



Analogues of Goldschmidt's thesis for fusion systems

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ANALOGUES OF GOLDSCHMIDT'S THESIS FOR FUSION SYSTEMS

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ABSTRACT. We extend the results of David Goldschmidt's thesis concerning fusion in finite groups to saturated fusion systems.

1. INTRODUCTION

Just recently, David Goldschmidt published his doctoral thesis [6] which had gone unpublished since 1968. In it he shows that if G is a finite simple group and $T \in \text{Syl}_2(G)$, then the exponent of $Z(T)$ (and hence of T) is bounded by a function of the nilpotence class of T . He also includes in the write-up a fusion factorization result for an arbitrary finite group involving $\mathcal{U}^1 Z$ and the Thompson subgroup. In this paper, we generalize these results to arbitrary saturated fusion systems. Throughout this paper, unless otherwise indicated, p denotes an arbitrary prime number, n a nonnegative integer, and P a nontrivial finite p -group.

Theorem 1. *Suppose P is of nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a saturated fusion system on P with $O_p(\mathcal{F}) = 1$. Then $Z(P)$ has exponent at most p^n .*

This bound is sharp for all n and p ; see Example 1 in Section 3. This also gives a bound on the exponent of P itself, which we certainly do not expect to be sharp.

Corollary 1. *Suppose P is of nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a saturated fusion system on P with $O_p(\mathcal{F}) = 1$. Then P has exponent at most $p^{n^2(p-1)+n}$.*

Proof. By Theorem 1, $Z(P)$ has exponent at most p^n . We claim that then every upper central quotient also has exponent at most p^n , and the proof is by induction. Let $k \geq 1$, and let $x \in Z^{k+1}(P)$. If x^{p^n} does not lie in $Z^k(P)$, then there exists $t \in P$ such that $[x^{p^n}, t]$ does not lie in $Z^{k-1}(P)$. But by a standard commutator identity, $[x^{p^n}, t] \equiv [x, t]^{p^n} \equiv 1$ modulo $Z^{k-1}(P)$, since by induction $Z^k(P)/Z^{k-1}(P)$ has exponent at most p^n . This contradiction establishes the claim. The nilpotence class of P is at most $n(p-1)+1$ by hypothesis, so the exponent of P is at most $p^{n(n(p-1)+1)}$. \square

Theorem 1 follows from the following, which we prove as Theorem 5 below.

Theorem 2. *Suppose P has nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a saturated fusion system on P . Then $\mathcal{U}^n(Z(P))$ is normal in \mathcal{F} .*

In the course of proving this last result in the group case for $p = 2$, Goldschmidt reduces to the situation in which a putative counterexample G has a weakly embedded 2-local

subgroup. Then his post-thesis classification [5] of such groups gives a contradiction. However, any weakly embedded 2-local M controls 2-fusion, and so the 2-subgroup $O_2(M)$ will show up as a normal subgroup in the fusion system, a shadow of the weakly embedded phenomenon. This allows the corresponding fusion result to hold for an arbitrary prime.

We note that Theorem 2 has the following corollary in the category of groups.

Theorem 3. *Let P be a nonabelian Sylow p -subgroup of a finite group G . Suppose that P has nilpotence class at most $n(p - 1) + 1$ and that G has no nontrivial strongly closed abelian p -subgroup. Then $Z(P)$ has exponent at most p^n .*

Proof. We can form the saturated fusion system $\mathcal{F}_P(G)$, and Theorem 2 then says that $\mathcal{U}^n(Z(P))$ is strongly \mathcal{F} -closed (see Proposition 1 below), that is, strongly closed in P with respect to G . Thus, $\mathcal{U}^n(Z(P))$ must be trivial. \square

Using a recent theorem of Flores and Foote [4], in which they use the Classification of Finite Simple Groups to describe all finite groups having a strongly closed p -subgroup, we get the following direct generalization of Goldschmidt's main theorem.

Corollary 2. *Let P be a nonabelian Sylow p -subgroup of a finite simple group G . If P has nilpotence class at most $n(p - 1) + 1$, then $Z(P)$ has exponent at most p^n .*

Proof. Suppose to the contrary that $A := \mathcal{U}^n(Z(P)) \neq 1$. Then by Theorem 2, A is a nontrivial strongly closed abelian subgroup of P . By inspection of the simple groups arising in the conclusion of the main theorem in [4], either P is abelian or $Z(P)$ has exponent p . Since P is nonabelian, we must have $n \geq 1$ and the corollary follows. \square

However, if the hypotheses of Corollary 2 are weakened slightly to assume only that $F^*(G)$ is simple, then the statement is false for all odd primes p , as the following example shows. Let $H = \text{PSL}(2, q)$ with $q = r^p$ for some prime power r and with the p -part of $q - 1$ equal to p^e . Let σ be a field automorphism of \mathbf{F}_q of order p and $G = H\langle\sigma\rangle$. If P is a Sylow p -subgroup of G , then P has nilpotence class 2, while $Z(P)$ has exponent p^{e-1} , and we may take e as large as we like.

