$, then $\beta(0,q,r,s,t) = 2\alpha(q,r,s,t)$. Since $q,r,s \geqslant 1$, this is negative only if $r = s = 1$ and $\min(q,t)$ is even. Similarly if $t = 0$ and $p > 0$, then $\beta(p,q,r,s,t) < 0$ if and only if $q = r = 1$ and $\min(p,s)$ is even (Case 1).

If $p,t > 0$, then Theorems 3.3.8 and 3.3.9 and the above expressions for $\beta(p,q,r,s,t)$ imply that it can only be negative if both of the following hold.

- $\underbrace{r = s = 1 \text{ and } \min(q,t) \text{ is even}}_{\alpha(q,r,s,t)<0}$, or $\underbrace{(q,s) = (1,2) \text{ and } \min(r,t) \text{ is even}}_{\beta(q-1,r,s,t)<0}$.
- $\underbrace{r = q = 1 \text{ and } \min(s,p) \text{ is even}}_{\alpha(s,r,q,p)<0}$, or $\underbrace{(s,q) = (1,2) \text{ and } \min(r,p) \text{ is even}}_{\beta(s-1,r,q,p)<0}$.

If $r = s = q = 1$, then $\min(s,p) = \min(1,p)$ must be even, which contradicts $p,t > 0$. If $(r,s,q) \in \{(1,1,2),(1,2,1)\}$, then either $\min(r,t) = \min(1,t)$ or $\min(r,p) = \min(1,p)$ must be even. This again contradicts $p,t > 0$, and so $\beta(p,q,r,s,t)$ is not negative for $p,t > 0$. □

We conclude that the only graphs $\Gamma \in \mathcal{G}_4^p$ for which either $\alpha(\Gamma)$ or $\beta(\Gamma)$ is negative arise from the addition of two edges to graphs in $\mathcal{G}_2^p$, between x and the vertices at the ends of the long induced path.

3.3.5 Degree 5

In this subsection, we study graphs of the form $\Gamma(k, p, q, r, s, t) \in \mathcal{G}_5^P$ for positive integers k, p, q, r, s, and t (see Figure 3.10).

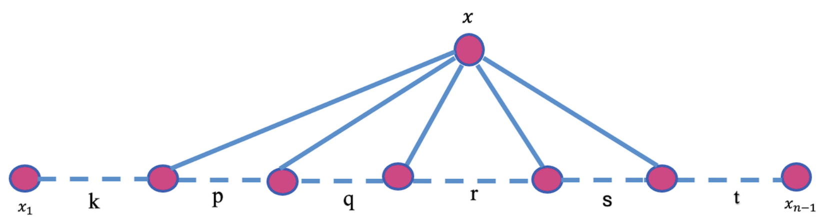


Figure 3.10: $\Gamma(k, p, q, r, s, t)$

Theorem 3.3.12. *$\alpha(k, p, q, r, s, t)$ is negative if and only if $k = t = 0$, $q = r = 1$, and $\min(p, s)$ is even.*

Proof. By Lemma 3.3.1, and since $\alpha(k, p, q, r, s, t) = \alpha(t, s, r, q, p, k)$:

$$\begin{aligned}\alpha(k, p, q, r, s, t) &= \frac{1}{2}F(k)\beta(p, q, r, s, t) + 2F(k-1)\alpha(p-1, q, r, s, t)\\ &= \frac{1}{2}F(t)\beta(s, r, q, p, k) + 2F(t-1)\alpha(s-1, r, q, p, k)\end{aligned}$$

Suppose $k = 0$. Then $\alpha(0, p, q, r, s, t) = \frac{1}{2}\beta(p, q, r, s, t)$. By Theorem 3.3.11 and since $p, q, r, s \geqslant 1$, then $\beta(p, q, r, s, t)$ is negative if and only if $t = 0, r = q = 1$, and $\min(s, p)$ is even. By symmetry, the same holds if $t = 0$.

Next, suppose $k, t > 0$. By Theorems 3.3.10 and 3.3.11, and since $p, q, r, s \geqslant 1$, the above expressions for $\alpha(k, p, q, r, s, t)$ imply that it can be negative only if both of the following hold.

- $\underbrace{t = 0, r = q = 1 \text{ and } \min(p, s) \text{ is even}}_{\beta(p,q,r,s,t)<0}$, or $\underbrace{t = 0, p = 1, r = 2 \text{ and } \min(s, q) \text{ is even}}_{\alpha(p-1,q,r,s,t)<0}$.
- $\underbrace{k = 0, r = q = 1 \text{ and } \min(s, p) \text{ is even}}_{\beta(s,r,q,p,k)<0}$, or $\underbrace{k = 0, s = 1, q = 2 \text{ and } \min(r, p) \text{ is even}}_{\alpha(s-1,r,q,p,k)<0}$.

$\beta(p, q, r, s, t) < 0$ implies $q = 1$ and $\alpha(s-1, r, q, p, k) < 0$ implies $q = 2$, and therefore they cannot hold simultaneously. Similarly, $\beta(s, r, q, p, k) < 0$ and $\alpha(p-1, q, r, s, t)$ cannot hold simultaneously due to contradicting requirements for r. If both $\alpha(p-1, q, r, s, t)$ and $\alpha(s-1, r, q, p)$ are negative, $\min(s, q) = \min(1, 2)$ is not even, another contradiction.

If $\beta(p, q, r, s, t) < 0$ and $\beta(s, r, q, p, k) < 0$, this means that $k = t = 0$. Therefore,

$$\alpha(0, p, q, r, s, 0) = \frac{1}{2}\beta(p, q, r, s, 0) = \frac{1}{2}\beta(0, s, r, q, p) = \alpha(s, r, q, p) = \alpha(p, q, r, s),$$

which is negative if and only if $q = r = 1$ and $\min(p, s)$ is even. □

Theorem 3.3.13. $\beta(k, p, q, r, s, t)$ *is positive for all* $k, p, q, r, s, t \in \mathbb{N}$.

Proof. By Lemma 3.3.1, and since $\beta(k, p, q, r, s, t) = \beta(t, s, r, q, p, k)$:

$$\begin{aligned}\beta(k, p, q, r, s, t) &= 2F(k)\alpha(p, q, r, s, t) + 2F(k-1)\beta(p-1, q, r, s, t)\\ &= 2F(t)\alpha(s, r, q, p, k) + 2F(t-1)\beta(s-1, r, q, p, k)\end{aligned}$$

If $k = 0$, then $\beta(0, p, q, r, s, t) = 2\alpha(p, q, r, s, t)$. By Theorem 3.3.10, and since $p \geqslant 1$, it follows that $\alpha(p, q, r, s, t)$ is never negative. By symmetry, $\beta(k, p, q, r, s, t)$ is never negative if $t = 0$.

Suppose $k, t > 0$. By Theorems 3.3.10 and 3.3.11, and since $p, q, r, s \geqslant 1$, the above expressions for $\beta(k, p, q, r, s, t)$ imply that it can be negative only if both of the following hold.

- $\underbrace{p = r = s = 1 \text{ and } \min(q, t) \text{ is even.}}_{\beta(p-1,q,r,s,t)<0}$
- $\underbrace{p = q = s = 1 \text{ and } \min(r, k) \text{ is even.}}_{\beta(s-1,r,q,p,k)<0}$

But since $r = q = 1$ in this case, $\min(q, t) = 1$ and $\min(r, k) = 1$, which contradicts the requirement that both are even. Therefore $\beta(k, p, q, r, s, t) > 0$ for all $k, p, q, r, s, t \in \mathbb{N}$. □

We conclude that the only graphs $\Gamma \in \mathcal{G}_5^P$ for which $\alpha(\Gamma)$ is negative arise from the addition of two edges to graphs in $\mathcal{G}_3^P$, between x and the vertices at the ends of the long induced path.

3.3.6 Degree $\geqslant 6$

In this final subsection, we prove that any graph of the form $\Gamma(t_0, t_1, \ldots, t_d) \in \mathcal{G}_d^P$ is represented by more matrices of rank n than rank $n-1$, for $d \geqslant 6$.

Theorem 3.3.14. *$\alpha(t_0, t_1, \cdots, t_d)$ and $\beta(t_0, t_1, \cdots, t_d)$ are positive if $d \geqslant 6$.*

Proof. Induction on d. From Lemma 3.3.1, we have the following.

$$\alpha(t_0, t_1, \ldots, t_d) = \frac{1}{2}F(t_0)\beta(t_1, t_2, , \ldots, t_d) + 2F(t_0 - 1)\alpha(t_1 - 1, t_2, \ldots, t_d)$$
$$\beta(t_0, t_1, \ldots, t_d) = 2F(t_0)\alpha(t_1, t_2, \ldots, t_d) + 2F(t_0 - 1)\beta(t_1 - 1, t_2, \ldots, t_d)$$

Suppose $d = 6$. Theorem 3.3.13 implies that $\beta(t_1, t_2, \cdots, t_d)$ and $\beta(t_1 - 1, t_2, \cdots, t_d)$ are never negative. Since $t_1 \geqslant 1$, Theorem 3.3.12 implies that $\alpha(t_1 - 1, t_2 \cdots, t_d)$ and $\alpha(t_1, t_2 \cdots, t_d)$ are never negative. Therefore $\alpha(t_0, t_1, \cdots, t_d)$ and $\beta(t_0, t_1, \cdots, t_d)$ are positive for $d = 6$.

Now suppose $\alpha(t_1, t_2, \cdots, t_d)$ and $\beta(t_1, t_2, \cdots, t_d)$ are positive for some $d \geqslant 6$. By the induction hypothesis, each term on the right-hand side of the above equations is positive. Therefore $\alpha(t_0, t_1, \cdots, t_d)$ and $\beta(t_0, t_1, \cdots, t_d)$ are positive if $d \geqslant 6$. □

The following theorem summarizes all cases possible studied in the graph families $\mathcal{G}_d^P$ that satisfy $\alpha(\Gamma) < 0$.

Theorem (3.3.2). *The following are all graphs in $\mathcal{G}^P$ represented by more matrices of rank $n-1$ than rank n (see Figure 3.11).*

- $\Gamma(0, s, t)$ *with* $\min(s, t)$ *even.*
- $\Gamma(s, 2, t)$ *with* $\min(s, t)$ *even.*
- $\Gamma(0, s, 1, t)$ *with* $\min(s, t)$ *odd.*
- $\Gamma(s, 1, 1, t)$ *with* $\min(s, t)$ *even.*
- $\Gamma(0, s, 2, t, 0)$ *with* $\min(s, t)$ *even.*
- $\Gamma(0, s, 1, 1, t, 0)$ *with* $\min(s, t)$ *even.*

The following figure summarises all graphs in $\mathcal{G}^P$ represented by more completions of rank $n-1$ than rank n.

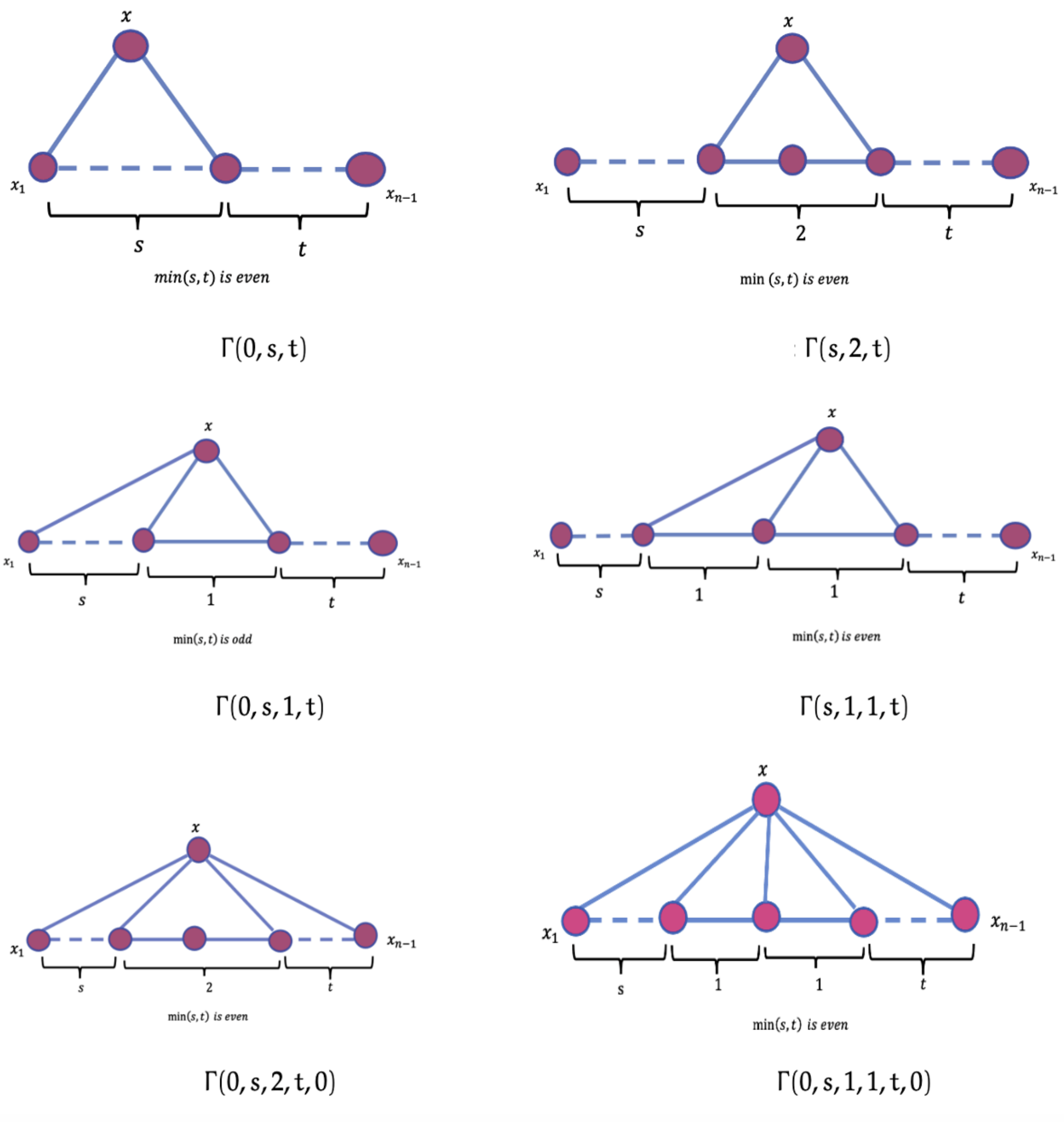


Figure 3.11: Graphs in $\mathcal{G}^P$ represented by more completions of rank $n-1$ than rank n

While $\mathcal{G}^P$ contains many examples of graphs Γ on n vertices for which $R_{n-1}(\Gamma) > R_n(\Gamma)$, many such graphs exist outside of $\mathcal{G}^P$. In the following chapters, we consider graphs containing a long induced cycle.

Chapter 4

Rank distribution for the cycle graph C_n over $\mathbb{F}_2$

For a positive integer n, let C_n be the cycle graph with vertices $x_1, \ldots, x_n$ and edges x_1x_n and x_ix_{i+1} for $1 \leqslant i \leqslant n-1$. Over any field $\mathbb{F}$, every matrix that represents C_n with respect to this vertex ordering has nonzero entries on the super-diagonal and sub-diagonal, as well as positions $(1, n)$ and $(n, 1)$. Therefore, it has rank at least $n-2$. It is routine to check that if the nonzero off-diagonal entries of a matrix representing C_n are all 1, then the diagonal entries may be completed to obtain a matrix of either rank n, rank $n-1$, or rank $n-2$. The cycle graph C_n has a special role in the minimum rank problem for graphs; for $n \geqslant 3$ and for every field $\mathbb{F}$, C_n is the graph whose minimum rank over $\mathbb{F}$ is $n-2$.

The indeterminate matrix that represents C_n over $\mathbb{F}_2$ is denoted $M(C_n)$ and has entries equal to 1 in the first super diagonal, the first sub diagonal, and in the $(1, n)$ and $(n, 1)$ positions, indeterminates on the main diagonal, and zeros elsewhere.

$$M(C_n) = \begin{bmatrix} d_1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & d_2 & 1 & \ddots & & 0 \\ 0 & 1 & d_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & & \ddots & 1 & d_{n-1} & 1 \\ 1 & 0 & \cdots & 0 & 1 & d_n \end{bmatrix}$$

We see that the indeterminate sub-matrix of $M(C_n)$, after deleting the first row and column, is $M(P_{n-1})$. Since every completion of $M(P_{n-1})$ has either rank $n-1$ or $n-2$, the rank of every completion of $M(C_n)$ is either $n-2$, $n-1$, or n. Our goal in this section is to determine $R_n(C_n)$, $R_{n-1}(C_n)$, and $R_{n-2}(C_n)$, the number of $\mathbb{F}_2$-completions of $M(C_n)$ of ranks n, $n-1$ and $n-2$, respectively.

