The Foundations of *Computable General Equilibrium* Theory

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Abstract

A constructive and recursion theoretic analysis of the standard Computable General Equilibrium (CGE) model of economic theory is undertaken. It is shown, contrary to widely expressed views and textbook versions of the CGE model, that the standard CGE model is neither computable nor constructive in the strict mathematical senses.

Key Words: General Equilibrium theory, CGE Models, Mathematical economics, Computability, Constructivity

JEL Classification Codes: C62, C63, D50, D58

1 Introduction

The economic foundations of *Computable General Equilibrium* (**CGE**) models lie in *Uzawa's Equivalence Theorem* ([17], [15],ch. 11, pp. 135-8, [4], p.719, ff); the mathematical foundations are underpinned by *topological fix point theorems* (Brouwer, Kakutani, etc.). The claim that such models are *computable* or *constructive* rests on mathematical foundations of an algorithmic nature: i.e., on recursion theory or some variety of constructive mathematics. It is a widely held belief that CGE models are both *constructive* and *computable*[1]. That the latter property is held to be true of **CGE** models is evident even from the generic name given to this class of models; that the former characterization is a feature of such models is claimed in standard expositions and applications of **CGE** models. For example in the well known, and pedagogically elegant, textbook by two of the more prominent advocates of applied **CGE** modelling in policy contexts, John Shoven and John Whalley ([14]), the following explicit claim is made:

> "The major result of postwar mathematical general equilibrium theory has been to demonstrate the existence of such an equilibrium by showing the applicability of mathematical fixed point theorems to economic models. ... Since applying general equilibrium models to policy issues involves computing equilibria, these fixed point theorems are important: It is essential to know that an equilibrium exists for a given model before attempting to compute that equilibrium. .....
>
> ...
>
> The weakness of such applications is twofold. First, they provide *non-constructive rather than constructive proofs of the existence of equilibrium*; that is, they show that equilibria exist but do not provide techniques by which equilibria can actually be determined. Second, existence per se has no policy significance. .... Thus, fixed point theorems are only relevant in testing the logical consistency of models prior to the models' use in comparative static policy analysis; such theorems do not provide insights as to how economic behavior will actually change when policies change. *They can only be employed in this way if they can be made constructive* (i.e., be used to find actual equilibria). *The extension of the Brouwer and Kakutani fixed point theorems in this direction is what underlies the work of Scarf .... on fixed point algorithms ....*"
>
> ibid, pp12, 20-1; italics added

[1] The references to *constructive* and *computable* are to their strict mathematical senses. The mathematical meaning of *computable* (or *effectively calculable*) is unambiguous in recursion theory (and under the *Church-Turing thesis*). *Constructivity* can be interpreted in a variety of ways, even in strict mathematical senses. However, the **CGE** Model is non-constructive under *any* mathematical interpretation of *constructive* and not only in the most 'puritanical' versions, such as Brouwer's (based on a particular type of intuitionistic logic).

However, in Scarf's classic book of 1973 there is the following characteristically careful caveat to any unqualified claims to *constructivity* of the algorithm he had devised:

> "In applying the algorithm it is, in general, *impossible* to select an ever finer sequence of grids and a convergent sequence of subsimplices. An algorithm for a digital computer must be basically finite and cannot involve an infinite sequence of successive refinements. .... *The passage to the limit is the nonconstructive aspect of Brouwer's theorem*, and we have no assurance that the subsimplex determined by a fine grid of vectors on S contains or is even close to a true fixed point of the mapping."
>
> [12], p.52; italics added

The main goal in this paper is to sort out this and other ambiguities by clarifying the precise roles played by computability and constructivity in the *theory* of **CGE** models. The paper is organized as follows. In the next section the Uzawa equivalence theorem is analysed from the point of view of computability theory. In section 3 the (non-) constructive content of the combinatorial proof of the Brouwer fix point theorem is made explicit and other, related, issues on constructivity in proofs of this theorem are also discussed. Following this, in the brief concluding section, further clarifying remarks on the mathematical and algorithmic foundations of the *theory* of CGE models are made together with some suggestions on going beyond reliance on topological fix point theorems in the proof of equilibrium existence.