Recall the Thompson subgroup $J(P)$, defined as the group generated by the abelian subgroups of P of maximum order. We also prove the following

Theorem 4. *Let \mathcal{F} be a saturated fusion system on P . Then*

$$\mathcal{F} = \langle C_{\mathcal{F}}(\mathcal{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle.$$

2. DEFINITIONS AND NOTATION

We collect in this section the necessary information on fusion systems. Since there are by now many good sources of this knowledge [2], in particular in background sections of papers [3, 7] to which this one is similar, we will content ourselves to be brief.

Let P be a finite p -group. A *category on P* is a category \mathcal{F} with objects the subgroups of P and whose morphism sets $\text{Hom}_{\mathcal{F}}(Q, R)$ consist of injective group homomorphisms subject to the requirement that every morphism in \mathcal{F} is a composition of an isomorphism in \mathcal{F} and an inclusion.

Let \mathcal{F} be a category on the p -group P . Let Q and R be subgroups of P . We write $\text{Aut}_{\mathcal{F}}(Q)$ for $\text{Hom}_{\mathcal{F}}(Q, Q)$, $\text{Hom}_P(Q, R)$ for the set of group homomorphisms in \mathcal{F} from Q to R induced by conjugation by elements of P , and $\text{Out}_{\mathcal{F}}(Q)$ for $\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q)$.

We say Q is

- *fully \mathcal{F} -normalized* if $|N_P(Q)| \geq |N_P(Q')|$ for all Q' which are \mathcal{F} -isomorphic to Q ,
- *fully \mathcal{F} -centralized* if $|C_P(Q)| \geq |C_P(Q')|$ for all Q' which are \mathcal{F} -isomorphic to Q ,
- *\mathcal{F} -centric* if $C_P(Q') \leq Q'$ for all Q' which are \mathcal{F} -isomorphic to Q , and
- *\mathcal{F} -radical* if $O_p(\text{Out}_{\mathcal{F}}(Q)) = 1$.

For a morphism $\varphi : Q \rightarrow P$ in \mathcal{F} , let

$$N_{\varphi} = \{x \in N_P(Q) \mid \exists y \in N_P(\varphi(Q)), \forall z \in Q, \varphi(xzx^{-1}) = y\varphi(z)y^{-1}\}$$

Note that we have $QC_P(Q) \leq N_{\varphi}$ for all $\varphi : Q \rightarrow P$ in \mathcal{F} .

A *saturated fusion system* on P is a category \mathcal{F} on P whose morphism sets contain all group homomorphisms induced by conjugation by elements of P , and which satisfies the following two axioms.

- (Sylow axiom) $\text{Aut}_P(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$, and
- (Extension axiom) for every isomorphism $\varphi : Q \rightarrow Q'$ with Q' fully \mathcal{F} -normalized, there exists a morphism $\tilde{\varphi} : N_{\varphi} \rightarrow P$ such that $\tilde{\varphi}|_Q = \varphi$.

For the remainder of the paper, \mathcal{F} will denote a saturated fusion system on the finite p -group P , even though we will often drop the adjective ‘‘saturated’’.

For $Q \leq P$, we define the following local subcategories of \mathcal{F} . The *normalizer* $N_{\mathcal{F}}(Q)$ of Q in \mathcal{F} is the category on $N_P(Q)$ such that for any $R_1, R_2 \leq N_P(Q)$, $\text{Hom}_{N_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \rightarrow R_2$ in \mathcal{F} for which there is an extension $\tilde{\varphi} : QR_1 \rightarrow QR_2$ of φ in \mathcal{F} such that $\tilde{\varphi}(Q) = Q$. The *centralizer* $C_{\mathcal{F}}(Q)$ of Q in \mathcal{F} is the category on $C_P(Q)$ such that for any $R_1, R_2 \leq C_P(Q)$, $\text{Hom}_{C_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \rightarrow R_2$ in \mathcal{F} for which there is an extension $\tilde{\varphi} : QR_1 \rightarrow QR_2$ of φ in \mathcal{F} such that $\tilde{\varphi}|_Q = \text{id}_Q$. Lastly, we define $N_P(Q)C_{\mathcal{F}}(Q)$ as we do the normalizer of Q , but only allow those $\varphi : R_1 \rightarrow R_2$ whose extensions $\tilde{\varphi}$ restrict to automorphisms in $\text{Aut}_P(Q)$.