The next theorem gives a complete description of the $\mathbb{F}_2$-rank distribution of the cycle C_n.

Theorem 4.0.1.

$$R_n(C_n) = \frac{1}{3}(2^n + (-1)^{n+1}),\ R_{n-1}(C_n) = 2^{n-1},\ R_{n-2}(C_n) = \frac{1}{3}(2^{n-1} + (-1)^n)$$

Proof. Let A' be a completion of $M(P_{n-1})$ and let M' be the indeterminate matrix obtained from $M(C_n)$ by completing the last $n-1$ indeterminates so that the lower right of the submatrix is A'. Let the vector v consist of the last $n-1$ entries of the first row of M', which has 1 as its first and last entries and otherwise consists of zeros. Recall that U_{n-1} consists of all vectors in $\mathbb{F}_2^{n-1}$ with first and last entries equal to 1 and no consecutive entries both equal to zero. Every element of U_{n-1} is orthogonal to v.

If $\text{rank}(A') = n-1$, then v is in the rowspace of A'. By Theorem 3.0.1, one choice for the upper left entry of M' gives a completion of $M(C_n)$ of rank $n-1$, and the other gives a completion of rank n.

If $\text{rank}(A') = n-2$, then v is again in the rowspace of A' since v is orthogonal to the element of U_{n-1} that spans the nullspace of A'. By Theorem 3.0.1, one choice for the upper left entry gives a completion of $M(C_n)$ of rank $n-1$, and the other gives a completion of rank $n-2$.

So every matrix that represents P_{n-1} and has rank $n-1$ contributes one to both $R_n(C_n)$ and $R_{n-1}(C_n)$, and every matrix that represents P_{n-1} and has rank $n-2$ contributes one to both $R_{n-1}(C_n)$ and $R_{n-2}(C_n)$. From Theorem 2.0.2, we conclude

- $R_n(C_n) = R_n(P_{n-1}) = \frac{1}{3}(2^n + (-1)^{n+1})$.
- $R_{n-1}(C_n) = R_n(P_{n-1}) + R_{n-1}(P_{n-1}) = 2^{n-1}$.
- $R_{n-2}(C_n) = R_{n-1}(P_{n-1}) = \frac{1}{3}(2^{n-1} + (-1)^n)$

□

Thus half of all $\mathbb{F}_2$-matrices representing the cycle C_n have rank $n-1$, approximately one-third have rank n, and approximately one-sixth have rank $n-2$.

Chapter 5

Graphs with a long cycle as an induced subgraph

This chapter looks at the class $\mathcal{G}^C$ of connected graphs containing an induced cycle on all but one vertex (called the extra vertex). The cycle C_n itself is not in $\mathcal{G}^C$, but Theorem 4.0.1, which gives the $\mathbb{F}_2$-rank distribution of C_n, is the starting point for our work here. Just as the rank distribution of the path was the starting point for Chapter 3, we now adapt Theorem 4.0.1 to study this wider class of graphs. For a graph Γ of order n in $\mathcal{G}^C$, we begin by exploring how the rank of an $\mathbb{F}_2$-matrix representing Γ relates to the rank of a submatrix representing an induced subgraph isomorphic to C_{n-1}.

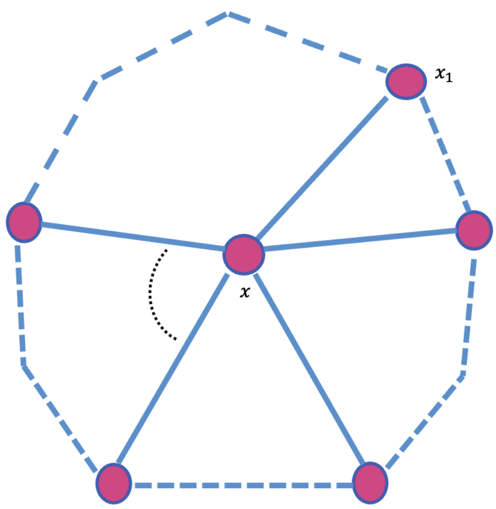


Figure 5.1: A graph Γ with a long induced cycle C_{n-1}

For any graph Γ of order n in $\mathcal{G}^C$, we label the vertices of Γ as $x, x_1, \ldots, x_{n-1}$, where the subgraph induced on the set of vertices $\{x_1, \ldots, x_{n-1}\}$ forms a cycle C_{n-1} with the edge set $E = \{x_i x_{i+1}, 1 \leqslant i \leqslant n-2\} \cup \{x_{n-1} x_1\}$. We write $M(\Gamma)$ for the indeterminate matrix that generically represents Γ with respect to this vertex ordering. Then

$$M(\Gamma) = \left(\begin{array}{c|c} d_0 & v^\top \\ \hline v & M(C_{n-1}) \end{array}\right),$$

where the upper left entry d_0 is an indeterminate, and the vector $v \in \mathbb{F}_2^{n-1}$ describes the incidences of the vertex x. Since every $\mathbb{F}_2$-completion of $M(C_{n-1})$ has rank at least $n-3$, every $\mathbb{F}_2$-matrix representing Γ has one of four possible ranks: $n-3, n-2, n-1,$ or n.

The following theorem details how the rank of a completion of $M(C_{n-1})$ determines the ranks of its two extensions to completions of $M(\Gamma)$.

Theorem 5.0.1. *Let* A *be a completion of* $M(C_{n-1})$ *and let* A_0 *and* A_1 *be the completions of* $M(\Gamma)$ *respectively given by*

$$A_0 = \left(\begin{array}{c|c} 0 & v^\top \\ \hline v & A \end{array}\right), \quad A_1 = \left(\begin{array}{c|c} 1 & v^\top \\ \hline v & A \end{array}\right).$$

Then

1. *If* $\text{rank}(A) = n-1$, *then one of* A_0 *and* A_1 *has rank* $n-1$ *and the other has rank* n.
2. *If* $\text{rank}(A) = n-2$ *and* $v^\top \notin \text{rowspace}(A)$, *then both* A_0 *and* A_1 *have rank* n. *If* $\text{rank}(A) = n-2$ *and* $v^\top \in \text{rowspace}(A)$, *then one of* A_0 *and* A_1 *has rank* $n-2$ *and the other has rank* $n-1$.
3. *If* $\text{rank}(A) = n-3$ *and* $v^\top \notin \text{rowspace}(A)$, *then both* A_0 *and* A_1 *have rank* $n-1$. *If* $\text{rank}(A) = n-3$ *and* $v^\top \in \text{rowspace}(A)$, *then one of* A_0 *and* A_1 *has rank* $n-2$ *and the other has rank* $n-3$.

Proof. 1. Let M denote the indeterminate matrix obtained from $M(\Gamma)$ by completing $M(C_{n-1})$ to A. Suppose that $\text{rank}(A) = n-1$. Then A_0 and A_1 both have rank at least $n-1$. Let A' denote the $(n-1) \times n$ submatrix of M consisting of rows 2 through n, which are linearly independent in $\mathbb{F}_2^n$. The rows of A form a basis for $\mathbb{F}_2^{n-1}$, and there is a unique $w \in \mathbb{F}_2^{n-1}$ for which $w^\top A = v^\top$. Then $w^\top A'$ is either equal to the first row of A_0 or of A_1, and exactly one of A_0 and A_1 has rank $n-1$. The other has rank n, since its first row is not a linear combination of subsequent rows.

2. Suppose that $\text{rank}(A) = n-2$ and that $v^\top$ is not in the rowspace of A. Then, since A is symmetric, v is not in the columnspace of A. Therefore $(v|A)$ is an $(n-1) \times n$ matrix of rank $n-1$. Since $v^\top$ is not in the rowspace of A, it follows that $(d_0|v^\top)$ is not in the rowspace of $(v|A)$ for either choice of d_0. So extending A to either A_0 or A_1 increases the rank from $n-2$ to n.

   If $v^\top$ is in the rowspace of A, then $v^\top$ is a linear combination of the rows of A. Either $(0|v^\top)$ or $(1|v^\top)$ is a linear combination of the rows of the $(n-1) \times n$ matrix $(v|A)$, which has rank $n-2$. Hence, at least one of A_0 and A_1 has rank $n-2$. If both $(0|v^\top)$ and $(1|v^\top)$ are in the rowspace of $(v|A)$, then $e_1^\top = (1|v^\top) - (0|v^\top)$ is in the rowspace of $(v|A)$, where $e_1 \in \mathbb{F}_2^n$. So both A_0 and A_1 have rank $n-2$ if and only if $u^\top(v|A) = e_1^\top$ for some $u \in \mathbb{F}_2^{n-1}$. Since v is in the columnspace of A, any u that satisfies $u^\top A = 0$ also satisfies $u^\top v = 0$. Therefore $e_1^\top$ is not a linear combination of the rows of $(v|A)$, hence one completion has rank $n-2$ and one has rank $n-1$.

3. Suppose that $\text{rank}(A) = n-3$. Let U be the nullspace of A, which has dimension 2. If $v \notin \text{rowspace}(A)$, then both A_0 and A_1 have rank $n-1$. If $v \in \text{rowspace}(A)$, then $v \in U^\perp$. In this case one, of A_0 and A_1 has rank $n-3$ and the other has rank $n-2$. The arguments are similar to those of 2. above.

This completes the proof. □

Theorem 5.0.1 allows us to infer the possible rank distribution of $M(\Gamma)$ over $\mathbb{F}_2$ from the known rank distribution of $M(C_{n-1})$. When the matrix A in Theorem 5.0.1 has full rank $n-1$, adding a new row and column produces two possibilities: one of A_0 and A_1 has rank $n-1$ and the other has rank n. This means the numbers of completions giving rank $n-1$ and rank n are the same, an outcome that depends crucially on the choice of $\mathbb{F}_2$ as the ground field, as in Theorem 3.0.1. Since our goal is to study situations where one rank occurs more often than the other, we do not need to consider this case further. When the matrix A of Theorem 5.0.1 has rank $n-2$, its right nullspace is one-dimensional. Over $\mathbb{F}_2$, this nullspace contains exactly one nonzero vector, which we denote by u. In this case, it is useful to determine whether the vector v is orthogonal to u or not. If $v^\top u = 0$, then $M(\Gamma)$ has two possible completions, one of rank $n-1$ and one of rank $n-2$. If $v^\top u = 1$, then both choices for d_0 result in a rank n completion of $M(\Gamma)$. If the matrix A has rank $n-3$, then its right nullspace U is 2-dimensional. In this case:

1. If $v \in U^\perp$, then $M(\Gamma)$ has exactly one completion of rank $n-2$ and one completion of rank $n-3$.

2. If $v \notin U^\perp$. In this situation, both choices of d_0 yield completions of rank $n-1$.

As a result, we restrict our attention to the completions of $M(\Gamma)$ for which the submatrix of the lower right $(n-1)\times(n-1)$ has rank $n-2$ or rank $n-3$.

For a graph Γ in the class $\mathcal{G}^C$, we write $A(\Gamma)$ and $B(\Gamma)$ respectively for the numbers of matrices of rank n and $n-1$ that represent Γ over $\mathbb{F}_2$ with respect to the order of the vertex of Γ, and for which the lower right $(n-1)\times(n-1)$ submatrix corresponding to the cycle C_{n-1} has rank $n-2$ or $n-3$. From Theorem 5.0.1, it follows that

$$R_n(\Gamma) - R_{n-1}(\Gamma) = A(\Gamma) - B(\Gamma).$$

We let $\alpha(\Gamma) = A(\Gamma) - B(\Gamma)$. The goal of this chapter is to identify those $\Gamma \in \mathcal{G}^C$ for which $\alpha(\Gamma)$ is negative.

5.1 Vectors in the nullspace of completions of $M(C_n)$

We now examine which nonzero column vectors over $\mathbb{F}_2$ can occur in the right nullspace of a matrix representing C_n in $M_n(\mathbb{F}_2)$. Note that, unlike in the case of $M(P_n)$, such a matrix may have nullspace of dimension greater than one.

Definition 5.1.1. *Two entries of a vector are said to be* cyclically consecutive *if they are consecutive or if they are the first and last entries.*

Lemma 5.1.2. *Suppose that $Au = 0$, for a matrix $A \in M_n(\mathbb{F}_2)$ representing C_n and a non-zero column vector $u \in \mathbb{F}_2^n$. Then, no pair of cyclically consecutive entries in u are both zero.*

Proof. We write $d_1, d_2, \dots, d_n$ for the diagonal entries of A. Let u be a vector with entries $u_1, u_2, \dots, u_n$ such that $Au = 0$. Then

- $d_1u_1 + u_2 + u_n = 0$
- $u_{i-1} + d_iu_i + u_{i+1} = 0$, for $2 \leqslant i \leqslant n-1$
- $u_1 + u_{n-1} + d_nu_n = 0$

If $u_1 = u_n = 0$, then from the first of the above equations, it follows that $u_2 = 0$. Applying the second equation to the successive triples (u_{i-1}, u_i, u_{i+1}) from $i = 2$, it follows that $u_i = 0$ for all i. This implies that all entries $u_1, u_2, \dots, u_n$ are zero. However, this contradicts u being a non-zero vector. Therefore, u_n and u_1 are not both zero.

Suppose now that $u_i = u_{i+1} = 0$ for some i with $2 \leqslant i \leqslant n-1$. Then u_{i-1} and u_{i+2} are also equal to zero, since $u_{i-1} + d_iu_i + u_{i+1} = 0$ and $u_i + d_{i+1}u_{i+1} + u_{i+2} = 0$. Repeating this argument, it follows that $u = 0$, and the zero vector is the only vector in the right nullspace of a completion of $M(C_n)$ that has zero entries in cyclically consecutive positions. □

For a positive integer n, we write T_n for the set of vectors in $\mathbb{F}_2^n$ that have no pair of cyclically consecutive entries, both equal to zero. Lemma 5.1.2 shows that every non-zero vector that is in the nullspace of a completion of $M(C_n)$ belongs to T_n.

Remark. *Let $T = T_n \cup \{0\}$. Then T is not a subspace of $\mathbb{F}_2^n$, since the sum of two vectors in T may produce a nonzero vector with cyclically consecutive entries equal to zero, which is not in T. Furthermore, T does not contain any 3-dimensional subspace of $\mathbb{F}_2^n$. If $U \subseteq \mathbb{F}_2^n$ with $\dim U = 3$, then it contains seven nonzero vectors. Looking at any two consecutive entries of a nonzero vector in T, only three patterns are possible, namely $(1,1)$,$(1,0)$, and $(0,1)$. Since the number of vectors is greater than this, two vectors must share the same pattern. The sum of these vectors is a nonzero vector with a consecutive pair of zeros in those positions, which is therefore not in T.*

In the next lemma, we show that T_n is exactly the set of nonzero vectors that occur in the nullspace of some rank $n-1$ completions of $M(C_n)$, and we determine the number of such completions with a particular 1-dimensional nullspace.

In the following statement, we interpret u_0 and u_{n+1} respectively as u_n and u_1.

Theorem 5.1.3. *Let $u \in T_n$ and let $z(u)$ be the number of zero entries in u. Let A be completion of $M(C_n)$ with diagonal entries $d_1, \dots, d_n$. Then $Au = 0$ if and only if $d_i = u_{i-1} + u_{i+1}$ for all i with $u_i = 1$. The number of completions of $M(C_n)$ satisfying $Au = 0$ is $2^{z(u)}$.*

Proof. $Au = 0$ if and only if $u_{i-1} + d_i u_i + u_{i+1} = 0$, for $1 \leqslant i \leqslant n$. If $u_i = 1$, then $d_i = u_{i-1} + u_{i+1}$. If $u_i = 0$, then $u_{i-1} = u_{i+1} = 1$, since u has no cyclically consecutive zero entries. Therefore $u_{i-1} + d_i u_i + u_{i+1} = 0$ is satisfied for either choice of value for d_i. Hence the number of choices for $(d_1, \dots, d_n)$ to ensure $Au = 0$ is $2^{z(u)}$. □

Similarly to Lemmas 3.1.1 and 3.1.2, results analogous to Lemma 5.1.2 and Theorem 5.1.3 exist over $\mathbb{F}_q$ for any prime power q.

To sum up, we have shown that the only nonzero vectors that belong to the nullspace of a completion of $M(C_n)$ are exactly the vectors in T_n. This gives a complete description of the possible nullspace vectors for the matrix representing C_n.

5.2 Completions of $M(C_n)$ and their ranks

We now consider the ranks of the completions of $M(C_n)$ whose nullspace contains a particular $u \in T_n$.

