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ANALOGUES OF GOLDSCHMIDT'S THESIS FOR FUSION SYSTEMS

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Abstract. We extend the results of David Goldschmidt's thesis concerning fusion in finite groups to saturated fusion systems.

1. Introduction

Just recently, David Goldschmidt published his doctoral thesis [6] which had gone unpublished since 1968. In it he shows that if G is a finite simple group and $T \in \mathrm{Syl}_2(G)$, then the exponent of $Z(T)$ (and hence of T) is bounded by a function of the nilpotence class of T. He also includes in the write-up a fusion factorization result for an arbitrary finite group involving $\mho^1 Z$ and the Thompson subgroup. In this paper, we generalize these results to arbitrary saturated fusion systems. Throughout this paper, unless otherwise indicated, p denotes an arbitrary prime number, n a nonnegative integer, and P a nontrivial finite p-group.

Theorem 1. *Suppose P is of nilpotence class at most $n(p-1)+1$ and $\mathcal{F}$ is a saturated fusion system on P with $O_p(\mathcal{F}) = 1$. Then $Z(P)$ has exponent at most p^n.*

This bound is sharp for all n and p; see Example 1 in Section 3. This also gives a bound on the exponent of P itself, which we certainly do not expect to be sharp.

Corollary 1. *Suppose P is of nilpotence class at most $n(p-1)+1$ and $\mathcal{F}$ is a saturated fusion system on P with $O_p(\mathcal{F}) = 1$. Then P has exponent at most $p^{n^2(p-1)+n}$.*

Proof. By Theorem 1, $Z(P)$ has exponent at most p^n. We claim that then every upper central quotient also has exponent at most p^n, and the proof is by induction. Let $k \geqslant 1$, and let $x \in Z^{k+1}(P)$. If x^{p^n} does not lie in $Z^k(P)$, then there exists $t \in P$ such that $[x^{p^n}, t]$ does not lie in $Z^{k-1}(P)$. But by a standard commutator identity, $[x^{p^n}, t] \equiv [x, t]^{p^n} \equiv 1$ modulo $Z^{k-1}(P)$, since by induction $Z^k(P)/Z^{k-1}(P)$ has exponent at most p^n. This contradiction establishes the claim. The nilpotence class of P is at most $n(p-1)+1$ by hypothesis, so the exponent of P is at most $p^{n(n(p-1)+1)}$. $\square$

Theorem 1 follows from the following, which we prove as Theorem 5 below.

Theorem 2. *Suppose P has nilpotence class at most $n(p-1)+1$ and $\mathcal{F}$ is a saturated fusion system on P. Then $\mho^n(Z(P))$ is normal in $\mathcal{F}$.*

In the course of proving this last result in the group case for $p = 2$, Goldschmidt reduces to the situation in which a putative counterexample G has a weakly embedded 2-local

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subgroup. Then his post-thesis classification [5] of such groups gives a contradiction. However, any weakly embedded 2-local M controls 2-fusion, and so the 2-subgroup $O_2(M)$ will show up as a normal subgroup in the fusion system, a shadow of the weakly embedded phenomenon. This allows the corresponding fusion result to hold for an arbitrary prime.

We note that Theorem 2 has the following corollary in the category of groups.

Theorem 3. *Let P be a nonabelian Sylow p-subgroup of a finite group G. Suppose that P has nilpotence class at most $n(p-1)+1$ and that G has no nontrivial strongly closed abelian p-subgroup. Then $Z(P)$ has exponent at most p^n.*

Proof. We can form the saturated fusion system $\mathcal{F}_P(G)$, and Theorem 2 then says that $\mho^n(Z(P))$ is strongly $\mathcal{F}$-closed (see Proposition 1 below), that is, strongly closed in P with respect to G. Thus, $\mho^n(Z(P))$ must be trivial. □

Using a recent theorem of Flores and Foote [4], in which they use the Classification of Finite Simple Groups to describe all finite groups having a strongly closed p-subgroup, we get the following direct generalization of Goldschmidt's main theorem.