If Q is fully \mathcal{F} -normalized, then $N_{\mathcal{F}}(Q)$ is a saturated fusion system. And if Q is fully \mathcal{F} -centralized, then both $C_{\mathcal{F}}(Q)$ and $N_P(Q)C_{\mathcal{F}}(Q)$ are saturated fusion systems.

A *characteristic functor* is a mapping from finite p -groups to finite p -groups which takes Q to a characteristic subgroup $W(Q)$ of Q such that for any group isomorphism $\varphi : Q \rightarrow Q'$, $\varphi(W(Q)) = W(Q')$. We say that a characteristic functor is *positive* provided $W(Q) \neq 1$ whenever $Q \neq 1$. The *center functor*, sending a finite p -group P to its center, is a positive characteristic p -functor.

A *conjugation family* for \mathcal{F} is a set \mathcal{C} of nonidentity subgroups of P such that \mathcal{F} is generated by compositions and restrictions of morphisms in $\text{Aut}_{\mathcal{F}}(Q)$ as Q ranges over \mathcal{C} . Alperin’s fusion theorem for saturated fusion systems says that the set of \mathcal{F} -centric, \mathcal{F} -radical subgroups is a conjugation family for \mathcal{F} , and we call this the *Alperin conjugation family*.

Recall that a subgroup W of P is said to be *weakly \mathcal{F} -closed* if for each $\varphi \in \text{Hom}_{\mathcal{F}}(W, P)$, $\varphi(W) = W$. The subgroup W is *strongly \mathcal{F} -closed* if for each subgroup W' of W and each

$\varphi \in \text{Hom}_{\mathcal{F}}(W', P)$, $\varphi(W') \leq W$. We say W is *normal in \mathcal{F}* if $\mathcal{F} = N_{\mathcal{F}}(W)$, and denote by $O_p(\mathcal{F})$ the largest such subgroup of P .

3. PROOFS

The following proposition is slightly misstated in [1, Proposition 1.6], where a normal W is claimed to be contained in every radical subgroup. For this reason, we state a correct version here, but the proof in [1] goes through with little modification.

Proposition 1. *Let \mathcal{F} be a fusion system on P and $W \leq P$. The following are equivalent.*

- (a) *W is normal in \mathcal{F} .*
- (b) *W is strongly \mathcal{F} -closed and is contained in every \mathcal{F} -centric, \mathcal{F} -radical subgroup of P .*
- (c) *W is weakly \mathcal{F} -closed and is contained in every subgroup of some conjugation family for \mathcal{F} .*

Lemma 1. *Suppose P has nilpotence class at most $n(p-1)+1$. If Q is a subgroup of P with $C_P(\mathcal{U}^n(Z(Q))) = Q$, then $Q = P$.*

Proof. This is Corollary 6 in [6]. □

Proposition 2. *Let W be a characteristic subfunctor of the center functor such that $W(P) \leq W(Q)$ for all $Q \leq P$ with $C_P(Q) \leq Q$. Then for any fusion system \mathcal{F} on P , either there exists a proper \mathcal{F} -centric subgroup Q of P such that $C_P(W(Q)) = Q$, or $W(P)$ is normal in \mathcal{F} .*

Proof. Suppose there is no proper \mathcal{F} -centric subgroup Q of P with $C_P(W(Q)) = Q$. We will show that $W(P)$ is weakly closed in \mathcal{F} . In this case, $W(P) \leq Z(P)$ is contained in every \mathcal{F} -centric subgroup of P , hence in every member of an Alperin conjugation family for \mathcal{F} . Thus, by Proposition 1, $W(P)$ is in fact normal in \mathcal{F} .

Let Q be a fully \mathcal{F} -normalized, \mathcal{F} -centric subgroup of P . Then by hypothesis, $W(P) \leq W(Q)$. Let $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$. By Alperin's fusion theorem, it suffices to show that $W(P)$ is invariant under α . We do this by induction on $|P : Q|$. If $Q = P$, then $\alpha(W(P)) = W(P)$ since $W(P)$ is a characteristic subgroup of P , so suppose that $Q < P$. Then $C_P(W(Q)) > Q$. Let $\beta : W(Q) \rightarrow R$ be an isomorphism in \mathcal{F} with R fully \mathcal{F} -normalized. Then by the extension axiom, β extends to a map $\tilde{\beta} : C_P(W(Q)) \rightarrow P$. By induction and Alperin's fusion theorem, we have that $\beta(W(P)) = \tilde{\beta}(W(P)) = W(P)$. But $\beta\alpha|_{W(Q)}$ also extends to $C_P(W(Q))$, and $\beta\alpha(W(P)) = W(P)$ by the same reasoning. Therefore $\alpha(W(P)) = \beta^{-1}\beta\alpha(W(P)) = W(P)$, and this completes the proof. □

We are now ready to prove Theorem 2.