Theorem 5.2.1. *Let $u \in T_n$ with $z(u) \geqslant 1$. Then half of the $2^{z(u)}$ completions of $M(C_n)$ whose nullspace contain u have rank $n-1$ and half have rank $n-2$.*

Proof. We consider how many 2-dimensional subspaces of $\mathbb{F}_2^n$ that are contained in T_n include u. If u is in such a 2-dimensional subspace, then it contains exactly 3 non-zero vectors u, v, and w, where $u + v = w$. Consider the matrix M with u, v, w as columns. Since no column of M has a pair of consecutive zero entries, no row of M consists entirely of zero entries. This is because the rows above and below would then have each entry equal to 1, contradicting $w = u + v$. It follows that every row has a single zero entry and two entries equal to 1. In each of the $z(u)$ positions where u has a zero entry, the corresponding row of M is $(0, 1, 1)$. Between two successive appearances of the row $(0, 1, 1)$, the rows of M alternate between $(1, 0, 1)$ and $(1, 1, 0)$. The matrix M is fully determined by the choice of either $(1, 0, 1)$ or $(1, 1, 0)$ for each row immediately after a row that is equal to $(0, 1, 1)$. Therefore there are $2^{z(u)}$ choices for the matrix M. Swapping the second and third column of M results in a distinct matrix corresponding to the same 2-dimensional subspace. Therefore u is in exactly $2^{z(u)-1}$ subspaces of $\mathbb{F}_2^n$ that are contained in T_n.

Let U be one of these subspaces, and let $\{u, v\}$ be a basis of U. We now show that U is the nullspace of exactly one completion of $M(C_n)$.

We consider the choice of diagonal entries for a completion A of $M(C_n)$ that satisfies $Au = Av = 0$. By Theorem 5.1.3, the equation $Au = 0$ determines the diagonal entries of A in those positions where the entries of u are 1, and imposes no further constraints. Similarly, the equation $Av = 0$ determines the diagonal entries of A in those positions where v has entry 1 in the same position in u. Since there is no position with zero entry in both u and v, it follows that there is at most one completion A of $M(C_n)$ whose nullspace contains U.

It remains to show that in any position where the entries of u and v are both 1, the values of the corresponding diagonal entry determined by $Au = 0$ and by $Av = 0$ coincide. Let i be such a position, so that $u_i = v_i = 1$. By Theorem 5.1.3, the values for the ith diagonal entry d_i of A determined by the equations $Au = 0$ and $Av = 0$ are respectively given by $d_i = u_{i-1} + u_{i+1}$ and $d_i = v_{i-1} + v_{i+1}$.

Since $u+v$ has no cyclically consecutive zero entries, $u_{i-1} \neq v_{i-1}$ and $u_{i+1} \neq v_{i+1}$. It follows that $u_{i-1} + u_{i+1} = v_{i-1} + v_{i+1}$.

We conclude that U is the nullspace of exactly one completion of $M(C_n)$. Therefore u belongs to the nullspace of $2^{z(u)-1}$ completions of $M(C_n)$ of rank $n-2$, and hence to the same number of completions of rank $n-1$.

□

Note that the only vector $u \in \mathbb{F}_2^n$ with $z(u) = 0$ is the all-ones vector $u = (1, 1, \dots, 1)^\top$, which we denote by $\mathbf{j}$. We now consider a completion A of $M(C_n)$ with $\mathbf{j}$ in its nullspace. By Theorem 5.1.3, each diagonal entry of A is $1 + 1 = 0$. Hence, exactly one completion of $M(C_n)$ has $\mathbf{j}$ in its nullspace.

Theorem 5.2.2. *Let* $\mathbf{j} = (1, 1, \dots, 1)^\top \in \mathbb{F}_2^n$ *and let* A *be the unique completion of* $M(C_n)$ *with* $\mathbf{j}$ *in its nullspace. Then* A *has rank* $n-1$ *if* n *is odd and rank* $n-2$ *if* n *is even.*

Proof. We consider whether $\mathbf{j}$ extends to a 2-dimensional subspace contained in T_n. Suppose $\mathbf{j}$ is the sum of a pair of non-zero vectors $v, w \in T_n$. By Lemma 5.1.2, v and w must have no cyclically consecutive zeros. The only way for v and w to sum to $\mathbf{j}$ and avoid cyclically consecutive zeros is for their entries to alternate. Without loss of generality, suppose the first entry of v is 1 and the first of w is 0.

If n is odd, then $w \notin T_n$, since it begins and ends with 0. In this case, no such v and w exist, and the nullspace of A is $\langle \mathbf{j} \rangle$. Hence, A has rank $n-1$ if n is odd.

If n is even, then $w \in T_n$. Both v and w belong to the nullspace of A, since

$v_{i-1} + v_{i+1} = w_{i-1} + w_{i+1} = 0$ for all $i \in \{1, \ldots, n\}$. In this case, the nullspace of A is the 2-dimensional space $\langle \mathbf{j}, v \rangle$. Hence, A has rank $n-2$ if n is even.

□

5.3 The rank distribution of graphs in $\mathcal{G}^C$

For column vectors in $\mathbb{F}_2^n$, we write $\perp$ for the relation of orthogonality with respect to the standard scalar product over $\mathbb{F}_2$. Recall that T_n is the set of vectors in $\mathbb{F}_2^n$ with no pair of cyclically consecutive entries which are both equal to zero.

Theorem 5.3.1. *Let $\Gamma \in \mathcal{G}^C$ have order n, let $v \in \mathbb{F}_2^{n-1}$ consist of the last $n-1$ entries of the first column of $M(\Gamma)$, and $\mathbf{j} \in \mathbb{F}_2^{n-1}$ be the vector with all entries equal to 1. Then*

1. $A(\Gamma) = \gamma + \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \not\perp v}} 2^{z(u)}, \quad \gamma = \begin{cases} 2, & \text{if } n \text{ is even and } \mathbf{j} \not\perp v, \\ 0, & \text{otherwise.} \end{cases}$

2. $B(\Gamma) = \delta + \sum_{u \in T_{n-1} \setminus \{\mathbf{j}\}} 2^{z(u)-1}, \quad \delta = \begin{cases} 1, & \text{if } n \text{ is even and } \mathbf{j} \perp v \\ 1, & \text{if } n \text{ is odd and } \mathbf{j} \not\perp v \\ 0, & \text{otherwise} \end{cases}$

Proof.

1. By definition, $A(\Gamma)$ only counts a completion of rank n of $M(\Gamma)$ if its lower-right $(n-1) \times (n-1)$ submatrix A has rank $n-2$ or $n-3$. For fixed A, Theorem 5.0.1 implies that there are 2 rank n completions with A as lower-right submatrix if $\text{rank}(A) = n-2$ and $v \notin \text{rowspace}(A)$, and that no other completions have rank n. Let $u \in T_{n-1} \setminus \{\mathbf{j}\}$. Theorem 5.2.1 implies that there are $2^{z(u)-1}$ rank $n-2$ completions of $M(C_{n-1})$, and $u \not\perp v$ if and only if $v \notin \text{rowspace}(A)$. Therefore each $u \in T_{n-1} \setminus \{\mathbf{j}\}$ corresponds to $2^{z(u)}$ rank n completions of $M(\Gamma)$ if and only if $u \not\perp v$. Theorem 5.2.2 implies the completion of $M(C_{n-1})$ with $\mathbf{j}$ as its non-zero nullspace vector has rank $n-2$ if and only if n is even. Therefore $\mathbf{j}$ corresponds to two rank n completions of $M(\Gamma)$ if and only if n is even and $\mathbf{j} \not\perp v$.

$$A(\Gamma) = \gamma + \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \not\perp v}} 2^{z(u)}$$

2. By the definition, $B(\Gamma)$ only counts a completion of rank $n-1$ of $M(\Gamma)$ if its lower-right $(n-1) \times (n-1)$ submatrix A has rank $n-2$ or $n-3$. Let $B_1(\Gamma)$ denote the number of completions of $M(\Gamma)$ of rank $n-1$ where $\text{rank}(A) = n-2$ and $B_2(\Gamma)$ denote the number of completions of $M(\Gamma)$ of rank $n-1$ where $\text{rank}(A) = n-3$.

$$B(\Gamma) = B_1(\Gamma) + B_2(\Gamma)$$

To determine $B_1(\Gamma)$, suppose A is a rank $n-2$ completion of $M(C_{n-1})$. $M(\Gamma)$ has 2 completions with A as its lower right $(n-1) \times (n-1)$ submatrix. If $v \notin \text{rowspace}(A)$, both of these have rank n. If $v \in \text{rowspace}(A)$, then Theorem 5.0.1 (2) implies that one completion has rank $n-1$ and the other has rank $n-2$. Note that $B_1(\Gamma)$ is the number of rank $n-2$ completions of $M(C_{n-1})$ whose non-zero nullspace vector is orthogonal to v.

Let $u \in T_{n-1} \setminus \{\mathbf{j}\}$. Theorem 5.2.1 implies that there are $2^{z(u)-1}$ rank $n-2$ completions of $M(C_{n-1})$ and $u \perp v$ if and only if $v \in \text{rowspace}(A)$. Consequently, each $u \in T_{n-1} \setminus \{\mathbf{j}\}$ corresponds to $2^{z(u)-1}$ rank $n-1$ completions of $M(\Gamma)$ if and only if $u \perp v$. Theorem 5.2.2 implies that the completion of $M(C_{n-1})$ with $\mathbf{j}$ as its non-zero nullspace vector has rank $n-2$ if and only if n is even. Thus, $\mathbf{j}$ corresponds to one rank $n-1$ completion of $M(\Gamma)$ if and only if n is even and $\mathbf{j} \perp v$. Let $\delta_1 = 1$ if n is even and $\mathbf{j} \perp v$, and equal 0 otherwise.

$$B_1(\Gamma) = \delta_1 \; + \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \perp v}} 2^{z(u)-1}$$

To determine $B_2(\Gamma)$, suppose A is a rank $n-3$ completion of $M(C_{n-1})$. $M(\Gamma)$ has 2 completions with A as its lower right $(n-1) \times (n-1)$ submatrix. If $v \in \text{rowspace}(A)$, neither of these have rank $n-1$. If $v \notin \text{rowspace}(A)$, then Theorem 5.0.1 (2) implies that both completions have rank $n-1$. Consequently, $B_2(\Gamma)$ is twice the number of rank $n-3$ completions of $M(C_{n-1})$ whose nullspace is not a subset of $v^{\perp}$. Since each 2-dimensional space contained in T_{n-1} is the nullspace of a unique such completion, $B_2(\Gamma)$ is twice the number of 2-dimensional spaces contained in T_{n-1} that are not subsets of $v^{\perp}$.

Let u be a vector in $T_{n-1} \setminus \{\mathbf{j}\}$ for which $u \not\perp v$. By Theorem 5.2.1, the number of 2-dimensional subspaces $U \subset T_{n-1}$ containing u is $2^{z(u)-1}$, and each such U is the nullspace of a unique completion A of $M(C_{n-1})$. Let the other non-zero elements of U be w and $u+w$. Then $v^{\top}u = 1$, and therefore $v^{\top}(u+w) = 1 + v^{\top}w$. So v is orthogonal to exactly one of w or $u+w$, and is therefore not orthogonal to exactly two elements of U. Therefore summing $2^{z(u)-1}$ over $u \in T_{n-1} \setminus \{\mathbf{j}\}$ counts each eligible U exactly twice, with one possible exception: if n is odd and $\mathbf{j} \not\perp v$, then the subspace containing $\mathbf{j}$ is counted only once in this sum. Let $\delta_2 = 1$ if n is odd and $\mathbf{j} \not\perp v$, and equal 0 otherwise.

$$B_2(\Gamma) = \; \delta_2 \;\; + \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \not\perp v}} 2^{z(u)-1}$$

Note that $\delta = \delta_1 + \delta_2$.

□

Recall the following corollary, where $S_{n-1} \subseteq \mathbb{F}_2^{n-1}$ is the set of vectors with no pair of consecutive zero entries, and $z(u)$ is the number of zero entries of the vector u.

Corollary 3.1.5. *For every positive integer* n.

$$F(n) = \sum_{u \in S_{n-1}} 2^{z(u)}.$$

Recall from Lemma 3.1.4 that U_{n+1} is the set of vectors in $\mathbb{F}_2^{n+1}$ with no consecutive zero entries whose first and last entries are equal to 1. By observation, S_{n-1} is in bijective correspondence with U_{n+1}, via a correspondence that deletes the first and last entry (both 1) from an element of U_{n+1}, or (in the other direction) appends a 1 at the first and last entry of an element of S_{n-1}. This correspondence preserves the number of zero entries, so

$$F(n) = \sum_{u \in U_{n+1}} 2^{z(u)} = \sum_{u \in S_{n-1}} 2^{z(u)} = \tfrac{1}{3}\big(2^{n+1} + (-1)^n\big).$$

Lemma 5.3.2. *Let* Γ *be a graph of order* n *in* $\mathcal{G}^C$*. Then*

$$B(\Gamma) = 2^{n-2} + \frac{1}{2}(-1)^{n-1} + \frac{1}{2}(-1)^{n+d},$$

where d *is the degree of the extra vertex of* Γ *that does not belong to the long cycle.*

Proof. By Theorem 5.3.1, we have the following.

$$B(\Gamma) = \delta + \sum_{u \in T_{n-1} \setminus \{\mathbf{j}\}} 2^{z(u)-1}, \quad \delta = \begin{cases} 1, & \text{if } n \text{ is even and } \mathbf{j} \perp v \\ 1, & \text{if } n \text{ is odd and } \mathbf{j} \not\perp v \\ 0, & \text{otherwise} \end{cases}$$

Recall S_n is the set of vectors in $\mathbb{F}_2^n$ with no consecutive zero entries. Note that $T_{n-1} \subset S_{n-1}$, where T_n is the set of vectors in $\mathbb{F}_2^n$ with no cyclically consecutive zeros. Any vector $u \in S_{n-1} \setminus T_{n-1}$ has 0 as the first and last entries, and 1 as the second and second-last entries. Deleting all four of these entries in u results in a vector $u' \in S_{n-5}$. Furthermore, every $u' \in S_{n-5}$ can be extended uniquely to give a vector $u \in S_{n-1} \setminus T_{n-1}$, and $z(u) = z(u') + 2$. By Corollary 3.1.5, we have

$$\sum_{u \in T_{n-1}} 2^{z(u)-1} = \sum_{u \in S_{n-1}} 2^{z(u)-1} - 4 \sum_{u \in S_{n-5}} 2^{z(u)-1} = \frac{1}{2}(F(n) - 4F(n-4)).$$

Returning to the expression for $B(\Gamma)$, we have

$$\begin{aligned} B(\Gamma) &= \delta + \sum_{u \in T_{n-1} \setminus \{\mathbf{j}\}} 2^{z(u)-1} \\ &= (\delta - 2^{z(\mathbf{j})-1}) + \frac{1}{2}F(n) - 2F(n-4) \\ &= (\delta - \frac{1}{2}) + \frac{1}{2} \cdot \frac{1}{3}\left(2^{n+1} + (-1)^n\right) - 2 \cdot \frac{1}{3}\left(2^{n-3} + (-1)^{n-4}\right) \\ &= (\delta - \frac{1}{2}) + 2^{n-2} + \frac{1}{2}(-1)^{n-1} \end{aligned}$$

Since

$$\delta - \frac{1}{2} = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is even and } \mathbf{j} \perp v \\ \frac{1}{2}, & \text{if } n \text{ is odd and } \mathbf{j} \not\perp v \\ \frac{-1}{2}, & \text{otherwise} \end{cases},$$

the term $(\delta - \frac{1}{2})$ simplifies to $\frac{1}{2}(-1)^{n+d}$. Substituting this into the previous expression gives

$$B(\Gamma) = 2^{n-2} + \frac{1}{2}(-1)^{n-1} + \frac{1}{2}(-1)^{n+d}$$

□

So, given a graph $\Gamma \in \mathcal{G}_d^C$ of order n, the value of $B(\Gamma)$ depends on whether d is even or odd:

- If d is even, the last two terms cancel, and we obtain

$$B(\Gamma) = 2^{n-2}.$$

- If d is odd, then

$$B(\Gamma) = \begin{cases} 2^{n-2} + 1, & \text{if } n \text{ is odd,} \\ 2^{n-2} - 1, & \text{if } n \text{ is even.} \end{cases}$$

In particular, $B(\Gamma)$ does not depend on the structure of Γ: it is determined only by n and d.

