# *Proximity by Numbers*

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# Proximity by numbers*

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**Abstract**

Imagine that everyone in a group chooses a real number and then these numbers are combined to produce a group number. Suppose that when everyone moves strictly closer to some individual's number, the group number either stays where it is or moves closer to this number. We call this the proximity condition. Restricting attention to group choice rules that are homogeneous of degree one and constant-preserving, we show that the only rules satisfying this property are dictatorships.

## 1 Introduction

An interesting formal problem is how to combine numbers into a number, so that this number aggregates or synthesises the original numbers. Aczél and Roberts (1989) call these procedures "merging functions". Examples of merging functions are those generating the arithmetic mean, the geometric mean etc.

One obvious application of this idea is to public good provision. Members of a group have views as to how much of a public good should be produced, and these views need to be combined to make a group choice.

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Unlike Quesada (2007), we do not restrict attention to a discrete set of numbers. In our model, any real number is admissible for individuals and the group. However, like him, we derive an impossibility result: the only group choice rules satisfying our axioms are dictatorships.

Consider the following normative criterion. Suppose everyone moves strictly closer to some individual's number. In other words, everyone in the group has become more sympathetic to this individual's view. It seems reasonable to expect that the group number should either stay where it is, or move closer to this individual's number. To move further away would be bizarre. Surely the group choice should not penalise this person.

We call this the "proximity condition". It originates in a paper by MacIntyre (1998) on topological social choice. MacIntyre calls it "monotonicity".

Despite its intuitive appeal, we demonstrate in this note that it must be rejected as a requirement of merging. This is because it is inconsistent with three axioms that we take to be true. First of all, merging should not be dictatorial. Secondly, a merging function should be homogeneous of degree one. Thirdly, adding a constant to each individual's number should add that constant to the group number.

It turns out that it is impossible for a merging function to satisfy the proximity condition and these three axioms.

# 2 Model

Let $N$ be a finite set $\{1, 2, \ldots, n\}$ of individuals with $n \geq 3$. A group choice rule (or merging function) is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ where $\mathbb{R}$ denotes the real line. Given any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, let $(x_1, \ldots, x_n) = \mathbf{x}$, $(y_1, \ldots, y_n) = \mathbf{y}$ and so on. And let $x$ denote $f(\mathbf{x})$, $y$ denote $f(\mathbf{y})$ and so on.

The following are conditions that group choice rules may satisfy.

**Homogeneous of degree one.** For all $\alpha \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$, $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$.

**Constant-preserving.** For all $\alpha \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$, $f(\mathbf{x} + (\alpha, \alpha, .., \alpha)) = f(\mathbf{x}) + \alpha$.

**Proximity.** For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $i \in N$, $x_i = y_i$ and $|x_i - y_j| < |x_i - x_j|$ for all $j \in N - \{i\}$ implies $|x_i - y| \leq |x_i - x|$.

**Dictatorship.** There exists $i \in N$ such that for all $\mathbf{x} \in \mathbb{R}^n$, $f(\mathbf{x}) = x_i$.

## 3 Theorem

**Theorem.** *A group choice rule rule satisfies homogeneity of degree one, is constant-preserving and satisfies the proximity condition if and only if it is a dictatorship.*

The ($\Leftarrow$) direction is trivial, so we only prove ($\Rightarrow$).

**Lemma 1.** *Let $f$ be a group choice rule that satisfies homogeneity of degree one, is constant-preserving and satisfies the proximity condition. If there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\#\{x_1, \ldots, x_n\} = n$ and $x \in \{x_1, \ldots, x_n\}$ then $f$ is dictatorial.*

*Proof.* Assume that there exists $\mathbf{x} \in \mathbb{R}^n$ and $i \in N$ such that $\#\{x_1, \ldots, x_n\} = n$ and $x = x_i$. Take any $\mathbf{y} \in \mathbb{R}^n$. There must exist $\alpha \in \mathbb{R}$ such that for all $j \in N - \{i\}$, $|x_i - x_j| > |\alpha y_i - \alpha y_j|$. There must also exist $\beta \in \mathbb{R}$ such that $\alpha y_i + \beta = x_i$. For all $k \in N$, let $z_k$ denote $\alpha y_k + \beta$. We know that $x_i = z_i$ and that $|x_i - x_j| > |x_i - z_j|$ for all $j \in N - \{i\}$ and so it must be that $|x_i - x| \geq |x_i - z|$. Given that $x = x_i$, we have then $z = x_i$. Since $f$ is homogeneous of degree one and constant-preserving, it must be true that $\alpha y + \beta = z$. In other words, $\alpha y + \beta = x_i$. Therefore $y = (x_i - \beta)/\alpha$. We know that $\alpha y_i + \beta = x_i$ and so we have $y = y_i$. □