Theorem 5. *Suppose P has nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a fusion system on P . Then $\mathcal{U}^n(Z(P))$ is normal in \mathcal{F} .*

Proof. Let $W = \mathcal{U}^n Z$. If $C_P(Q) \leq Q \leq P$, then $Z(P) \leq Z(Q)$ and so $W(P) = \mathcal{U}^n(Z(P)) \leq \mathcal{U}^n(Z(Q)) = W(Q)$. Thus W satisfies the hypotheses of Proposition 2, and Lemma 1 says that there is no proper subgroup of P with $C_P(W(Q)) = Q$. Therefore by Proposition 2, $\mathcal{U}^n(Z(P))$ is normal in \mathcal{F} . □

Theorem 1 now follows immediately from Theorem 2. The following example generalizes a remark of Goldschmidt's in [6], and shows that the bound on the exponent of $Z(P)$ given in Theorem 1 is sharp.

Example 1. Let p be an odd prime, let $G = \text{SL}(p+1, q)$ with $|q-1|_p = p^n$, and let P be a Sylow p -subgroup of G . Then P is isomorphic to $C_{p^n} \wr C_p$. Let x be the wreathing element, a p -cycle permutation matrix, generating the C_p on top. Then $P' = [P, P]$ is isomorphic to $p-1$ copies of C_{p^n} . Let $P_0 = \langle P', x \rangle$. As $Z(P)$ has exponent p^n , the bound in Theorem 1 is sharp provided the class of P is $n(p-1) + 1$. For this it suffices to show that P_0 has class $n(p-1)$, that is, P_0 is of maximal class.

By an inductive argument, we quickly reduce to the case where $n = 2$. Suppose $n = 2$ and let a_1, \dots, a_{p-1} be generators for the $p-1$ cyclic groups of order p^2 . Then x sends a_i to a_{i+1} for $1 \leq i \leq p-2$ and a_{p-1} to $a_1^{-1} \cdots a_{p-1}^{-1}$. Factoring by $\Omega_1(P')$ we have that $[P'/\Omega_1(P'), x; p-1] = 1$ so that $[P', x; p-1] \leq \Omega_1(P')$. By direct computation,

$$[a_1, x; p-1] = \prod_{k=0}^{p-2} a_{k+1}^{(-1)^k \binom{p-1}{k}^{-1}}.$$

The sum of the exponents of the a_i in $[a_1, x; p-1]$ is

$$-p+1 + \sum_{k=0}^{p-2} (-1)^k \binom{p-1}{k} = -p+1 + (1-1)^{p-1} - \binom{p-1}{p-1} = -p.$$

This means that $[a_1, x; p-1]$ lies outside the sum-zero submodule (which is the unique maximal submodule) for the action of x on $\Omega_1(P')$, and so $[P', x; p-1] = \Omega_1(P')$. It follows that P_0 has class $2(p-1)$, as claimed.

Therefore P has class $n(p-1) + 1$ while $Z(P)$ has exponent p^n , and so the bound of Theorem 1 is sharp.

We now turn to the proof of Theorem 4. We will need a version of the Frattini argument due to Onofrei and Stancu [8, Proposition 3.7].

Proposition 3. *Let \mathcal{F} be a fusion system on P and suppose $Q \leq P$ is normal in \mathcal{F} . Then*

$$\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle.$$

Lemma 2. *Suppose P is a p -group, $Q \leq P$, and $C_P(\mathcal{U}^1(Z(Q))) = Q$. Then $J(P) \leq Q$.*

Proof. This is Lemma 8 in [6]. □

The *Thompson ordering* on subgroups of P is defined by

$$Q \leq_P Q' \text{ iff } |N_P(Q)| \leq |N_P(Q')| \text{ or } |N_P(Q)| = |N_P(Q')| \text{ and } |Q| \leq |Q'|.$$

We are now ready to prove

Theorem 6. *Let \mathcal{F} be a fusion system on P . Then*

$$\mathcal{F} = \langle C_{\mathcal{F}}(\mathcal{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle.$$

Proof. Write $\mathcal{F}' = \langle C_{\mathcal{F}}(\mathcal{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle$. Since each \mathcal{F} -centric subgroup of P contains $Z(P)$, it suffices by Alperin's fusion theorem to prove that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ for all $Q \leq P$ with $Z(P) \leq Q$. We do this by induction on the Thompson ordering. If $Q = P$, then $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$, since $J(P)$ is a characteristic subgroup of P , so suppose that $Q <_P P$ with $Z(P) \leq Q$ and that $N_{\mathcal{F}}(Q') \subseteq \mathcal{F}'$ for all $Q' >_P Q$ with $Z(P) \leq Q'$.