Recall $\alpha(\Gamma) = A(\Gamma) - B(\Gamma)$ counts the difference in the number of completions of rank n and $n-1$ representing a graph $\Gamma \in \mathcal{G}_d^C$ of order n over $\mathbb{F}_2$. From Theorem 5.3.1, we derive the following expression for $\alpha(\Gamma)$.

$$\begin{aligned} \alpha(\Gamma) &= A(\Gamma) - B(\Gamma) \\ &= \gamma + \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \not\perp v}} 2^{z(u)} - \left(\delta + \sum_{u \in T_{n-1} \setminus \{\mathbf{j}\}} 2^{z(u)-1} \right) \\ &= (\gamma - \delta) + \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \not\perp v}} \left(2^{z(u)} - 2^{z(u)-1} \right) - \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \perp v}} 2^{z(u)-1} \end{aligned}$$

This gives us the following expression for $\alpha(\Gamma)$.

$$\alpha(\Gamma) = (\gamma - \delta) + \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \not\perp v}} 2^{z(u)-1} - \sum_{\substack{u \in T_{n-1} \setminus \{\mathbf{j}\} \\ u \perp v}} 2^{z(u)-1} \tag{5.1}$$

$$\text{where } \gamma - \delta = \begin{cases} 2, & \text{if } n \text{ even and } \mathbf{j} \not\perp v \\ -1, & \text{if } (n \text{ even and } \mathbf{j} \perp v) \text{ or } (n \text{ odd and } \mathbf{j} \not\perp v) \\ 0, & \text{if } n \text{ odd and } \mathbf{j} \perp v \end{cases}$$

The results in this section explain how the vectors in T_{n-1} determine the number of completions of ranks n, $n-1$, and $n-2$. Moreover, $A(\Gamma)$ depends on the specific choice of ν, while $B(\Gamma)$ is essentially independent of the structure of Γ, depending only on n. The expression for $\alpha(\Gamma)$ captures how the counts of different ranks are affected by ν.

5.4 Investigating the sign of α for graphs in $\mathcal{G}^C$

In this section, we analyze which Γ in certain subclasses of $\mathcal{G}^C$ satisfy $\alpha(\Gamma) < 0$. In the case of graphs with a long induced cycle, we do not have a step-by-step method to move from one degree to the next as we had in Chapter 3. Instead, we consider a family of examples that highlight an important difference between the behaviour of the parameter α in the classes $\mathcal{G}^P$ and $\mathcal{G}^C$. We write $\mathcal{G}^C_d$ for the class of graphs in $\mathcal{G}^C$ in which the vertex that does not belong to the induced cycle has degree d. If $\Gamma \in \mathcal{G}^P_d$ and $\alpha(\Gamma) < 0$, then the maximum possible degree of a vertex in Γ is 5, by Theorem 3.3.12. In this section we show that there is no upper bound on the maximum degree of a vertex in a graph $\Gamma \in \mathcal{G}^C_d$ with $\alpha(\Gamma) < 0$.

The first three subsections of this section analyze the cases where the extra vertex x has degree 1, 2, or 3. Subsection 5.4.4 presents the main result of the section, Theorem 5.4.8, which describes an infinite family of graphs in $\mathcal{G}^C$ containing a graph with extra vertex of degree d for any $d \geqslant 3$ and with negative α.

For $d \geqslant 1$ and positive integers $t_1, t_2, \ldots, t_d$, we define $\Gamma(t_1, t_2, \ldots, t_d)$ to be the graph in $\mathcal{G}^C_d$ constructed as follows. Let C be a cycle on $t_1 + t_2 + \ldots + t_d$ vertices, and let x be the extra vertex not in C. The neighbours of the extra vertex x divide the cycle into d intervals of length $t_1, t_2, \ldots, t_d$, each containing at least one edge. Without loss of generality, let x_1 be the first neighbour of x. Then there are t_1 edges between x_1 and the second neighbour of x, then t_2 edges until the third neighbour of x, and so on around the cycle.

In this section, we aim to determine a condition solely in terms of $t_1, t_2, \ldots, t_d$ for whether $\alpha(\Gamma)$ is positive or negative. We find expressions for $A(\Gamma)$ and $B(\Gamma)$ in terms of the parameters t_i and the function F. However, it is not always possible to decide the sign of $\alpha(\Gamma)$ from these expressions. We present complete results in the cases where $\deg(x) \in \{1, 2, 3\}$. These cases will be the focus of the following subsections.

5.4.1 Degree 1

In this subsection, we study the class $\mathcal{G}^C_1$ consisting of all graphs in $\mathcal{G}^C$ where the vertex x not belonging to the long induced cycle has degree 1. For $t \geqslant 3$, we consider graphs of the form $\Gamma(t)$. Note that $\Gamma(t)$ is the unique graph in $\mathcal{G}^C_1$ of order $n = t + 1$. Its vertex set is $\{x, x_1, x_2, \ldots, x_t\}$, and its edges are xx_1 together with the cycle edges $x_1x_2, x_2x_3, \ldots$, and x_tx_1 (see Figure 5.2).

Let $v \in \mathbb{F}_2^t$ describe the incidences at the vertex x on the cycle C_t. Here, we have $v = (1, 0, \dots, 0)$. Let T_t be the set of vectors with no cyclically consecutive zeros defined in Lemma 5.1.2. We write $A(t)$, $B(t)$, and $\alpha(t)$ respectively for $A(\Gamma(t))$, $B(\Gamma(t))$, and $\alpha(\Gamma(t))$.

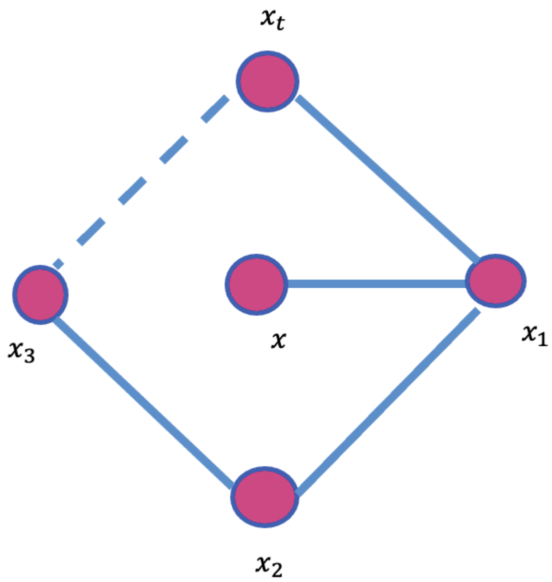


Figure 5.2: $\Gamma(t)$

Recall that $F(n)$ is defined in Definition 2.0.3 by

$$F(n) = \frac{1}{3}\left(2^{n+1} + (-1)^n\right).$$

In the next Theorem 5.4.1, we use $F(n)$ to determine $A(t)$ and $B(t)$.

Theorem 5.4.1. *Let $t \geqslant 3$ be a positive integer.*

1. $A(t) = \frac{2}{3}\left(2^t - (-1)^t\right)$
2. $B(t) = 2^{t-1} + (-1)^t$

Proof. 1. By Theorem 5.3.1,

$$A(t) = \gamma + \sum_{\substack{u \in T_t \setminus \{j\} \\ u \not\leq v}} 2^{z(u)}$$

where $z(u)$ denotes the number of zero entries in u, and γ is as follows.

$$\gamma = \begin{cases} 2, & \text{if } t \text{ is odd} \\ 0, & \text{if } t \text{ is even} \end{cases}$$

For any vector $u = (u_1, \dots, u_t) \in T_t$,

$$u \perp v \iff u_1 = 0 \quad \text{and} \quad u \not\perp v \iff u_1 = 1.$$

Hence, the vectors in T_t that are not orthogonal to v are exactly those with first entry equal to 1. Equivalently, such vectors can be written as

$$u = (1, u'), \quad \text{where } u' \in S_{t-1}.$$

Then by Corollary 3.1.5,

$$\sum_{\substack{u \in T_t \\ u \not\perp v}} 2^{z(u)} = \sum_{u' \in S_{t-1}} 2^{z(u')} = F(t).$$

Since the $\mathbf{j} = (1, \dots, 1)$ is excluded from the sum in the expression for $A(t)$,

$$\sum_{\substack{u \in T_t \setminus \{\mathbf{j}\} \\ u \not\perp v}} 2^{z(u)} = F(t) - 1.$$

Hence,

$$A(t) = \gamma + F(t) - 1 = \begin{cases} F(t) + 1, & \text{if t is odd} \\ F(t) - 1, & \text{if t is even} \end{cases}$$

This can be written as follows.

$$A(t) = F(t) + (-1)^{t+1} = \frac{1}{3}\left(2^{t+1} + (-1)^t + 3(-1)^{t+1}\right)$$

2. Since $d = \deg(x) = 1$ and $t = n - 1$, Lemma 5.3.2 implies

$$\begin{aligned} B(t) &= 2^{(t+1)-2} + \tfrac{1}{2}(-1)^{(t+1)-1} + \tfrac{1}{2}(-1)^{(t+1)+1} \\ &= 2^{t-1} + \tfrac{1}{2}(-1)^t + \tfrac{1}{2}(-1)^t \\ &= 2^{t-1} + (-1)^t. \end{aligned}$$

□

Corollary 5.4.2. *Let* $t \geqslant 3$. *Then* $\alpha(t) > 0$.

Proof.

$$\begin{aligned} \alpha(t) &= A(t) - B(t) \\ &= \frac{2}{3}\left(2^t - (-1)^{t+1}\right) - \left(2^{t-1} + (-1)^t\right) \\ &= \frac{1}{3}\left(4 \cdot 2^{t-1} + 2(-1)^{t+1} - 3 \cdot 2^{t-1} + 3(-1)^{t+1}\right) \\ &= \frac{1}{3}\left(2^{t-1} + 5(-1)^{t-1}\right), \end{aligned}$$

which is clearly positive for all $t \geqslant 3$.

□

5.4.2 Degree 2

In this subsection, we study the class $\mathcal{G}_2^C$ of graphs in $\mathcal{G}^C$ where the vertex x not belonging to the long induced cycle has degree 2.

For positive integers $s \leqslant t$, we consider the unique graph $\Gamma(s,t)$ in $\mathcal{G}_2^C$ for which the neighbours of x have paths of lengths s and t between them in the long induced cycle. The vertex set of $\Gamma(s,t)$ is $\{x, x_1, x_2, \dots, x_{s+t}\}$, and its edge set is $\{xx_1, xx_{s+1}\} \cup \{x_1x_2, x_2x_3, \dots, x_{s+t-1}x_{s+t}, x_{s+t}x_1\}$ (see Figure 5.3).

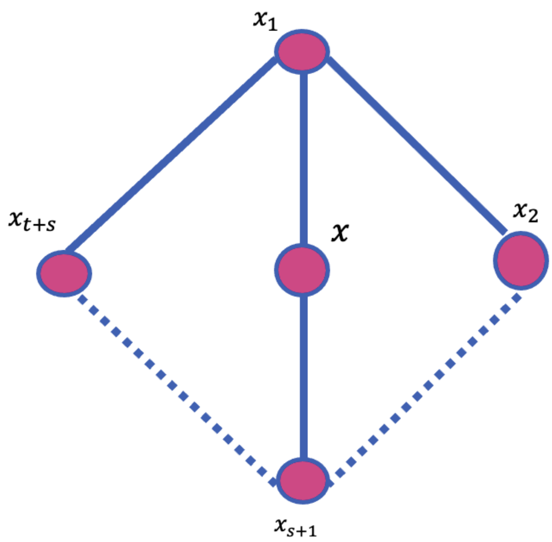


Figure 5.3: $\Gamma(s,t)$

Let $v \in \mathbb{F}_2^{s+t}$ describe the incidences at the vertex x on the cycle C_{s+t}. Here, we have $v = (1,0,\dots,0,1,0,\dots,0)$, where the $v_1 = v_{s+1} = 1$. Let T_{s+t} be the set of vectors with no cyclically consecutive zeros defined in Lemma 5.1.2. We write $A(s,t)$, $B(s,t)$, and $\alpha(s,t)$ respectively for $A(\Gamma(s,t))$, $B(\Gamma(s,t))$, and $\alpha(\Gamma(s,t))$.

Theorem 5.4.3. *Let $s \leqslant t$ be positive integers. Then*

1. $A(s,t) = \frac{4}{9}\left(2^{s+t} - 2^s(-1)^t - 2^t(-1)^s + (-1)^{s+t}\right)$.
2. $B(s,t) = 2^{s+t-1}$.

Proof. Let $v \in \mathbb{F}_2^{s+t}$ be the vector with $v_1 = v_{s+1} = 1$ and $v_i = 0$ otherwise.

By Theorem 5.3.1, it follows that.

$$A(s,t) = \gamma + \sum_{\substack{u \in T_{s+t} \setminus \{\mathbf{j}\} \\ u \not\leq v}} 2^{z(u)}, \quad \gamma = \begin{cases} 2, & \text{if } s+t+1 \text{ is even and } \mathbf{j} \not\leq v, \\ 0, & \text{otherwise.} \end{cases}$$

1. For any vector $u = (u_1, \dots u_{s+t}) \in T_{s+t}$,

$$u \not\perp v \iff (u_1, u_{s+1}) \in \{(1,0),(0,1)\}$$

Hence, the vectors in T_{s+t} that are not orthogonal to v are exactly those for which $u_1 \neq u_{s+1}$. Equivalently, such vectors can be written as

$$u = (1, u', 1, 0, 1, u'') \quad \text{or} \quad u = (0, 1, u', 1, u'', 1)$$

where $u' \in S_{s-2}$ and $u'' \in S_{t-2}$. Note that $2^{z(u)} = 2 \cdot 2^{z(u')} \cdot 2^{z(u'')}$. Then, by Corollary 3.1.5, we have

$$\begin{aligned}\sum_{\substack{u \in T_{s+t} \\ u \not\perp v}} 2^{z(u)} &= 2\left(\sum_{u' \in S_{s-2}} 2^{z(u')}\right)\left(\sum_{u'' \in S_{t-2}} 2^{z(u'')}\right) + 2\left(\sum_{u' \in S_{s-2}} 2^{z(u')}\right)\left(\sum_{u'' \in S_{t-2}} 2^{z(u'')}\right) \\ &= 4F(s-1)F(t-1).\end{aligned}$$

Since $\mathbf{j} \perp v$, the above expression for $A(s,t)$ is the same whether the sum includes $\mathbf{j}$ or not, and $\gamma = 0$. Consequently, we can express $A(s,t)$ as follows.

$$\begin{aligned}A(s,t) &= 4F(s-1)F(t-1) \\ &= 4 \cdot \frac{1}{3}\left(2^s - (-1)^s\right) \cdot \frac{1}{3}\left(2^t - (-1)^t\right) \\ &= \frac{4}{9}\left(2^{s+t} - 2^s(-1)^t - 2^t(-1)^s + (-1)^{s+t}\right)\end{aligned}$$

2. Since $d = 2$ and $n = s + t + 1$, Lemma 5.3.2 implies $B(s,t) = 2^{n-2} = 2^{t+s-1}$.

□

Theorem 5.4.4. *Let $s \leqslant t$ be positive integers. Then $\alpha(s,t)$ is positive if and only if $s = 1$ and $t \neq 2$, or $s = 3$ and t is odd.*

Proof. By Theorem 5.4.3, we have

$$A(s,t) = \frac{4}{9}\left(2^{s+t} - 2^s(-1)^t - 2^t(-1)^s + (-1)^{s+t}\right), \quad \text{and} \quad B(s,t) = 2^{s+t-1}.$$

We can determine $\alpha(s,t)$ as follows.

$$\begin{aligned}\alpha(s,t) &= \frac{4}{9}\left(2^{s+t} - 2^s(-1)^t - 2^t(-1)^s + (-1)^{s+t}\right) - \left(2^{s+t-1}\right) \\ &= \frac{4}{9}\left(2^s(-1)^{t+1} + 2^t(-1)^{s+1} + (-1)^{s+t} - 2^{t+s-3}\right)\end{aligned}$$

1. If $s = 1$:

$$\begin{aligned}\alpha(1,t) &= \frac{4}{9}\left(2(-1)^{t+1} + 2^t + (-1)^{t+1} - 2^{t-2}\right) \\ &= \frac{4}{9}\left(3 \cdot 2^{t-2} + 3(-1)^{t+1}\right) = \frac{4}{3}\left(2^{t-2} + (-1)^{t+1}\right)\end{aligned}$$

If t is odd, then $\alpha(1,t) > 0$. If t is even, then $\alpha(1,t)$ is positive for all $t \neq 2$, since the 2^{t-2} term dominates. For $t = 2$, however, $\alpha(1,t) = 0$.

2. If $s = 2$:

$$\alpha(2,t) = \frac{4}{9}\left(4(-1)^{t+1} - 2^t + (-1)^t - 2^{t-1}\right)$$
$$= \frac{4}{9}\left(3(-1)^{t+1} - 3 \cdot 2^{t-1}\right) = \frac{4}{3}\left((-1)^{t+1} - 2^{t-1}\right)$$

Hence $\alpha(2,1) = 0$ and $\alpha(2,t) < 0$ for all $t \geqslant 2$.