Corollary 2. *Let P be a nonabelian Sylow p-subgroup of a finite simple group G. If P has nilpotence class at most $n(p-1)+1$, then $Z(P)$ has exponent at most p^n.*

Proof. Suppose to the contrary that $A := \mho^n(Z(P)) \neq 1$. Then by Theorem 2, A is a nontrivial strongly closed abelian subgroup of P. By inspection of the simple groups arising in the conclusion of the main theorem in [4], either P is abelian or $Z(P)$ has exponent p. Since P is nonabelian, we must have $n \geqslant 1$ and the corollary follows. □

However, if the hypotheses of Corollary 2 are weakened slightly to assume only that $F^*(G)$ is simple, then the statement is false for all odd primes p, as the following example shows. Let $H = \mathrm{PSL}(2,q)$ with $q = r^p$ for some prime power r and with the p-part of $q-1$ equal to p^e. Let σ be a field automorphism of $\mathbf{F}_q$ of order p and $G = H\langle\sigma\rangle$. If P is a Sylow p-subgroup of G, then P has nilpotence class 2, while $Z(P)$ has exponent p^{e-1}, and we may take e as large as we like.

Recall the Thompson subgroup $J(P)$, defined as the group generated by the abelian subgroups of P of maximum order. We also prove the following

Theorem 4. *Let $\mathcal{F}$ be a saturated fusion system on P. Then*

$$\mathcal{F} = \langle\ C_{\mathcal{F}}(\mho^1(Z(P)),\ N_{\mathcal{F}}(J(P))\ \rangle.$$

2. Definitions and notation

We collect in this section the necessary information on fusion systems. Since there are by now many good sources of this knowledge [2], in particular in background sections of papers [3, 7] to which this one is similar, we will content ourselves to be brief.

Let P be a finite p-group. A *category on P* is a category $\mathcal{F}$ with objects the subgroups of P and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ consist of injective group homomorphisms subject to the requirement that every morphism in $\mathcal{F}$ is a composition of an isomorphism in $\mathcal{F}$ and an inclusion.

Let $\mathcal{F}$ be a category on the p-group P. Let Q and R be subgroups of P. We write $\operatorname{Aut}_{\mathcal{F}}(Q)$ for $\operatorname{Hom}_{\mathcal{F}}(Q,Q)$, $\operatorname{Hom}_P(Q,R)$ for the set of group homomorphisms in $\mathcal{F}$ from Q to R induced by conjugation by elements of P, and $\operatorname{Out}_{\mathcal{F}}(Q)$ for $\operatorname{Aut}_{\mathcal{F}}(Q)/\operatorname{Aut}_Q(Q)$.

We say Q is

- *fully $\mathcal{F}$-normalized* if $|N_P(Q)| \geqslant |N_P(Q')|$ for all Q' which are $\mathcal{F}$-isomorphic to Q,
- *fully $\mathcal{F}$-centralized* if $|C_P(Q)| \geqslant |C_P(Q')|$ for all Q' which are $\mathcal{F}$-isomorphic to Q,
- *$\mathcal{F}$-centric* if $C_P(Q') \leqslant Q'$ for all Q' which are $\mathcal{F}$-isomorphic to Q, and
- *$\mathcal{F}$-radical* if $O_p(\operatorname{Out}_{\mathcal{F}}(Q)) = 1$.

For a morphism $\varphi : Q \to P$ in $\mathcal{F}$, let

$$N_\varphi = \{\, x \in N_P(Q) \mid \exists y \in N_P(\varphi(Q)),\, \forall z \in Q,\, \varphi(xzx^{-1}) = y\varphi(z)y^{-1}\}$$

Note that we have $QC_P(Q) \leqslant N_\varphi$ for all $\varphi : Q \to P$ in $\mathcal{F}$.

A *saturated fusion system* on P is a category $\mathcal{F}$ on P whose morphism sets contain all group homomorphisms induced by conjugation by elements of P, and which satisfies the following two axioms.