First we reduce to the case where Q is fully \mathcal{F} -normalized. Suppose Q is not fully \mathcal{F} -normalized. By [7, Lemma 2.2], there exists $\alpha : N_P(Q) \rightarrow P$ such that $\alpha(Q)$ is fully \mathcal{F} -normalized. Note that $\alpha(Q) >_P Q$, and since $R >_P Q$ for every $R \leq P$ with $|N_P(Q)| \leq |R|$, we have by induction and Alperin's fusion theorem that α is in \mathcal{F}' . Also note that $\alpha(N_P(Q)) \leq N_P(\alpha(Q))$; we still denote by α the induced morphism $N_P(Q) \rightarrow N_P(\alpha(Q))$. Let $\varphi : R_1 \rightarrow R_2$ be a morphism in $N_{\mathcal{F}}(Q)$, and let $\tilde{\varphi}$ be an extension to $QR_1 \leq N_P(Q)$. Then $\alpha\tilde{\varphi}\alpha^{-1} : \alpha(Q)\alpha(R_1) \rightarrow \alpha(Q)\alpha(R_2)$ restricts to an automorphism of $\alpha(Q)$, whence is contained in \mathcal{F}' by induction. But α is in \mathcal{F}' , so φ is in \mathcal{F}' too. Thus $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$, so henceforth we assume Q is fully \mathcal{F} -normalized.

For brevity, set $W = \mathcal{U}^1(Z(Q))$, $N = N_P(Q)$, and $C = C_N(W)$. Then $C \trianglelefteq N$, so that $N_P(C) \geq N$. Suppose first that $C = Q$. Then by Lemma 2, we have $J(N) \leq Q$. As $J(N) \trianglelefteq N_P(N)$, either $J(N) >_P Q$ or $N = P$. In the first case, since $Z(P) \leq J(N)$ and $J(N) = J(Q)$ is a characteristic subgroup of Q , we apply induction to get $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(N)) \subseteq \mathcal{F}'$. In the second case we have $J(P) \leq Q$, so $J(P) = J(Q)$, and hence $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$ here as well.

Assume now that $C > Q$. Then $C >_P Q$ because $C \trianglelefteq N$. Looking to see that $W \trianglelefteq N_{\mathcal{F}}(Q)$, we apply Proposition 3 in this normalizer to get

$$N_{\mathcal{F}}(Q) = \langle NC_{N_{\mathcal{F}}(Q)}(W), N_{N_{\mathcal{F}}(Q)}(C) \rangle.$$

Since C contains $Z(P)$, we have by induction that $N_{N_{\mathcal{F}}(Q)}(C) \subseteq N_{\mathcal{F}}(C) \subseteq \mathcal{F}'$, so to complete the proof, it suffices to show that $NC_{N_{\mathcal{F}}(Q)}(W) \subseteq C_{\mathcal{F}}(\mathcal{U}^1(Z(P)))$. To see this, let $R_1, R_2 \leq N$, and let $\varphi : R_1 \rightarrow R_2$ be a morphism in $NC_{N_{\mathcal{F}}(Q)}(W)$. Then there exists $x \in N$ such that φ extends to an \mathcal{F} -map $\tilde{\varphi} : WR_1 \rightarrow WR_2$ with $\tilde{\varphi}|_W = c_x$, the conjugation map induced by x . But since Q contains $Z(P)$, it follows that $W = \mathcal{U}^1(Z(Q)) \geq \mathcal{U}^1(Z(P))$, and so $\tilde{\varphi}|_{\mathcal{U}^1(Z(P))} = c_x|_{\mathcal{U}^1(Z(P))} = \text{id}_{\mathcal{U}^1(Z(P))}$. Therefore, $\varphi \in C_{\mathcal{F}}(\mathcal{U}^1(Z(P)))$, as was to be shown. We conclude that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ and the result follows. \square

Remark 1. In [3, Theorem 4.1], the authors prove in part that for any fusion system \mathcal{F} on P , $\mathcal{U}^1(Z(P)) \cap Z(N_{\mathcal{F}}(J(P))) \leq Z(\mathcal{F})$ by reducing to the group case. Theorem 4 gives a reduction-free proof of this fact.

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