3. If $s = 3$:

$$\alpha(3,t) = \frac{4}{9}\left(8(-1)^{t+1} + 2^t + (-1)^{t+1} - 2^t\right) = 4(-1)^{t+1}$$

Therefore $\alpha(3,t) < 0$ when t is even, and $\alpha(3,t) > 0$ when t is odd.

4. If $s \geqslant 4$.

First, suppose $t = s = 4$. Then $\alpha(s,t) = \frac{4}{9}(1 - 16 - 16 - 32) < 0$.

Now suppose $t \geqslant 5$ and $s \geqslant 4$. Then

$$\alpha(s,t) \leqslant \frac{4}{9}\left(2^s + 2^t + 1 - 2 \cdot 2^{t+s-4}\right)$$
$$= \frac{4}{9}\left(2^s - 2^{t+s-4} + 2^t - 2^{t+s-4} + 1\right)$$
$$\leqslant \frac{4}{9}\left(2^s - 2^{5+s-4} + 2^t - 2^{t+4-4} + 1\right) \leqslant \frac{4}{9}\left(1 - 2^s\right) < 0.$$

□

Remark. *There exist graphs* $\Gamma(s,t)$ *with* $\alpha(s,t) < 0$ *or* $\alpha(s,t) > 0$ *when* $n \equiv 2$ *or* $3 \pmod 4$.

Example 5.4.5. Let Γ_1 and Γ_2 be the two graphs in $\mathcal{G}_2^C$ of order 6 in Figure 5.4.

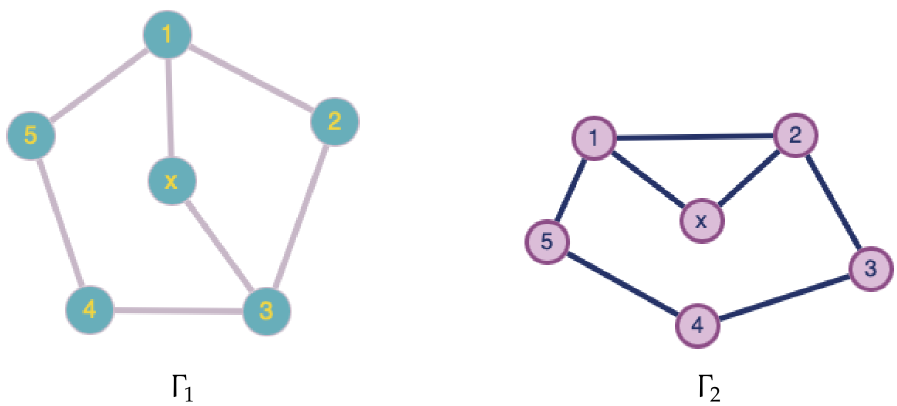


Figure 5.4: Two graphs of order 6, each containing an induced C_5.

Recall the rank distribution of the cycle C_5 over $\mathbb{F}_2$, given in Table 5.1.

Rank	3	4	5
# $\mathbb{F}_2$-completions for C_5	5	16	11

Table 5.1: Rank distribution of $\mathbb{F}_2$-completions of $M(C_5)$.

Table 5.2 shows the rank distributions of Γ_1 and Γ_2. Recall that for a graph $\Gamma \in \mathcal{G}^C$ of order n, each completion A of $M(C_{n-1})$ extends to two completions of $M(\Gamma)$. The relationship between the rank of A and the ranks of the completions of $M(\Gamma)$ is described in Theorem 5.0.1. The colours in the tables keep track of which matrices of each rank representing C_5 contribute to the counts for matrices representing Γ_1 and Γ_2.

Rank	3	4	5	6
# $\mathbb{F}_2$-completions for Γ_1	2	10+2=12	11+10+6=27	11 + 12 =23
# $\mathbb{F}_2$-completions for Γ_2	0	6	11+6 + 10=27	11 +20=31

Table 5.2: Rank distributions of $\mathbb{F}_2$-completions of $M(\Gamma_1)$ and $M(\Gamma_2)$.

We see (written in orange) that each rank n matrix representing C_5 corresponds to a matrix of rank $n-1$ and a matrix of rank n for both graphs. The other values (written in teal and pink) correspond to $A(\Gamma)$ and $B(\Gamma)$. In particular, note that $B(\Gamma) = 2^{s+t-1} = 2^4$ for both $\Gamma = \Gamma_1$ and $\Gamma = \Gamma_2$, as per Theorem 5.4.3. The behaviour of the ranks of the remaining matrices (those not in orange and not corresponding to rank$(\Gamma) = 5$) is what determines the sign of $\alpha(s,t)$.

5.4.3 Degree 3

We now study the class $\mathcal{G}_3^C$, consisting of all graphs in $\mathcal{G}^C$ where the unique vertex not belonging to the long induced cycle has degree 3.

Let r, s, t be positive integers with $r \leqslant s \leqslant t$. Recall that $\Gamma(r,s,t)$ is the graph in $\mathcal{G}_3^C$ consisting of a cycle $(x_1 x_2 \dots x_{r+s+t} x_1)$ and an additional vertex x with neighbours x_1, x_{r+1}, and x_{r+t+1} (see Figure 5.5).

Let $\nu \in \mathbb{F}_2^{r+s+t}$ be the vector with entries equal to 1 in positions $1, r+1, r+s+1$ and equal to 0 in all other positions. Thus ν describes the incidences at the vertex x on the cycle C_{r+s+t}.

Let T_{r+s+t} be the set of vectors in $\mathbb{F}_2^{r+s+t}$ with no cyclically consecutive zero entries, as in Lemma 5.1.2. We write $A(r,s,t)$ and $B(r,s,t)$ respectively for $A(\Gamma(r,s,t))$ and $B(\Gamma(r,s,t))$, where $\Gamma(r,s,t)$ is the graph in $\mathcal{G}_3^C$ whose long induced cycle has $r+s+t$ vertices.

In the following Theorem, we find the form of $A(r,s,t)$ and $B(r,s,t)$.

Theorem 5.4.6. *Let r, s and t be positive integers with $r \leqslant s \leqslant t$. Then*

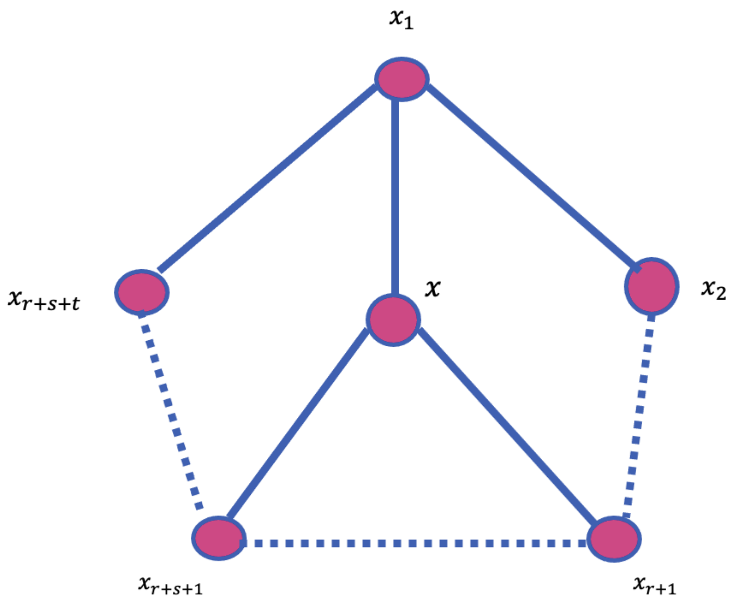


Figure 5.5: $\Gamma(r,s,t)$

1. $A(r,s,t) = F(r)F(s)F(t) + 4F(r-1)F(s-2)F(t-1) + 4F(r-1)F(s-1)F(t-2) + 4\,F(r-2)F(s-1)F(t-1) + (-1)^{r+s+t+1}$

2. $B(r,s,t) = 2^{r+s+t-1} + (-1)^{r+s+t}$

Proof. 1. Let $r,s,t \geqslant 1$, and let

$$v = (1,0,\dots,0,1,0,\dots,0,1,0,\dots,0) \in \mathbb{F}_2^{r+s+t}$$

where the entries at positions 1, $r+1$, and $r+s+1$ are 1. These correspond respectively to the vertices x_1, x_{r+1}, and x_{r+s+1} of the cycle C_{r+s+t}.

For any vector $u = (u_1,\dots,u_{r+s+t}) \in T_{r+s+t}$, the inner product over $\mathbb{F}_2$ satisfies

$$u \cdot v = u_1 + u_{r+1} + u_{r+s+1}.$$

Then

$$u \not\perp v \iff (u_1, u_{r+1}, u_{r+s+1}) \in \{(1,1,1),(1,0,0),(0,1,0),(0,0,1)\}.$$

We write T_1, T_2, T_3 and T_4 for the subsets of T_{r+s+t} consisting of those vectors whose triples of entries in positions $1, r+1$ and $r+s+1$ are respectively $(1,1,1),(1,0,0),(0,1,0)$ and $(0,0,1)$. By Theorem 5.3.1

$$A(r,s,t) = \sum_{\substack{u\in T_{r+s+t}\\ u\not\perp v}} 2^{z(u)} = \sum_{i=1}^{4}\left(\sum_{u\in T_i} 2^{z(u)}\right) + (-1)^{r+s+t+1}.$$

The terms $(-1)^{r+s+t+1}$ arises from the term γ in Theorem 5.3.1 and the fact that the sum here is taken over all $u \not\perp v$ in T_{n-1}, including $\mathbf{j}$.

An element of T_1 is determined by independently choosing elements of S_{r-1}, S_{s-1} and S_{t-1}, respectively for the strings from position 2 to position r, from position $r+2$ to position $r+s$, and from position $r+s+2$ to position $r+s+t$. Hence

$$\sum_{u \in T_1} 2^{z(u)} = \sum_{u' \in S_{r-1}} 2^{z(u')} \cdot \sum_{u'' \in S_{s-1}} 2^{z(u'')} \cdot \sum_{u''' \in S_{t-1}} 2^{z(u''')}.$$

By Corollary 3.1.5, this expression is equal to $F(r)F(s)F(t)$.

Every element of T_2 has the entry 1 in positions $r, r+2, r+s$ and $r+s+2$. An element of T_2 is determined by independently choosing elements of S_{r-2}, S_{s-3} and S_{t-2}, respectively for the strings from position 2 to position $r-1$, from position $r+3$ to position $r+s-1$ and from position $r+s+3$ to position $r+s+t$. Since an element of T_1 has zero entries in positions $r+1$ and $r+s+1$, we have

$$\begin{aligned} \sum_{u \in T_2} 2^{z(u)} &= 4 \sum_{u' \in S_{r-2}} 2^{z(u')} \cdot \sum_{u'' \in S_{s-3}} 2^{z(u'')} \cdot \sum_{u''' \in S_{t-2}} 2^{z(u''')} \\ &= 4F(r-1)F(s-2)F(t-1), \end{aligned}$$

again applying Corollary 3.1.5. By similar arguments,

$$\sum_{u \in T_3} 2^{z(u)} = 4F(r-1)F(s-1)F(t-2), \quad \text{and} \quad \sum_{u \in T_4} 2^{z(u)} = 4F(r-2)F(s-1)F(t-1).$$

This completes the proof of 1.

2. We use Lemma 5.3.2 to determine $B(r, s, t)$ in degree 3 as follows.

$$\begin{aligned} B(r,s,t) &= 2^{r+s+t-1} + \tfrac{1}{2}(-1)^{r+s+t} + \tfrac{1}{2}(-1)^{r+s+t} \\ &= 2^{r+s+t-1} + (-1)^{r+s+t} \end{aligned}$$

□

We write $\alpha(r, s, t)$ for $\alpha(\Gamma(r, s, t))$. We now use the expression of $\alpha(r, s, t)$ in the following theorem to determine the sign of α.

Theorem 5.4.7. *Let r, s, t be positive integers with $r \leqslant s \leqslant t$. Then $\alpha(r, s, t) < 0$ if and only if one of the following holds.*

1. $r = 1$ *and* $s \neq 2$.
2. $r = 1, s = 2$ *and* t *is odd.*
3. $r = 3$ *and* s *is odd.*

Proof. We note that $\alpha(r,s,t) > 0$ if and only if $A(r,s,t) > B(r,s,t)$, i.e. $A(r,s,t) > 2^{r+s+t-1} + (-1)^{r+s+t}$. Expanding the expression for $A(r,s,t)$ in Theorem 5.4.6, we have

$$\begin{aligned} A(r,s,t) &= \frac{28}{27}2^{r+s+t-1} + \frac{(-1)^t}{27}2^{r+s+2} + \frac{(-1)^s}{27}2^{r+t+2} + \frac{(-1)^r}{27}2^{s+t+2} \\ &\quad + \frac{(-1)^{s+t+1}}{27}2^{r+2} + \frac{(-1)^{r+t+1}}{27}2^{s+2} + \frac{(-1)^{r+s+1}}{27}2^{t+2} - \frac{14}{27}(-1)^{r+s+t} \end{aligned}$$

Subtracting $B(r,s,t) = 2^{r+s+t-1} + (-1)^{r+s+t}$ from the above expression (and multiplying by 27), we observe that the sign of $\alpha(r,s,t)$ is the same as that of

$$\begin{aligned} \omega(r,s,t) &:= 2^{r+s+t-1} + (-1)^t 2^{r+s+2} + (-1)^s 2^{r+t+2} + (-1)^r 2^{s+t+2} \\ &\quad + (-1)^{s+t+1}2^{r+2} + (-1)^{r+t+1}2^{s+2} + (-1)^{r+s+1}2^{t+2} + 41(-1)^{r+s+t+1} \end{aligned} \tag{5.2}$$

Step 1 Our first step is to show that $\omega(r,s,t)$, hence $\alpha(r,s,t)$, is positive if $r \geqslant 4$. First suppose $r = 4$.

$$\omega(4,s,t) = 3 \times 2^{t+s+2} + 60(-1)^t 2^s + 60(-1)^s 2^t + 105(-1)^{t+s+1}.$$

We write $3 \times 2^{t+s+2} = 2^{t+s+2} + 2^{t+s+2} + 2^{t+s+2}$. Since $s \geqslant 4$ and $t \geqslant 4$, we have $2^{t+s+2} \geqslant 64 \times 2^t > 60 \times 2^t$, $2^{t+s+2} \geqslant 64 \times 2^s > 60 \times 2^s$ and $2^{t+s+2} > 105$. Hence $\omega(4,s,t)$ and $\alpha(4,s,t)$ are positive for all $t \geqslant s \geqslant 4$.

Now suppose $r \geqslant 5$. Then r, s, t are all at least 5, and it follows that the expressions $2^{r+s+2}, 2^{r+t+2}$ and 2^{s+t+2} are all bounded above by $\frac{1}{4}2^{r+s+t-1}$. Similarly, the expressions 2^{r+2}, 2^{s+2} and 2^{t+2} are all bounded above by $\frac{1}{2^7}2^{r+s+t-1}$. It follows that for $r \geqslant 5$,

$$\omega(r,s,t) \geqslant 2^{r+s+t-1} - \frac{3}{4}2^{r+s+t-1} - \frac{3}{128}2^{r+s+t-1} - 41,$$

which is certainly positive for $t \geqslant s \geqslant r \geqslant 5$. This completes Step 1.

We proceed to consider the cases $r = 1, r = 2$ and $r = 3$.

Step 2 Let $r = 1$. From 5.2 it follows that

$$\omega(1,s,t) = -3 \times 2^{s+t} + 12(-1)^s 2^t + 12(-1)^t 2^s + 33(-1)^{t+s}.$$

Since $2^t \geqslant 2^s$, $\omega(1,s,t)$ is clearly negative if s is odd. Suppose that s is *even* and write $s = 2k$. Then

$$\omega(1,2k,t) = -3 \times 2^{2k+t} + 12 \times 2^t + 12(-1)^t 2^{2k} + 33(-1)^t.$$

If $k > 1$, then $t \geqslant 4$, $2^{2k} \geqslant 16$ and

$$2^{2k+t} \geqslant 16 \times 2^t = 12 \times 2^t + 4 \times 2^t > 12 \times 2^t + 33.$$

Also if $k > 1$, then $t \geqslant 4$ and $2 \times 2^{2k+t} > 2^{t+1}2^{2k} > 12 \times 2^{2k}$. It follows that

$$\omega(1, 2k, t) = \underbrace{-2 \times 2^{2k+t} + 12(-1)^t 2^{2k}}_{<0} + \underbrace{(-2^{2k+t} + 12 \times 2^t + 33(-1)^t}_{<0}.$$

Hence $\omega(1, s, t) < 0$ provided that $s \neq 2$.