2 Uncomputability and Undecidability in the Uzawa Equivalence Theorem

The Uzawa Equivalence theorem is the fulcrum around which the *theory* of **CGE** modelling revolves. This key theorem[2] provides the theoretical justification for relying on the use of the algorithms that have been devised for determining general economic equilibria as fix points using essentially non-constructive topological arguments. The essential content of the theorem is the mathematical equivalence between a precise statement of *Walras' Existence Theorem* (**WET**) and Brouwer's (or any other relevant[3]) Fix-Point Theorem. To study the algorithmic - i.e., computable and constructive - content of the theorem, it is

[2] To the best of my knowledge it is only in Ross Starr's excellent textbook ([15], op.cit) that this fundamental theorem is stated and proved in an illuminating way. A theorem that is crucial not only from a theoretical point of view, but also one that underpins all the applied work that is based on **CGE** models should, surely, be more widely available at the advanced textbook level?

[3] For simplicity, I shall restrict the discussion of the equivalence, in this paper, to that between Walras' Existence Theorem and only Brouwer's Fixed-Point Theorem. The analysis applies, however, *pari passu* to the equivalence with the Kakutani or any other topological fixed point theorem.

necessary to analyse the assumptions underpinning **WET**, the nature of the proof of economic equilibrium existence in **WET** and the nature of the proof of equivalence. By the 'nature of the proof' I mean, of course, the constructive[4] content in the logical procedures used in the demonstrations- whether, for example, the law of double negation or the law of the excluded middle (LEM: *tertium non datur*) is invoked in non-finitary instances. This latter point will be crucial in the next section. Therefore, I shall, first, state an elementary version of **WET** (cf., [17], p. 60 or [15], p. 136).

Theorem 1 *Walras" Existence Theorem (**WET**)*

Let the excess demand function, $X(p) = [x_1(p), ..., x_n(p)]$*, be a mapping from the price simplex,* S*, to the* $\mathbb{R}^N$

commodity space; i.e., $X(p) : S \to \mathbb{R}^N$

where:

i). $X(p)$ *is continuous for all prices,* $p \in S$

ii). $X(p)$ *is homogeneous of degree* 0*;*

iii). $p.X(p) = 0, \forall p \in S$ *(Walras' Law holds:* $\sum_{i=1}^{n} p_i x_i(p) = 0,\ \forall p \in S$*)*[5]

Then:

$\exists p^* \in S$*, s.t.,* $X(p^*) \leq 0$*, with* $p_i^* = 0, \forall i$*, s.t.,* $X_i(p^*) < 0$

The finesse in this half of the equivalence theorem, i.e., that **WET** implies the Brouwer fix point theorem, is to show the feasibility of devising[6] a continuous excess demand function, $X(p)$, satisfying Walras' Law (and homogeneity), from an arbitrary continuous function, say $f(.) : S \to S$, such that the equilibrium price vector implied by $X(p)$ is also the fix point for $f(.)$, from which it is 'constructed'[7].

The question remains, however, of the recursion theoretic status of $X(p)$. Is this function computable for arbitrary $p \in S$? Obviously, if it is, then there is no need to use the alleged constructive procedure to determine the Brouwer fix point (or any of the other usual topological fix points that are invoked in general equilibrium theory and CGE Modelling) to locate the economic equilibrium implied by WET. If it is not, $p.X(p)$ is meaningless from a recursion theoretic point of view for uncomputable $X(p)$.

[4] Since computability theory freely invokes classical logical principles in proof procedures such an analysis is not required for any of the proofs from a recursion theoretic point of view.

[5] As far as possible I attempt to retain fidelity to Uzawa's original notation and structure even although more general formulations are possible. .

[6] I have to seek recourse to words such as 'devise' to avoid the illegitimate use of mathematically loaded terms like 'construction', 'choice', 'choose', etc., that the literature on **CGE** modelling is replete with, signifying, illegitimately, possibilities of meaningful - i.e., algorithmic - 'construction', 'choice', etc. For example, Uzawa, at this point, states: "We *construct* an excess demand function.." (op.cit, p.61; italics added; Starr, at a comparable stage of the proof states: "If we have *constructed* [the excess demand function] cleverly enough..." (op.cit., p.137; italics added). Neither of these claims are valid from the point of view of any kind of algorithmic procedure.

[7] I have placed this use of the word within inverted commas because 'constructive' and 'construction' are supposed to mean something very specific in a mathematical sense, in this paper. See also related remarks in the previous footnote.