**Lemma 2.** *Let $f$ be a group choice rule that satisfies homogeneity of degree one, is constant-preserving and satisfies the proximity condition. For all $j, k \in N$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}$ and $\mathbf{y}$ differ only at the $j$th component, $|y_j - y_k| < |x_j - y_k|$ implies $|y - y_k| \leq |x - y_k|$.*

*Proof.* Take any $j \in N$ and two $n$-tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ that differ only at the $j$th component. Assume, by way of contradiction, that there exists some $k \in N$ such that $|x_j - x_k| > |y_j - x_k|$ and $|x - x_k| < |y - x_k|$. Let $\alpha$ be any number strictly between 1 and $x/y$. Consider the profile $\mathbf{z} \in \mathbb{R}^n$ with $z_i = \alpha y_i + (1 - \alpha) y_k$ for all $i \in N$. We know then that $z = \alpha y + (1 - \alpha) y_k$. Comparing $\mathbf{z}$ with $\mathbf{y}$, note that $z_k = y_k$ and that $|y_i - y_k| > |z_i - y_k|$ for all $i \in N - \{k\}$. The proximity condition implies then that $|y - y_k| \geq |z - y_k|$. However, this contradicts $z = \alpha y + (1 - \alpha) y_k$. □

Proof of theorem. Take an $n$-tuple $\mathbf{x} \in \mathbb{R}^n$ with $x_1 < x_2 < \ldots < x_n$ and $x_m = \frac{1}{2}(x_1 + x_n)$ for some $m \in N$. We consider three cases.

*Case 1.* $x \in \{x_1, \ldots, x_n\}$. Lemma 1 implies that $f$ is dictatorial.

*Case 2.* $x \notin \{x_1, \ldots, x_n\}$ and ($x_1 < x < x_m$ or $x_m < x < x_n$). Assume without loss of generality that $x_1 < x < x_m$. Consider the $n$-tuple $\mathbf{y}$ that is identical to $\mathbf{x}$ except that $y_n = x$. Lemma 2 implies that $|x - x_1| \geq |y - x_1|$

and $|x - x_m| \geq |y - x_m|$ which can be true only if $y = x$. Therefore $y_n = y$ and Lemma 1 implies that $f$ is dictatorial. This contradicts $x \notin \{x_1, \ldots, x_n\}$.

*Case 3.* $x \notin \{x_1, \ldots, x_n\}$ and ($x > x_n$ or $x < x_1$). Assume that $x > x_n$ (a similar argument applies when $x < x_1$). Let $\otimes : \mathbb{R}^n \to \mathbb{R}^n$ be the operation defined as follows. For all $\mathbf{z} \in \mathbb{R}^n$ and all $i \in N$, and letting $(\otimes z_1, \ldots, \otimes z_n) = \otimes \mathbf{z}$,

$$\otimes z_i = \begin{cases} \max(\mathbf{z}) + x_n - x_m & \text{if } z_i = \min(\mathbf{z}) \\ z_i & \text{otherwise.} \end{cases}$$

Let $(\otimes x_1, \ldots, \otimes x_n) = \otimes \mathbf{x}$ and let $f(\otimes \mathbf{x})$ be denoted by $\otimes x$. Given that $|\otimes x_1 - x_n| < |x_1 - x_n|$ and $\otimes x_j = x_j$ for all $j \in N - \{1\}$, Lemma 2 implies that $|\otimes x - x_n| \leq |x - x_n|$. Also, given that $|\otimes x_1 - x_2| > |x_1 - x_2|$, Lemma 2 implies that $|\otimes x - x_2| \geq |x - x_2|$. So, with $|\otimes x - x_n| \leq |x - x_n|$ and $|\otimes x - x_2| \geq |x - x_2|$ and $x_2 < x_n$, it must be true that $\otimes x = x$. Repeating the above argument, we can see that if $\otimes x > \max(\otimes \mathbf{x})$ then we have $\otimes(\otimes x) = \otimes x = x$. Consider the sequence of $n$-tuples $\mathbf{x}$, $\otimes \mathbf{x}$, $\otimes(\otimes \mathbf{x})$, $\otimes(\otimes(\otimes \mathbf{x}))$, and so on. This sequence must then contain an $n$-tuple $\mathbf{x}^*$ such that $x^* \in \{x_1^*, \ldots, x_n^*\}$ or $x_1^* > x^* > x_n^*$. Referring to Case 1 and Case 2 we can see then that it is implied that $f$ is dictatorial. This contradicts $x > x_n$.

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