Finally we consider $\omega(1, 2, t)$ which is an unusual case: $\omega(1, 2, t) = 81(-1)^t$, which is negative if t is odd and positive when t is even. It follows that $\alpha(1, 2, t) = 3$ for all even values of $t \geqslant 2$ and $\alpha(1, 2, t) = -3$ for all odd values of $t > 2$. This completes Step 2 which includes items 1. and 2. in the statement of the theorem.

Step 3 Let $r = 2$. From 5.2 we note that

$$\omega(2, s, t) = 6 \times 2^{s+t} + 12(-1)^t 2^s + 12(-1)^s 2^t - 57(-1)^{t+s}.$$

It is clear that $\omega(2, 2, t) = 36 \times 2^t + 9(-1)^{t+1}$ is positive for all $t \geqslant 2$.
Suppose $s \geqslant 3$. Then $t \geqslant 3$ and $2^{s+t+1} \geqslant 16 \times 2^s$. Also $2^{s+t+1} \geqslant 16 \times 2^t$, and $2^{s+t+1} > 57$. It follows that $6 \times 2^{s+t} > 12 \times 2^s + 12 \times 2^t + 30$ and hence $\alpha(2, s, t) > 0$ for all $t \geqslant s \geqslant 2$.

Step 4 Let $r = 3$. From 5.2 again, we find that $\omega(3, s, t)$ has an exceptionally simple form.

$$\omega(3, s, t) = 36(-1)^t 2^s + 36(-1)^s 2^t + 9(-1)^{s+t}.$$

Suppose first that $t = s$. Then $\omega(3, s, s) = 72(-1)^s 2^s + 9$, so $\omega(3, s, s)$ is negative if s is odd and positive if s is even. Alternatively if $t > s$, then $36 \times 2^t > 36 \times 2^s + 9$ and $\omega(3, s, t)$ has the same sign as $36(-1)^s 2^t$. We conclude that $\alpha(3, s, t) < 0$ if and only if s is odd.

□

5.4.4 Higher degree

We define the *alternating wheel graph* of order n, denoted by W_n^{alt}, to be the graph with vertex set $\{x_1, x_2, \ldots, x_{n-1}, x\}$, whose edges consist of the cycle $\{x_1x_2, x_2x_3, \ldots, x_{n-2}x_{n-1}, x_{n-1}x_1\}$, together with the additional edges xx_j for all odd indices **j**. In other words, the extra vertex x is adjacent to every second vertex of the cycle, following an alternating pattern (see Figure 5.6).

We look at the alternating wheel graph because it gives us a clear picture of how graphs with long induced cycles behave differently from graphs with long induced paths. In particular, for the alternating wheel graph, the sign of α follows an interesting pattern depending on n mod 4, which is our reason for introducing this family. The main purpose of this section is to prove the following theorem.

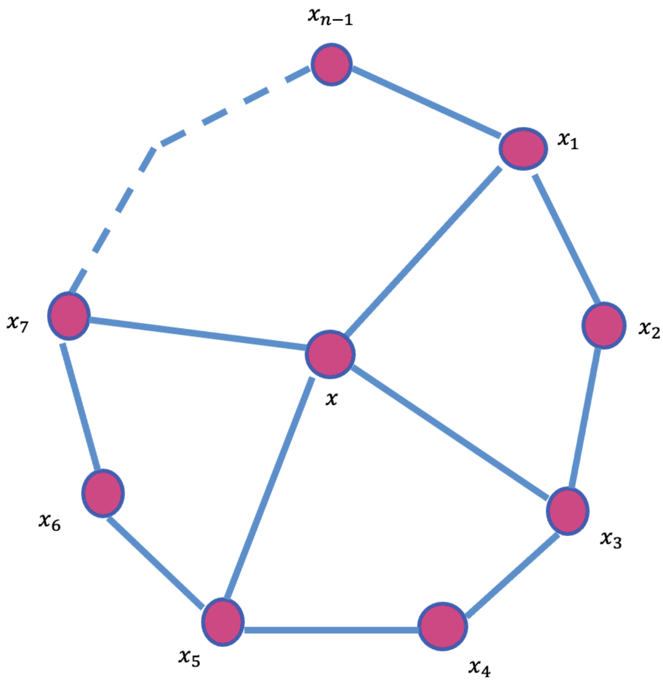


Figure 5.6: Alternating wheel

Theorem 5.4.8. *Let* $n \geqslant 9$*. Then*

$$\alpha(W_n^{\text{alt}}) < 0 \quad \textit{if } n \equiv 0 \textit{ or } 1 \mod 4, \qquad \alpha(W_n^{\text{alt}}) > 0 \quad \textit{if } n \equiv 2 \textit{ or } 3 \mod 4.$$

In particular, this (along with Theorem 5.4.4) means that for any positive even integer d, there exists $\Gamma \in \mathcal{G}^C$ with an extra vertex of degree d and $\alpha(\Gamma)$ negative. Similarly, for odd $d \geqslant 5$, there exist graphs $\Gamma \in \mathcal{G}^C$ with an extra vertex of degree d and $\alpha(\Gamma)$ positive. Theorem 5.4.7 demonstrates the existence of graphs in $\mathcal{G}_3^C$ with negative α. For each $d \geqslant 2$, we expect that $\mathcal{G}_d^C$ contains graphs with both positive and negative α. Theorems 3.3.2 and 5.4.8 illustrate the contrast in behaviour between the classes $\mathcal{G}^P$ and $\mathcal{G}^C$.

As a first step to proving the theorem, we define A_n and B_n by

$$A_n = \sum_{\substack{u \in T_{n-1} \\ u \not\perp v(n-1)}} 2^{z(u)}, \qquad B_n = \sum_{u \in T_{n-1}} 2^{z(u)-1}, \tag{5.3}$$

where T_{n-1} is the set of all vectors in $\mathbb{F}_2^{n-1}$ with no cyclically consecutive zeros and

$$v(n-1) = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots \end{pmatrix}^\top \in \mathbb{F}_2^{n-1}.$$

The vector $v(n-1)$ records the incidences of the vertex x in W_n^{alt}. Note that A_n and B_n closely resemble $A(W_n^{\mathrm{alt}})$ and $B(W_n^{\mathrm{alt}})$, differing only in that they do not account for the special role of the vector $\mathbf{j}$, as detailed in Theorem 5.3.1. We will account for this later. We then define $\alpha_n = A_n - B_n$. Note similarly that α_n is almost the same as $\alpha\left(W_n^{\mathrm{alt}}\right)$, differing by at most 2 due to the terms γ and δ as stated in Theorem 5.3.1.

We define $D_n = 2B_n - A_n$. Therefore

$$D_n = \sum_{\substack{u \in T_{n-1} \\ u \perp v}} 2^{z(u)}.$$

We choose to work with A_n and D_n rather than A_n and B_n because A_n and D_n are sums over two sets of vectors which partition T_{n-1}, namely those that are not orthogonal to and those that are orthogonal to v, respectively. This makes it easier to find a direct recurrence relation.

Next, define $\beta_n = A_n - D_n$, and note that α_n and β_n have the same sign because

$$D_n = 2B_n - A_n \iff 2A_n + D_n = A_n + 2B_n \iff 2(A_n - B_n) = A_n - D_n.$$

To set up recurrence relations for A_n and D_n, recall the set S_m consisting of all vectors in $\mathbb{F}_2^m$ with no two consecutive entries equal to 0. Note that this is different from the set T_m, which consists of all such vectors with the additional condition that the first and last entries are not both equal to 0. For the alternating wheel graph W_n^{alt}, the relevant set will be S_{n-1}. We also define the alternating vector

$$v(m) = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \end{bmatrix}^\top \in \mathbb{F}_2^m.$$

We now introduce two quantities A'_n and D'_n, which are closely related to A_n and D_n, but are instead defined over the set S_{n-1}.

$$A'_n = \sum_{\substack{u \in S_{n-1} \\ u \not\perp v(n-1)}} 2^{z(u)}, \qquad D'_n = \sum_{\substack{u \in S_{n-1} \\ u \perp v(n-1)}} 2^{z(u)}. \tag{5.4}$$

To obtain the recurrence relations for A'_n and D'_n, we split their sums according to whether the last entry of each vector in S_{n-1} is 1 or 0. This approach allows us to relate A'_n and D'_n to the corresponding quantities for smaller values of n.

Lemma 5.4.9. *Let $n \geqslant 4$. Then*

$$\begin{aligned} &\textit{if } n \textit{ is even:} \quad A'_n = D'_{n-1} + 2A'_{n-2}, \qquad D'_n = A'_{n-1} + 2D'_{n-2}, \\ &\textit{if } n \textit{ is odd:} \quad A'_n = A'_{n-1} + 2D'_{n-2}, \qquad D'_n = D'_{n-1} + 2A'_{n-2}. \end{aligned}$$

Proof. Every $u \in S_{n-1}$ has either 1 or 0 as its last entry. Since vectors in S_{n-1} do not contain consecutive zeros, we can write S_{n-1} as the disjoint union of $S_{n-1}[1]$ and $S_{n-1}[0]$, where

$$S_{n-1}[1] = \{(u', 1) : u' \in S_{n-2}\}, \qquad S_{n-1}[0] = \{(u'', 1, 0) : u'' \in S_{n-3}\},$$

We now define $A'_n[1], A'_n[0], D'_n[1], D'_n[0]$ as follows.

$$A'_n[1] = \sum_{\substack{u \in S_{n-1}[1] \\ u \not\perp v(n-1)}} 2^{z(u)}, \qquad A'_n[0] = \sum_{\substack{u \in S_{n-1}[0] \\ u \not\perp v(n-1)}} 2^{z(u)},$$

$$D'_n[1] = \sum_{\substack{u \in S_{n-1}[1] \\ u \perp v(n-1)}} 2^{z(u)}, \qquad D'_n[0] = \sum_{\substack{u \in S_{n-1}[0] \\ u \perp v(n-1)}} 2^{z(u)}.$$

It follows that

$$A'_n = A'_n[1] + A'_n[0], \qquad D'_n = D'_n[1] + D'_n[0].$$

We now consider separately the cases when n is even and when n is odd.

1. Suppose n is even and let $u \in S_{n-1}[1]$. Then $u = (u', 1)$ where $u' \in S_{n-2}$ and $z(u') = z(u)$. Since $v(n-1)$ has 1 in the last position, $u \perp v(n-1)$ if and only if $u' \not\perp v(n-2)$. It follows that

   $$A'_n[1] = \sum_{\substack{u' \in S_{n-2} \\ u' \perp v(n-2)}} 2^{z(u')} = D'_{n-1}, \qquad D'_n[1] = \sum_{\substack{u' \in S_{n-2} \\ u' \not\perp v(n-2)}} 2^{z(u')} = A'_{n-1}.$$

   Now let $u \in S_{n-1}[0]$, so $u = (u'', 1, 0)$ with $u'' \in S_{n-3}$ and $z(u) = z(u'') + 1$. Since $v(n-1)$ has 0 in the second-last position and 1 in last position, $u \perp v(m)$ if and only if $u'' \perp v(n-3)$. Therefore

   $$A'_n[0] = \sum_{\substack{u'' \in S_{n-3} \\ u'' \not\perp v(n-3)}} 2^{z(u'')+1} = 2A'_{n-2}, \qquad D'_n[0] = \sum_{\substack{u'' \in S_{n-3} \\ u'' \perp v(n-3)}} 2^{z(u'')+1} = 2D'_{n-2}.$$

   Combining these, for n even we obtain

   $$A'_n = D'_{n-1} + 2A'_{n-2}, \qquad D'_n = A'_{n-1} + 2D'_{n-2}.$$

2. Suppose n is odd and let $u \in S_{n-1}[1]$. Then $u = (u', 1)$ with $u' \in S_{n-2}$ and $z(u) = z(u')$. Since $v(n-1)$ has 0 in the last position, we have that $u \perp v(n-1)$ if and only if $u' \perp v(n-2)$. Thus

   $$A'_n[1] = \sum_{\substack{u' \in S_{n-2} \\ u' \not\perp v(n-2)}} 2^{z(u')} = A'_{n-1}, \qquad D'_n[1] = \sum_{\substack{u' \in S_{n-2} \\ u' \perp v(n-2)}} 2^{z(u')} = D'_{n-1}.$$

   Now let $u \in S_{n-1}[0]$, then $u = (u'', 1, 0)$ with $u'' \in S_{n-3}$ and $z(u) = z(u'') + 1$. Since $v(n-1)$ has 1 in the second-last position and 0 in the last position, we obtain that $u \perp v(n-1)$ if and only if $u'' \not\perp v(n-3)$. Hence

   $$A'_n[0] = \sum_{\substack{u'' \in S_{n-3} \\ u'' \perp v(n-3)}} 2^{z(u'')+1} = 2\,D'_{n-2}, \qquad D'_n[0] = \sum_{\substack{u'' \in S_{n-3} \\ u'' \not\perp v(n-3)}} 2^{z(u'')+1} = 2\,A'_{n-2}.$$

   Therefore, when n is odd, we obtain

   $$A'_n = A'_{n-1} + 2D'_{n-2}, \qquad D'_n = D'_{n-1} + 2A'_{n-2}.$$

This completes the proof. □

Define $\beta'_n = A'_n - D'_n$. We now use the previous result to solve for β'_n.

Lemma 5.4.10. *Let* $k \geqslant 1$

$$\beta'_{2k+4} = \frac{-51-5\sqrt{17}}{34}\left(\frac{-1+\sqrt{17}}{2}\right)^k + \frac{-51+5\sqrt{17}}{34}\left(\frac{-1-\sqrt{17}}{2}\right)^k$$
$$\beta'_{2k+3} = \frac{17-9\sqrt{17}}{34}\left(\frac{-1+\sqrt{17}}{2}\right)^k + \frac{17+9\sqrt{17}}{34}\left(\frac{-1-\sqrt{17}}{2}\right)^k$$

Proof. From Lemma 5.4.9, we gain that

$$\beta'_n = A'_n - D'_n = \begin{cases} -\beta'_{n-1} + 2\beta'_{n-2}, & n \text{ even} \\ \beta'_{n-1} - 2\beta'_{n-2}, & n \text{ odd} \end{cases}$$

Consequently, $\beta'_n = (-1)^{n-1}\beta'_{n-1} + 2(-1)^n\beta'_{n-2}$.

This recurrence can be also written in matrix form as follows.

$$\begin{pmatrix} \beta'_n \\ \beta'_{n-1} \end{pmatrix} = \begin{pmatrix} (-1)^{n-1} & 2(-1)^n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta'_{n-1} \\ \beta'_{n-2} \end{pmatrix}$$

where the transition matrix alternates between $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$ as n is even or odd. Let $k \geqslant 0$. Then

$$\begin{pmatrix} \beta'_{2k+4} \\ \beta'_{2k+3} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta'_{2k+2} \\ \beta'_{2k+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \beta'_{2k+2} \\ \beta'_{2k+1} \end{pmatrix}$$

Let $M = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$. Repeating this step gives

$$\begin{pmatrix} \beta'_{2k+4} \\ \beta'_{2k+3} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}^k \begin{pmatrix} \beta'_4 \\ \beta'_3 \end{pmatrix} = M^k \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

since $\beta'_4 = A'_4 - D'_4 = 4 - 7 = -3$ and $\beta'_3 = A'_3 - D'_3 = 3 - 2 = 1$.

The eigenvalues of the matrix M are $\lambda_1 = \frac{-1+\sqrt{17}}{2}$ and $\lambda_2 = \frac{-1-\sqrt{17}}{2}$. The matrix of the eigenvectors that correspond to these eigenvalues is

$$E = \begin{pmatrix} \frac{3+\sqrt{17}}{2} & \frac{3-\sqrt{17}}{2} \\ 1 & 1 \end{pmatrix}.$$

We diagonalize M and compute M^k to give explicit expressions for β'.

$$\begin{pmatrix} \beta'_{2k+4} \\ \beta'_{2k+3} \end{pmatrix} = \frac{1}{\sqrt{17}} \begin{pmatrix} \frac{3+\sqrt{17}}{2} & \frac{3-\sqrt{17}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \begin{pmatrix} 1 & -\frac{3-\sqrt{17}}{2} \\ -1 & \frac{3+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

The result follows.

$$\begin{pmatrix} \beta'_{2k+3} \\ \beta'_{2k+2} \end{pmatrix} = \begin{pmatrix} \frac{-51-5\sqrt{17}}{34}\lambda_1^k + \frac{-51+5\sqrt{17}}{34}\lambda_2^k \\ \frac{17-9\sqrt{17}}{34}\lambda_1^k + \frac{17+9\sqrt{17}}{34}\lambda_2^k \end{pmatrix}$$

□

We now turn to the relationship between A_n and D_n, and the corresponding quantities A'_n and D'_n.