- (Sylow axiom) $\operatorname{Aut}_P(P)$ is a Sylow p-subgroup of $\operatorname{Aut}_{\mathcal{F}}(P)$, and
- (Extension axiom) for every isomorphism $\varphi : Q \to Q'$ with Q' fully $\mathcal{F}$-normalized, there exists a morphism $\tilde{\varphi} : N_\varphi \to P$ such that $\tilde{\varphi}\,|_Q = \varphi$.

For the remainder of the paper, $\mathcal{F}$ will denote a saturated fusion system on the finite p-group P, even though we will often drop the adjective "saturated".

For $Q \leqslant P$, we define the following local subcategories of $\mathcal{F}$. The *normalizer* $N_{\mathcal{F}}(Q)$ of Q in $\mathcal{F}$ is the category on $N_P(Q)$ such that for any $R_1, R_2 \leqslant N_P(Q)$, $\operatorname{Hom}_{N_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \to R_2$ in $\mathcal{F}$ for which there is an extension $\tilde{\varphi} : QR_1 \to QR_2$ of φ in $\mathcal{F}$ such that $\tilde{\varphi}(Q) = Q$. The *centralizer* $C_{\mathcal{F}}(Q)$ of Q in $\mathcal{F}$ is the category on $C_P(Q)$ such that for any $R_1, R_2 \leqslant C_P(Q)$, $\operatorname{Hom}_{C_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \to R_2$ in $\mathcal{F}$ for which there is an extension $\tilde{\varphi} : QR_1 \to QR_2$ of φ in $\mathcal{F}$ such that $\tilde{\varphi}\,|_Q = \mathrm{id}_Q$. Lastly, we define $N_P(Q)C_{\mathcal{F}}(Q)$ as we do the normalizer of Q, but only allow those $\varphi : R_1 \to R_2$ whose extensions $\tilde{\varphi}$ restrict to automorphisms in $\operatorname{Aut}_P(Q)$.

If Q is fully $\mathcal{F}$-normalized, then $N_{\mathcal{F}}(Q)$ is a saturated fusion system. And if Q is fully $\mathcal{F}$-centralized, then both $C_{\mathcal{F}}(Q)$ and $N_P(Q)C_{\mathcal{F}}(Q))$ are saturated fusion systems.

A *characteristic functor* is a mapping from finite p-groups to finite p-groups which takes Q to a characteristic subgroup $W(Q)$ of Q such that for any group isomorphism $\varphi : Q \to Q'$, $\varphi(W(Q)) = W(Q')$. We say that a characteristic functor is *positive* provided $W(Q) \neq 1$ whenever $Q \neq 1$. The *center functor*, sending a finite p-group P to its center, is a positive characteristic p-functor.

A *conjugation family* for $\mathcal{F}$ is a set $\mathcal{C}$ of nonidentity subgroups of P such that $\mathcal{F}$ is generated by compositions and restrictions of morphisms in $\operatorname{Aut}_{\mathcal{F}}(Q)$ as Q ranges over $\mathcal{C}$. Alperin's fusion theorem for saturated fusion systems says that the set of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups is a conjugation family for $\mathcal{F}$, and we call this the *Alperin conjugation family.*

Recall that a subgroup W of P is said to be *weakly $\mathcal{F}$-closed* if for each $\varphi \in \operatorname{Hom}_{\mathcal{F}}(W, P)$, $\varphi(W) = W$. The subgroup W is *strongly $\mathcal{F}$-closed* if for each subgroup W' of W and each

$\varphi \in \mathrm{Hom}_{\mathcal{F}}(W', P)$, $\varphi(W') \leqslant W$. We say W is *normal in* $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(W)$, and denote by $O_p(\mathcal{F})$ the largest such subgroup of P.

3. Proofs

The following proposition is slightly misstated in [1, Proposition 1.6], where a normal W is claimed to be contained in every radical subgroup. For this reason, we state a correct version here, but the proof in [1] goes through with little modification.