Lemma 5.4.11. *Let $n \geqslant 6$. Then*

$$A_n = \begin{cases} A'_n - 4A'_{n-4}, & n \text{ even}, \\ A'_n - 4D'_{n-4}, & n \text{ odd}, \end{cases} \qquad D_n = \begin{cases} D'_n - 4D'_{n-4}, & n \text{ even}, \\ D'_n - 4A'_{n-4}, & n \text{ odd}. \end{cases}$$

Proof.

$$A_n = \sum_{\substack{u \in T_{n-1} \\ u \not\perp v(n-1)}} 2^{z(u)} = A'_n - \sum_{\substack{u \in S_{n-1} \setminus T_{n-1} \\ u \not\perp v(n-1)}} 2^{z(u)}$$

Let $u \in S_{n-1} \setminus T_{n-1}$. Then $u = (0, 1, u', 1, 0)$, where $u' \in S_{n-5}$.

Suppose n is even. Then the last two entries of $v(n-1)$ are $(0, 1)$. Note that $u \cdot v(n-1) = u' \cdot v(n-5)$. Furthermore $z(u) = z(u') + 2$.

$$A_n = A'_n - \sum_{\substack{u \in S_{n-1} \setminus T_{n-1} \\ u \not\perp v(n-1)}} 2^{z(u)} = A'_n - \sum_{\substack{u' \in S_{n-5} \\ u' \not\perp v(n-5)}} 2^{z(u)+2} = A_n - 4\, A'_{n-4}$$

Suppose n is odd, then the last two entries of $v(n-1)$ are $(1, 0)$. Then $u \cdot v(n-1) = u' \cdot v(n-5) + 1$. Again, $z(u) = z(u') + 2$.

$$A_n = A'_n - \sum_{\substack{u \in S_{n-1} \setminus T_{n-1} \\ u \perp v(n-1)}} 2^{z(u)} = A'_n - \sum_{\substack{u' \in S_{n-5} \\ u' \perp v(n-5)}} 2^{z(u)+2} = A_n - 4\, D'_{n-4}$$

Now for D_n.

$$D_n = \sum_{\substack{u \in T_{n-1} \\ u \perp v(n-1)}} 2^{z(u)} = D'_n - \sum_{\substack{u \in S_{n-1} \setminus T_{n-1} \\ u \perp v(n-1)}} 2^{z(u)}$$

- Suppose n is even. Then the last two entries of $v(n-1)$ are $(0,1)$, and $u \cdot v(n-1) = u' \cdot v(n-5)$. Again, $z(u) = z(u') + 2$.

$$D_n = D'_n - \sum_{\substack{u \in S_{n-1} \setminus T_{n-1} \\ u \perp v(n-1)}} 2^{z(u)} = D_n - 4 \sum_{\substack{u \in S_{n-1}, u_1 = u_{n-1} = 0 \\ u' \perp v(n-5)}} 2^{z(u)} = D_n - 4\, D'_{n-4}$$

- Suppose n is odd. Then the last two entries of $v(n-1)$ are $(1,0)$, and $u \cdot v(n-1) = u' \cdot v(n-5) + 1$. Again, $z(u) = z(u') + 2$.

$$D_n = D'_n - \sum_{\substack{u \in S_{n-1} \setminus T_{n-1} \\ u \not\perp v(n-1)}} 2^{z(u)} = D_n - 4 \sum_{\substack{u \in S_{n-1}, u_1 = u_{n-1} = 0 \\ u' \not\perp v(n-5)}} 2^{z(u)} = D_n - 4\, A'_{n-4}$$

□

Using the previous result, we derive expressions for β_n in terms of β'_n as follows.

If n is even:

$$\begin{aligned} \beta_n &= A_n - D_n \\ &= (A'_n - 4A'_{n-4}) - (D'_n - 4D'_{n-4}) \\ &= (A'_n - D'_n) - 4(A'_{n-4} - D'_{n-4}) \\ &= \beta'_n - 4\, \beta'_{n-4}. \end{aligned}$$

If n is odd:

$$\begin{aligned} \beta_n &= A_n - D_n \\ &= (A'_n - 4D'_{n-4}) - (D'_n - 4A'_{n-4}) \\ &= (A'_n - D'_n) + 4(A'_{n-4} - D'_{n-4}) \\ &= \beta'_n + 4\, \beta'_{n-4}. \end{aligned}$$

We summarise this result as follows.

Corollary 5.4.12. *Let $n \geqslant 2$. Then*

$$\beta_n \;=\; \beta'_n \;+\; 4(-1)^{n+1}\, \beta'_{n-4}.$$

We now use this corollary, together with Lemma 5.4.10 to determine the sign of β_n, for all $n \geqslant 6$.

Lemma 5.4.13. *Let $n \geqslant 9$. Then.*

- $\beta_n < 0$ *if* $n \equiv 0$ *or* $1 \pmod 4$.
- $\beta_n > 0$ *if* $n \equiv 2$ *or* $3 \pmod 4$.

Proof. Suppose n is odd. Lemma 5.4.10 implies $\beta'_{2k+3} = P_1\,\lambda_1^k + P_2\,\lambda_2^k$, where

$$\beta'_{2k+3} = \frac{17-9\sqrt{17}}{34}\lambda_1^k + \frac{17+9\sqrt{17}}{34}\lambda_2^k$$

$$P_1 = \frac{17-9\sqrt{17}}{34} \qquad P_2 = \frac{17+9\sqrt{17}}{34}$$

Corollary 5.4.12 implies

$$\begin{aligned}\beta_{2k+3} &= \beta'_{2k+3} + 4\,\beta'_{2k-1} \\ &= P_1\lambda_1^k + P_2\lambda_2^k + 4\big(P_1\lambda_1^{k-2} + P_2\lambda_2^{k-2}\big) \\ &= P_1\left(\lambda_1^2+4\right)\lambda_1^{k-2} + P_2\big(\lambda_2^2+4\big)\lambda_2^{k-2}.\end{aligned}$$

Let $Q_1 = P_1(\lambda_1^2+4)$ and $Q_2 = P_2(\lambda_2^2+4)$.

Then,

$$\begin{aligned}\beta_{2k+3} &= Q_1\,\lambda_1^{k-2} + Q_2\,\lambda_2^{k-2} \\ &= \left(\frac{13-5\sqrt{17}}{2}\right)\left(\frac{-1+\sqrt{17}}{2}\right)^{k-2} + \left(\frac{13+5\sqrt{17}}{2}\right)\left(\frac{-1-\sqrt{17}}{2}\right)^{k-2}.\end{aligned}$$

Since $\lambda_1 > 0$ and $Q_1 \approx -3.81 < 0$, the term $Q_1\lambda_1^{k-2}$ is negative for all k. Since $\lambda_2 < 0$ and $Q_2 \approx 16.81 > 0$, the sign of $Q_2\lambda_2^{k-2}$ depends on the parity of k.

Therefore:

- For even k: the first term in β_{2k+3} is negative and the second is positive and dominant for all $k \geqslant 0$. Hence $\beta_{2k+3} > 0$.
- For odd k, the second term is negative in addition to the first term, hence $\beta_{2k+3} < 0$.

It follows that β_n is positive if $n \equiv 3 \mod 4$ and β_n is negative if $n \equiv 1 \mod 4$.

Now suppose n is even. Lemma 5.4.10 implies $\beta'_{2k+4} = S_1\,\lambda_1^k + S_2\,\lambda_2^k$, where

$$S_1 = \frac{-51-5\sqrt{17}}{34}, \qquad S_2 = \frac{-51+5\sqrt{17}}{34}.$$

$$\lambda_1 = \frac{-1+\sqrt{17}}{2}, \qquad \lambda_2 = \frac{-1-\sqrt{17}}{2}.$$

Corollary 5.4.12 implies

$$\beta_{2k+4} = S_1\lambda_1^k + S_2\lambda_2^k - 4\big(S_1\lambda_1^{k-2} + S_2\lambda_2^{k-2}\big) = R_1\,\lambda_1^{k-2} + R_2\,\lambda_2^{k-2},$$

$$R_1 = S_1(\lambda_1^2 - 4) = \frac{17+23\sqrt{17}}{34} > 0, \qquad R_2 = S_2(\lambda_2^2 - 4) = \frac{17-23\sqrt{17}}{34} < 0$$

Substitute R_1 and R_2 in β_{2k+4}.

$$\begin{aligned}\beta_{2k+4} &= R_1\lambda_1^{k-2} + R_2\lambda_2^{k-2}\\ &= \left(\frac{17+23\sqrt{17}}{34}\right)\left(\frac{-1+\sqrt{17}}{2}\right)^{k-2} + \left(\frac{17-23\sqrt{17}}{34}\right)\left(\frac{-1-\sqrt{17}}{2}\right)^{k-2}.\end{aligned}$$

where $R_1 \approx 3.29$ and $R_2 \approx -2.29$.

Therefore:

- If k is even, the first term is positive and the second term is negative. Since $|\lambda_2| > \lambda_1$, the term with λ_2^{k-2} dominates for all $k \geqslant 4$. So $\beta_{2k+4} < 0$.
- If k is odd, then $\lambda_2^{k-2} < 0$, and the second term is positive. Therefore $\beta_{2k+4} > 0$ for all $k \geqslant 1$.

Consequently, for $n \geqslant 9$,

$$k \text{ even} \iff n \equiv 0 \pmod 4 \Rightarrow \beta_n < 0, \qquad k \text{ odd} \iff n \equiv 2 \pmod 4 \Rightarrow \beta_n > 0.$$

Therefore $\beta_n < 0$ if $n \equiv 0$ or $1 \pmod 4$ and $\beta_n > 0$ if $n \equiv 2$ or $3 \pmod 4$. □

Now, we use Lemma 5.4.13 to complete the proof of Theorem 5.4.8.

Theorem 5.4.8. Let $n \geqslant 9$. Then

$$\alpha(W_n^{\text{alt}}) < 0 \quad \text{if } n \equiv 0 \text{ or } 1 \mod 4, \qquad \alpha(W_n^{\text{alt}}) > 0 \quad \text{if } n \equiv 2 \text{ or } 3 \mod 4.$$

Proof. Recall

$$A_n = \sum_{\substack{u \in T_{n-1} \\ u \not\perp v(n-1)}} 2^{z(u)}, \qquad B_n = \sum_{u \in T_{n-1}} 2^{z(u)-1}, \qquad D_n = \sum_{\substack{u \in T_{n-1} \\ u \perp v(n-1)}} 2^{z(u)}.$$

Theorem 5.3.1 asserts the following relationships between A_n and $A\left(W_n^{alt}\right)$ and between B_n and $B\left(W_n^{alt}\right)$.

$$A\left(W_n^{alt}\right) = \gamma + \sum_{\substack{u \not\perp v(n-1) \\ u \in T_{n-1} \setminus \{\mathbf{j}\}}} 2^{z(u)} = \gamma + (A_n - \kappa),$$

$$\text{where} \quad \gamma = \begin{cases} 2, & \text{if } n \text{ is even and } \mathbf{j} \not\perp v \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \kappa = \begin{cases} 1, & \text{if } \mathbf{j} \not\perp v \\ 0, & \text{if } \mathbf{j} \perp v \end{cases}.$$

From Theorem 5.3.1 $\gamma = 2$ if $\mathbf{j} \not\perp v$ and n is even, and $\gamma = 0$ otherwise. It follows that $A\left(W_n^{alt}\right) \in \{A_n - 1, A_n, A_n + 1\}$ for all $n \geqslant 6$.

$$\begin{aligned} B\left(W_n^{alt}\right) &= \delta + \sum_{u \in T_{n-1} \setminus \{\ \mathbf{j}\}} 2^{z(u)-1} = \delta + \frac{1}{2}\left(\sum_{n \in T_{n-1}} 2^{z(u)} - 1\right) \\ &= \delta + \left(B_n - \frac{1}{2}\right) \end{aligned}$$

where

$$\delta = \begin{cases} 1, & \text{if } n \text{ is even and } \mathbf{j} \perp v \\ 1, & \text{if } n \text{ is odd and } \mathbf{j} \not\perp v \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \alpha\left(W_n^{alt}\right) &= A\left(W_n^{alt}\right) - B\left(W_n^{alt}\right) \\ &= (A_n - B_n) + \left((\gamma - \kappa) - (\delta - \frac{1}{2})\right) \end{aligned}$$

Since $A_n - B_n = \alpha_n$, then $\alpha(W_n^{alt}) = \alpha_n + \left((\gamma - \kappa) - (\delta - \frac{1}{2})\right)$

Since $\beta_n = 2\alpha_n$, $\alpha(W_n^{alt}) = \frac{1}{2}\beta_n + \left((\gamma - \kappa) - (\delta - \frac{1}{2})\right)$

The absolute value of $(\gamma - \kappa) - (\delta - \frac{1}{2})$ is no greater than $\frac{3}{2}$, since $\gamma - \kappa \in \{-1, 0, 1\}$ and $\delta \in \{0, 1\}$. Therefore, this term cannot change the sign of $\alpha_n(W_n^{alt})$ when $n \geqslant 9$.

Consequently,

$$\text{sign}\left(\alpha(W_n^{alt})\right) = \text{sign}(\alpha_n) = \text{sign}(\beta_n).$$

□

Chapter 6

Conclusion

The motivation for this research was the aspiration of characterizing connected graphs of order n, which are represented by more matrices of rank $n-1$ than of rank n over $\mathbb{F}_2$. In the general setting, this goal remains aspirational. However, the work presented in this thesis has achieved advances for two particular classes of graphs, and given some indication of the likely complexity of the general problem. Counts of symmetric matrices over $\mathbb{F}_2$ by rank, detailed in Chapter 1, suggest an expectation that the events $\alpha(\Gamma) > 0$ and $\alpha(\Gamma) < 0$ should occur with comparable frequency as Γ ranges through all connected simple graphs. The results of Chapters 3 and 5 explore this expectation for two classes of graphs, and Theorem 5.4.8 in particular is consistent with it. The results of Chapter 4 indicate that almost all graphs with an induced path on all but one vertex have more $\mathbb{F}_2$-representations of full rank than of rank $n-1$. The work presented in this thesis might be seen as initial steps in a more comprehensive study of the sign of the difference α between the numbers of rank $n-1$ and rank n representations across extensive classes of connected graphs, a topic that abounds with opportunities for further investigation and connects to the substantial research literature on rank problems for matrices representing graphs (see, for example, [5] and the references therein).

The following contributions and outcomes from this thesis are noted.

- The rank distribution of the path graph over $\mathbb{F}_2$ was determined in Chapter 2. One-third of all matrices representing the path graph have rank $n-1$, while the remainder have rank n.
- The study of the path graph was extended to connected graphs containing an induced path on all but one vertex. The vectors in the nullspaces of matrices representing the path graph were characterized. Based on this, Theorem 3.1.6 provides explicit expressions for the functions that count the number of matrices of rank $n-1$ and n representing graphs with a long induced path.
- The recurrence relations in Theorem 3.2.1 were developed to investigate the rank distribution of graphs in $\mathcal{G}_d^P$, providing a way to compute these values

from the corresponding results for $\mathcal{G}^P_{d-1}$.

- Theorem 3.3.2 presents the complete classification of graphs with a long induced path with more $\mathbb{F}_2$-representations of rank $n-1$ than rank n.
- The rank distribution of the cycle graph was determined in Chapter 4. Half of all matrices representing C_n have rank $n-1$, roughly one-third have rank n, and around one-sixth have rank $n-2$.
- The investigation was extended to graphs containing an induced cycle on all but one vertex. The relationship between the rank of the matrix representing this class of graph and the rank of its submatrix representing the long induced cycle is investigated and proved in Theorem 5.0.1.
- The 1-dimensional and 2-dimensional nullspaces of matrices representing the cycle graph were identified, and their properties were analysed. Theorem 5.3.1 provides explicit formulae, expressed in terms of these nullspace vectors, for the number of matrices of rank $n-1$ and n representing graphs in $\mathcal{G}^C$.
- Graphs with more $\mathbb{F}_2$-representations of rank $n-1$ than rank n in $\mathcal{G}^C_1$, $\mathcal{G}^C_2$, and $\mathcal{G}^C_3$ are classified in Theorems 5.4.2, 5.4.4, and 5.4.7.
- The sign behaviour of $\alpha(\Gamma)$ was determined for the class of graphs in $\mathcal{G}^C$ called alternating wheel graph, as stated in Theorem 5.4.8. It is proven that the sequence $\left(\alpha\left(W^{\text{alt}}_n\right)\right)_{n\geqslant 7}$ shows a periodic sign pattern with period 4. As a consequence, unlike in the case of $\mathcal{G}^P$, there is no upper bound on the degree of the extra vertex in $\Gamma \in \mathcal{G}^C$ with a negative $\alpha(\Gamma)$.
- Another future work could be considered other type of graphs with minimum rank $n-2$ such as 2-connected graph. Since as this type of graphs has high minimum rank. This property might be helpful to use to extend with graphs with

During our work on this study, we used Sagemath and Maple software for computation. It was used to test small cases, identify patterns, verify formulas, and formulate conjectures.

Some limitations of this study and possible directions for future research are also noted.

- Future research could investigate graphs with an induced path on all but two or more vertices, though such cases are likely to require a more refined combinatorial construction. For graphs with one extra vertex, the recurrence relations in Theorem 3.2.1 were derived from smaller graphs by increasing the degree of the unique extra vertex by 1. In contrast, when there are multiple extra vertices, it is far more difficult to identify a recurrence relation. Similarly, our approach of analysing the nullspace vectors of matrices representing the long induced path would become far more complicated, as their orthogonality to multiple incidence vectors would need to be considered.

- A limitation of this study lies in the increasing complexity when extending the analysis from path to cycle. The methods developed for the classes $\mathcal{G}_1^C, \mathcal{G}_2^C, \mathcal{G}_3^C$ do not extend to higher degrees in the same way as those for $\mathcal{G}_d^P$ in Chapter 3. As the degree of the extra vertex increases, the formulas for the rank distributions become substantially more complicated and we were unable to identify recurrence relations.

- Extending the results for graphs with an induced cycle to higher degrees remains an open problem. The approach that worked well for graphs with a long induced path could not be adapted here. As previously mentioned, we were unable to identify recurrence relations, but we also proved that there is no bound on the degree of the extra vertex in a graph in $\mathcal{G}^C$ with negative α.

- We studied graphs on n vertices containing either P_{n-1} or C_{n-1} as an induced subgraph. These classes were particularly amenable to the methods developed in this thesis, because the rank distributions of path graphs and cycle graphs are simple and well understood, and because such graphs have high minimum rank. These features might be exploited for further classes, for example graphs of order n with an induced subgraph on $n-1$ vertices consisting of disjoint paths. This might be feasible but would be complicated by the fact that a matrix representing a disjoint union of k paths has $k+1$ possible ranks.

- A natural direction for future work is to study the $\mathbb{F}_2$-rank distribution and sign of α for graphs that have minimum rank $n-2$ over $\mathbb{F}_2$. This class includes the cycles C_n and all (non-path) graphs with a long induced path but is not limited to these. Additionally, it may be possible to to extend the ideas of Chapter 5 to further examples of graphs of order n that have an induced subgraph whose minimum rank is $n-3$.

- The work presented in this thesis focuses on the sign of the parameter α rather than its magnitude. It would be of interest to know the maximum and minimum values of $\alpha(\Gamma)$ over all graphs Γ of fixed order n, and the graphs that achieve these extreme values.

- In the final section, we focused on the alternating wheel graphs, where the connections at the extra vertex follow a periodic pattern. Future work could study how the rank behaviour changes when the incidences at the extra vertex follow some other periodic pattern, since the regularity of the incidence vector appears to drive the observed sign changes in α. For example, do all periodic incidence patterns exhibit this behaviour, with α changing in sign periodically as n increases?

- A possible direction for future work is to consider the $\mathbb{F}_2$-rank distribution over all isomorphism classes of looped extensions of a graph Γ rather than all distinct matrices representing Γ, considering non-zero entries on the main diagonal to represent loops. The extreme case of the complete graph shows that these two viewpoints can yield quite different results. However, there is no difference for graphs with trivial automorphism group.

Bibliography

[1] Wayne Barrett, Jason Grout, and Raphael Loewy. The minimum rank problem over the finite field of order 2: minimum rank 3. *Linear Algebra Appl.*, 430(4):890–923, 2009.

[2] Wayne Barrett, Hein van der Holst, and Raphael Loewy. Graphs whose minimal rank is two: the finite fields case. *Electron. J. Linear Algebra*, 14:32–42, 2005.

[3] Calum Buchanan, Christopher Purcell, and Puck Rombach. Subgraph complementation and minimum rank. *Electron. J. Combin.*, 29(1):Paper No. 1.38, 20, 2022.

[4] Guoli Ding and Andreĭ Kotlov. On minimal rank over finite fields. *Electron. J. Linear Algebra*, 15:210–214, 2006.

[5] Shaun M. Fallat and Leslie Hogben. The minimum rank of symmetric matrices described by a graph: a survey. *Linear Algebra Appl.*, 426(2-3):558–582, 2007.

[6] S. D. Fisher and M. N. Alexander. Matrices over a finite field. *Amer. Math. Monthly*, 73(6):639–641, 1966.

[7] Shmuel Friedland and Raphael Loewy. On the minimum rank of a graph over finite fields. *Linear Algebra Appl.*, 436(6):1710–1720, 2012.

[8] Jason Grout. The minimum rank problem over finite fields. *Electron. J. Linear Algebra*, 20:691–716, 2010.

[9] Leslie Hogben. Minimum rank problems. *Linear Algebra Appl.*, 432(8):1961–1974, 2010.

[10] Jessie MacWilliams. Orthogonal matrices over finite fields. *Amer. Math. Monthly*, 76:152–164, 1969.

[11] W. A. Stein et al. *Sage Mathematics Software (Version 9.6)*. The Sage Development Team, 2022. `http://www.sagemath.org`.

Appendix

The following Sage program verifies the calculations in Subsections 3.3 and 5.4.

```
[1]: # Set k,m,p,q,r,s,t to be "symbolic" variables (without specified values)
var('k m p q r s t')
# Set F to be a symbolic function that takes one argument
F = function("F", nargs = 1)

# f(x, d) allows the program to caluclate F(x-d)
def f(x, d):
    # If x is a number, then just evaluate F(x-d) and return that
    if type(x) == type(1):
        return (1/3)*(2^(x-d+1) + (-1)^(x-d))
    # f(x,0) = F(x)
    if d == 0:
        return F(x)
    # f(x,1) = F(x-1)
    if d == 1:
        return (1/2)*(F(x) + (-1)^(x-1))
    # Otherwise, recursively define f(x,d) in terms of f(x,d-1)
    return (1/2)*(f(x,d-1) + (-1)^(x-d))

# Defining alpha, again keeping track of the deficits d
# For example, alpha([r,s,t], [0,1,2]) = (r, s-1, t-2)
def alpha(L, d=[]):
    # If the user inputs no deficits, then each deficit is 0
    if d == []:
        d = len(L)*[0]
    # If alpha is given two arguments, evaluate in terms of F
    if len(L) == 2:
        return 2*f(L[0], d[0])*f(L[1], d[1]) - 2*f(L[0],d[0]+1)*f(L[1],d[1]+1)
    # Otherwise, recursively define alpha in terms of alpha and beta with one␣
 ↪fewer argument
    return (1/2)*f(L[0], d[0])*beta(L[1:], d[1:]) + 2*f(L[0],d[0]+1)*alpha(L[1:
 ↪], [d[1]+1]+d[2:])

# Beta is defined similarly to alpha
def beta(L, d=[]):
    if d == []:
        d = len(L)*[0]
```

```
    if len(L) == 2:
        return -2*f(L[0], d[0])*f(L[1], d[1]) + 8*f(L[0],d[0]+1)*f(L[1],d[1]+1)
    return 2*f(L[0], d[0])*alpha(L[1:], d[1:]) + 2*f(L[0], d[0]+1)*beta(L[1:],␣
↪[d[1]+1]+d[2:])
```

[2]:
```
# Theorem 3.3.5
show(alpha([r,s,t]).expand())
```

$$-\frac{1}{4}\left(-1\right)^{r}\left(-1\right)^{s}\left(-1\right)^{t}+\frac{5}{4}\left(-1\right)^{s}\left(-1\right)^{t}F\left(r\right)-\frac{1}{4}\left(-1\right)^{r}\left(-1\right)^{t}F\left(s\right)-\frac{3}{4}\left(-1\right)^{t}F\left(r\right)F\left(s\right)+\frac{5}{4}\left(-1\right)^{r}\left(-1\right)^{s}F\left(t\right)-\frac{9}{4}\left(-1\right)^{s}F\left(r\right)F\left(t\right)-\frac{3}{4}\left(-1\right)^{r}F\left(s\right)F\left(t\right)+\frac{3}{4}F\left(r\right)F\left(s\right)F\left(t\right)$$

[3]:
```
# Theorem 3.3.5
show(alpha([r,2,t]).expand())
```

$$-\left(-1\right)^{r}\left(-1\right)^{t}-\left(-1\right)^{t}F\left(r\right)-\left(-1\right)^{r}F\left(t\right)$$

[4]:
```
# Theorem 3.3.6
show(beta([r,s,t]).expand())
```

$$\left(-1\right)^{r}\left(-1\right)^{s}\left(-1\right)^{t}-2\left(-1\right)^{s}\left(-1\right)^{t}F\left(r\right)+\left(-1\right)^{r}\left(-1\right)^{t}F\left(s\right)-2\left(-1\right)^{r}\left(-1\right)^{s}F\left(t\right)+3\left(-1\right)^{s}F\left(r\right)F\left(t\right)+3F\left(r\right)F\left(s\right)F\left(t\right)$$

[5]:
```
# Theorem 3.3.6
show(beta([r,1,t]).expand())
```

$$2\left(-1\right)^{t}F\left(r\right)+2\left(-1\right)^{r}F\left(t\right)$$

[6]:
```
# Theorem 3.3.7
show(alpha([q,2,s,t]).expand())
```

$$-\frac{3}{2}\left(-1\right)^{q}\left(-1\right)^{s}\left(-1\right)^{t}-\left(-1\right)^{s}\left(-1\right)^{t}F\left(q\right)+\frac{1}{2}\left(-1\right)^{q}\left(-1\right)^{t}F\left(s\right)+\frac{7}{2}\left(-1\right)^{q}\left(-1\right)^{s}F\left(t\right)-\frac{3}{2}\left(-1\right)^{q}F\left(s\right)F\left(t\right)+6F\left(q\right)F\left(s\right)F\left(t\right)$$

[7]:
```
# Theorem 3.3.7
show(alpha([1,2,s,t]).expand())
```

$$\frac{1}{2}\left(-1\right)^{s}\left(-1\right)^{t}-\frac{1}{2}\left(-1\right)^{t}F\left(s\right)-\frac{7}{2}\left(-1\right)^{s}F\left(t\right)+\frac{15}{2}F\left(s\right)F\left(t\right)$$

[8]:
```
# Theorem 3.3.7
show(alpha([q,1,s,t]).expand())
```

$$-\left(-1\right)^{q}\left(-1\right)^{s}\left(-1\right)^{t}-\frac{1}{2}\left(-1\right)^{s}\left(-1\right)^{t}F\left(q\right)+\left(-1\right)^{q}\left(-1\right)^{t}F\left(s\right)-\frac{3}{2}\left(-1\right)^{t}F\left(q\right)F\left(s\right)+\left(-1\right)^{q}\left(-1\right)^{s}F\left(t\right)+\frac{3}{2}\left(-1\right)^{s}F\left(q\right)F\left(t\right)+\frac{3}{2}F\left(q\right)F\left(s\right)F\left(t\right)$$

[9]:
```
# Theorem 3.3.7
show(alpha([q,1,1,t]).expand())
```

$2\,(-1)^{q}\,(-1)^{t}-(-1)^{t}\,F\,(q)-(-1)^{q}\,F\,(t)$

[10]:
```
# Theorem 3.3.8
show(beta([q,1,1,t]).expand())
```

$-2\,(-1)^{q}\,(-1)^{t}-2\,(-1)^{t}\,F\,(q)-2\,(-1)^{q}\,F\,(t)+12\,F\,(q)\,F\,(t)$

[11]:
```
# Theorem 3.3.8
show(beta([q,1,2,t]).expand())
```

$-2\,(-1)^{q}\,(-1)^{t}+2\,(-1)^{t}\,F\,(q)-10\,(-1)^{q}\,F\,(t)+12\,F\,(q)\,F\,(t)$

[12]:
```
# Theorem 3.3.8
show(beta([q,2,2,t]).expand())
```

$6\,(-1)^{q}\,(-1)^{t}-14\,(-1)^{t}\,F\,(q)-14\,(-1)^{q}\,F\,(t)+12\,F\,(q)\,F\,(t)$

[13]:
```
# Theorem 3.3.8
show(beta([1,2,2,t]).expand())
```

$-20\,(-1)^{t}+26\,F\,(t)$

[14]:
```
# Theorem 3.3.9
show(alpha([p,1,r,1,t]).expand())
```

$-(-1)^{r}\,(-1)^{t}\,F\,(p)-(-1)^{p}\,(-1)^{t}\,F\,(r)-(-1)^{p}\,(-1)^{r}\,F\,(t)-3\,(-1)^{r}\,F\,(p)\,F\,(t)+3\,F\,(p)\,F\,(r)\,F\,(t)$

[15]:
```
# Theorem 3.3.9
show(alpha([p,1,1,1,t]).expand())
```

$-(-1)^{p}\,(-1)^{t}+(-1)^{t}\,F\,(p)+(-1)^{p}\,F\,(t)+6\,F\,(p)\,F\,(t)$

In [7]:
```
# Define symbolic variables
var('s t')
# Define A(s,t)
def A(s, t):
    return (4/9) * (2**(s + t) - 2**s * (-1)**t - 2**t * (-1)**s + (-1)**(s
+ t))
# Define B(s,t)
def B(s, t):
    return 2**(s + t - 1)
# Define alpha(s,t)
def alpha(s, t):
    return A(s, t) - B(s, t)
# ====== INSERT YOUR VALUES HERE ======
s_value = 1 # <--- change this
t_value = t # <--- and this
# ====================================
#Theorem 5.4.4
show(simplify(alpha(s,t)))
```

$$-\frac{1}{9}\cdot 2^{t+2}(-1)^{s}-\frac{1}{9}\cdot 2^{s+2}(-1)^{t}-\frac{1}{9}\cdot 2^{s+t-1}+\frac{4}{9}\,(-1)^{s+t}$$

In [8]:
```
# Theorem 5.4.4
show(simplify(alpha(1,t)))
```

$$\frac{1}{9}\cdot 2^{t+3}-\frac{5}{9}\cdot 2^{t}+\frac{4}{9}\,(-1)^{t+1}-\frac{8}{9}\,(-1)^{t}$$

In [9]:
```
#Theorem 5.4.4
show(simplify(alpha(2,t)))
```

$$\frac{1}{9}\cdot 2^{t+4}-\frac{11}{9}\cdot 2^{t+1}+\frac{4}{9}\,(-1)^{t+2}-\frac{16}{9}\,(-1)^{t}$$

In [11]:
```
#Theorem 5.4.4
show(simplify(alpha(3,t)))
```

$$\frac{4}{9}\,(-1)^{t+3}-\frac{32}{9}\,(-1)^{t}$$

In [12]:
```
#Theorem 5.4.4
show(simplify(alpha(4,t)))
```

$$\frac{1}{9}\cdot 2^{t+6}-\frac{19}{9}\cdot 2^{t+2}+\frac{4}{9}\,(-1)^{t+4}-\frac{64}{9}\,(-1)^{t}$$

In [2]:
```
# Define symbolic variables
var('r s t')
# Define F(p) symbolically as a Python function
def F(p):
    return (1/3) * (2**(p+1) + (-1)**p)
# Define A(r, s, t)
def A(r, s, t):
    term1 = F(r) * F(s) * F(t)
    term2 = 4 * F(r-1) * F(s-2) * F(t-1)
    term3 = 4 * F(r-1) * F(s-1) * F(t-2)
    term4 = 4 * F(r-2) * F(s-1) * F(t-1)
    term5 = (-1)**(r + s + t + 1)
    return term1 + term2 + term3 + term4 + term5
# Define B(r, s, t)
def B(r, s, t):
    return 2**(r + s + t - 1) + (-1)**(r + s + t)
# Define alpha(r, s, t)
def alpha(r, s, t):
    return expand(A(r, s, t) - B(r, s, t))
# Symbolic form (it won't simplify fully, but shows structure)
show(alpha(1, s, t))
```

$$-\frac{1}{9} \cdot 2^s 2^t + \frac{4}{9} \cdot 2^t (-1)^s + \frac{4}{9} \cdot 2^s (-1)^t + \frac{11}{9} (-1)^s (-1)^t$$

In [6]:
```
# Theorem 5.4.7
show(alpha(1, 2, t))
```

$$3\,(-1)^t$$

In [5]:
```
# Theorem 5.4.7
show(alpha(3,s,t))
```

$$\frac{4}{3} \cdot 2^t (-1)^s + \frac{4}{3} \cdot 2^s (-1)^t + \frac{1}{3} (-1)^s (-1)^t$$