Proposition 1. *Let $\mathcal{F}$ be a fusion system on P and $W \leqslant P$. The following are equivalent.*

(a) *W is normal in $\mathcal{F}$.*
(b) *W is strongly $\mathcal{F}$-closed and is contained in every $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of P.*
(c) *W is weakly $\mathcal{F}$-closed and is contained in every subgroup of some conjugation family for $\mathcal{F}$.*

Lemma 1. *Suppose P has nilpotence class at most $n(p-1)+1$. If Q is a subgroup of P with $C_P(\mho^n(Z(Q))) = Q$, then $Q = P$.*

Proof. This is Corollary 6 in [6]. □

Proposition 2. *Let W be a characteristic subfunctor of the center functor such that $W(P) \leqslant W(Q)$ for all $Q \leqslant P$ with $C_P(Q) \leqslant Q$. Then for any fusion system $\mathcal{F}$ on P, either there exists a proper $\mathcal{F}$-centric subgroup Q of P such that $C_P(W(Q)) = Q$, or $W(P)$ is normal in $\mathcal{F}$.*

Proof. Suppose there is no proper $\mathcal{F}$-centric subgroup Q of P with $C_P(W(Q)) = Q$. We will show that $W(P)$ is weakly closed in $\mathcal{F}$. In this case, $W(P) \leqslant Z(P)$ is contained in every $\mathcal{F}$-centric subgroup of P, hence in every member of an Alperin conjugation family for $\mathcal{F}$. Thus, by Proposition 1, $W(P)$ is in fact normal in $\mathcal{F}$.

Let Q be a fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric subgroup of P. Then by hypothesis, $W(P) \leqslant W(Q)$. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$. By Alperin's fusion theorem, it suffices to show that $W(P)$ is invariant under α. We do this by induction on $|P : Q|$. If $Q = P$, then $\alpha(W(P)) = W(P)$ since $W(P)$ is a characteristic subgroup of P, so suppose that $Q < P$. Then $C_P(W(Q)) > Q$. Let $\beta : W(Q) \to R$ be an isomorphism in $\mathcal{F}$ with R fully $\mathcal{F}$-normalized. Then by the extension axiom, β extends to a map $\tilde{\beta} : C_P(W(Q)) \to P$. By induction and Alperin's fusion theorem, we have that $\beta(W(P)) = \tilde{\beta}(W(P)) = W(P)$. But $\beta\alpha|_{W(Q)}$ also extends to $C_P(W(Q))$, and $\beta\alpha(W(P)) = W(P)$ by the same reasoning. Therefore $\alpha(W(P)) = \beta^{-1}\beta\alpha(W(P)) = W(P)$, and this completes the proof. □

We are now ready to prove Theorem 2.

Theorem 5. *Suppose P has nilpotence class at most $n(p-1)+1$ and $\mathcal{F}$ is a fusion system on P. Then $\mho^n(Z(P))$ is normal in $\mathcal{F}$.*

Proof. Let $W = \mho^n Z$. If $C_P(Q) \leqslant Q \leqslant P$, then $Z(P) \leqslant Z(Q)$ and so $W(P) = \mho^n(Z(P)) \leqslant \mho^n(Z(Q)) = W(Q)$. Thus W satisfies the hypotheses of Proposition 2, and Lemma 1 says that there is no proper subgroup of P with $C_P(W(Q)) = Q$. Therefore by Proposition 2, $\mho^n(Z(P))$ is normal in $\mathcal{F}$. □

Theorem 1 now follows immediately from Theorem 2. The following example generalizes a remark of Goldschmidt's in [6], and shows that the bound on the exponent of $Z(P)$ given in Theorem 1 is sharp.

Example 1. Let p be an odd prime, let $G = \mathrm{SL}(p+1, q)$ with $|q-1|_p = p^n$, and let P be a Sylow p-subgroup of G. Then P is a isomorphic to $C_{p^n} \wr C_p$. Let x be the wreathing element, a p-cycle permutation matrix, generating the C_p on top. Then $P' = [P, P]$ is isomorphic to $p-1$ copies of C_{p^n}. Let $P_0 = \langle P', x\rangle$. As $Z(P)$ has exponent p^n, the bound in Theorem 1 is sharp provided the class of P is $n(p-1)+1$. For this it suffices to show that P_0 has class $n(p-1)$, that is, P_0 is of maximal class.

By an inductive argument, we quickly reduce to the case where $n = 2$. Suppose $n = 2$ and let $a_1, \ldots, a_{p-1}$ be generators for the $p-1$ cyclic groups of order p^2. Then x sends a_i to a_{i+1} for $1 \leqslant i \leqslant p-2$ and a_{p-1} to $a_1^{-1} \cdots a_{p-1}^{-1}$. Factoring by $\Omega_1(P')$ we have that $[P'/\Omega_1(P'), x; p-1] = 1$ so that $[P', x; p-1] \leqslant \Omega_1(P')$. By direct computation,

$$[a_1, x; p-1] = \prod_{k=0}^{p-2} a_{k+1}^{(-1)^k \binom{p-1}{k} - 1}.$$

The sum of the exponents of the a_i in $[a_1, x; p-1]$ is

$$-p + 1 + \sum_{k=0}^{p-2} (-1)^k \binom{p-1}{k} = -p + 1 + (1-1)^{p-1} - \binom{p-1}{p-1} = -p.$$

This means that $[a_1, x; p-1]$ lies outside the sum-zero submodule (which is the unique maximal submodule) for the action of x on $\Omega_1(P')$, and so $[P', x; p-1] = \Omega_1(P')$. It follows that P_0 has class $2(p-1)$, as claimed.

Therefore P has class $n(p-1)+1$ while $Z(P)$ has exponent p^n, and so the bound of Theorem 1 is sharp.

We now turn to the proof of Theorem 4. We will need a version of the Frattini argument due to Onofrei and Stancu [8, Proposition 3.7].

Proposition 3. *Let $\mathcal{F}$ be a fusion system on P and suppose $Q \leqslant P$ is normal in $\mathcal{F}$. Then*

$$\mathcal{F} = \langle\, PC_{\mathcal{F}}(Q),\ N_{\mathcal{F}}(QC_P(Q))\,\rangle.$$

Lemma 2. *Suppose P is a p-group, $Q \trianglelefteqslant P$, and $C_P(\mho^1(Z(Q))) = Q$. Then $J(P) \leqslant Q$.*

Proof. This is Lemma 8 in [6]. □

The *Thompson ordering* on subgroups of P is defined by

$$Q \leqslant_P Q' \quad \text{iff} \quad |N_P(Q)| \leqslant |N_P(Q')| \quad \text{or} \quad |N_P(Q)| = |N_P(Q')| \quad \text{and} \quad |Q| \leqslant |Q'|.$$

We are now ready to prove

Theorem 6. *Let $\mathcal{F}$ be a fusion system on P. Then*

$$\mathcal{F} = \langle\ C_{\mathcal{F}}(\mho^1(Z(P)),\ N_{\mathcal{F}}(J(P))\ \rangle.$$

Proof. Write $\mathcal{F}' = \langle\, C_{\mathcal{F}}(\mho^1(Z(P))),\ N_{\mathcal{F}}(J(P))\,\rangle$. Since each $\mathcal{F}$-centric subgroup of P contains $Z(P)$, it suffices by Alperin's fusion theorem to prove that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ for all $Q \leqslant P$ with $Z(P) \leqslant Q$. We do this by induction on the Thompson ordering. If $Q = P$, then $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$, since $J(P)$ is a characteristic subgroup of P, so suppose that $Q <_P P$ with $Z(P) \leqslant Q$ and that $N_{\mathcal{F}}(Q') \subseteq \mathcal{F}'$ for all $Q' >_P Q$ with $Z(P) \leqslant Q'$.

First we reduce to the case where Q is fully $\mathcal{F}$-normalized. Suppose Q is not fully $\mathcal{F}$-normalized. By [7, Lemma 2.2], there exists $\alpha : N_P(Q) \to P$ such that $\alpha(Q)$ is fully $\mathcal{F}$-normalized. Note that $\alpha(Q) >_P Q$, and since $R >_P Q$ for every $R \leqslant P$ with $|N_P(Q)| \leqslant |R|$, we have by induction and Alperin's fusion theorem that α is in $\mathcal{F}'$. Also note that $\alpha(N_P(Q)) \leqslant N_P(\alpha(Q))$; we still denote by α the induced morphism $N_P(Q) \to N_P(\alpha(Q))$. Let $\varphi : R_1 \to R_2$ be a morphism in $N_{\mathcal{F}}(Q)$, and let $\tilde{\varphi}$ be an extension to $QR_1 \leqslant N_P(Q)$. Then $\alpha\tilde{\varphi}\alpha^{-1} : \alpha(Q)\alpha(R_1) \to \alpha(Q)\alpha(R_2)$ restricts to an automorphism of $\alpha(Q)$, whence is contained in $\mathcal{F}'$ by induction. But α is in $\mathcal{F}'$, so φ is in $\mathcal{F}'$ too. Thus $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$, so henceforth we assume Q is fully $\mathcal{F}$-normalized.

For brevity, set $W = \mho^1(Z(Q))$, $N = N_P(Q)$, and $C = C_N(W)$. Then $C \trianglelefteqslant N$, so that $N_P(C) \geqslant N$. Suppose first that $C = Q$. Then by Lemma 2, we have $J(N) \leqslant Q$. As $J(N) \trianglelefteqslant N_P(N)$, either $J(N) >_P Q$ or $N = P$. In the first case, since $Z(P) \leqslant J(N)$ and $J(N) = J(Q)$ is a characteristic subgroup of Q, we apply induction to get $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(N)) \subseteq \mathcal{F}'$. In the second case we have $J(P) \leqslant Q$, so $J(P) = J(Q)$, and hence $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$ here as well.

Assume now that $C > Q$. Then $C >_P Q$ because $C \trianglelefteqslant N$. Looking to see that $W \trianglelefteqslant N_{\mathcal{F}}(Q)$, we apply Proposition 3 in this normalizer to get

$$N_{\mathcal{F}}(Q) = \langle\ NC_{N_{\mathcal{F}}(Q)}(W),\ N_{N_{\mathcal{F}}(Q)}(C)\ \rangle.$$

Since C contains $Z(P)$, we have by induction that $N_{N_{\mathcal{F}}(Q)}(C) \subseteq N_{\mathcal{F}}(C) \subseteq \mathcal{F}'$, so to complete the proof, it suffices to show that $NC_{N_{\mathcal{F}}(Q)}(W) \subseteq C_{\mathcal{F}}(\mho^1(Z(P)))$. To see this, let $R_1, R_2 \leqslant N$, and let $\varphi : R_1 \to R_2$ be a morphism in $NC_{N_{\mathcal{F}}(Q)}(W)$. Then there exists $x \in N$ such that φ extends to an $\mathcal{F}$-map $\tilde{\varphi} : WR_1 \to WR_2$ with $\tilde{\varphi}\,|_W = c_x$, the conjugation map induced by x. But since Q contains $Z(P)$, it follows that $W = \mho^1(Z(Q)) \geqslant \mho^1(Z(P))$, and so $\tilde{\varphi}\,|_{\mho^1(Z(P))} = c_x\,|_{\mho^1(Z(P))} = \mathrm{id}_{\mho^1(Z(P))}$. Therefore, $\varphi \in C_{\mathcal{F}}(\mho^1(Z(P)))$, as was to be shown. We conclude that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ and the result follows. $\square$

Remark 1. In [3, Theorem 4.1], the authors prove in part that for any fusion system $\mathcal{F}$ on P, $\mho^1(Z(P)) \cap Z(N_{\mathcal{F}}(J(P))) \leqslant Z(\mathcal{F})$ by reducing to the group case. Theorem 4 gives a reduction-free proof of this fact.

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