The key step in proceeding from a given, arbitrary, $f(.) : S \to S$ to an excess demand function $X(p)$ is the definition of an appropriate scalar:

$$\mu(p) = \frac{\sum_{i=1}^{n} p_i f_i[\frac{p}{\lambda(p)}]}{\sum_{i=1}^{n} p_i^2} = \frac{p.f(p)}{|p|^2} \tag{1}$$

Where:

$$\lambda(p) = \sum_{i=1}^{n} p_i \tag{2}$$

From (1) and (2), the following excess demand function, $X(p)$, is defined:

$$x_i(p) = f_i(\frac{p}{\lambda(p)}) - p_i \mu(p) \tag{3}$$

i.e.,

$$X(p) = f(p) - \mu(p)p \tag{4}$$

It is simple to show that (3) [or (4)] satisfies (i)-(iii) of Theorem 1 and, hence, $\exists p^*$s.t., $X(p^*) \leq 0$ (with equality unless $p^* = 0$). Elementary (non-constructive[8]) logic and economics then imply that $f(p^*) = p^*$. I claim that the procedure that leads to the definition of (3) [or, equivalently, (4)] to determine $p^*$is provably *undecidable*. In other words, the crucial scalar in (1) cannot be defined recursion theoretically (and, *a fortiori*, constructively) to effectivize a sequence of projections that would ensure convergence to the equilibrium price vector.

Clearly, given any $p \in S$, all the elements on the r.h.s of (1) and (2) *seem* to be well defined. However, $f(p)$ is not necessarily computable (nor meaningfully constructive) for arbitrary $p \in S$. Restricting the choice of $f(.)$ to the partial recursive functions may most obviously violate the assumption of *Walras' Law*. Therefore, even from a very elementary (classical) recursion theoretic standpoint it is easy to show the absence of a computable (and constructive) content to Theorem 1. I shall, however, opt for a slightly more direct and formal route to the demonstration of a recursion theoretic infelicity in the implication in Theorem 1. This will be a formal demonstration that it is impossible to devise an algorithm to define (3) [or (4)] for an arbitrary $f(p)$, such that the equilibrium p^* for the defined excess demand function is also the fix point of $f(.)$. If it were possible, then the famous *Halting Problem for Turing Machines* can be solved, which is an impossibility.

Theorem 2 *$X(p^*)$, as defined in (3) [or (4)] above is undecidable; i.e., cannot be determined algorithmically.*

[8] There is an implicit reliance on LEM (*tertium non datur*) in an infintary situation in the proof, too.

Proof. *Suppose, contrariwise, there is an algorithm which, given an arbitrary $f(.) : S \to S$, determines $X(p^*)$. This means, therefore, in view of (i)-(iii) of Theorem 1, that the given algorithm determines the equilibrium p^* implied by **WET**. In other words, given the arbitrary initial conditions $p \in S$ and $f(.) : S \to S$, the assumption of the existence of an algorithm to determine $X(p^*)$ implies that its* halting *configurations are decidable. But this violates the* undecidability of the Halting Problem for Turing Machines. *Hence, the assumption that there exists an algorithm to determine - i.e., to construct - $X(p^*)$ is untenable.* ■

Remark 3 *The algorithmically important content of the proof is the following. Starting with an arbitrary continuous function mapping the simplex into itself and an arbitrary price vector, the existence of an algorithm to determine $X(p^*)$ entails the feasibility of a procedure to choose price sequences in some determined way to check for p^* and to halt when such a price vector is found. Now, the two scalars, μ and λ are determined once $f(.)$ and p are given. But an arbitrary initial price vector p, except for flukes, will not be the equilibrium price vector p^*. Therefore the existence of an algorithm would imply that there is a systematic procedure to choose price vectors, determine the values of $f(.)$, μ and λ and the associated excess demand vector $X(p; \mu, \lambda)$. At each determination of such an excess demand vector, a projection of the given, arbitrary, $f(p)$, on the current $X(p)$, for the current p, will have to be tried. This procedure must continue till the projection for a price vector results in excess demands that vanish for some price. Unless severe recursive constraints are imposed on price sequences - constraints that will make very little economic sense - such a test is algorithmically infeasible. In other words, given an arbitrary, continuous, $f(.)$, there is no procedure - algorithm (constructive or recursion theoretic) - by which a sequence of price vectors, $p \in S$, can be systematically tested to find p^*.*

Remark 4 *In the previous remark, as in the discussion before stating Theorem 2, I have assumed away the difficulties with uncomputable functions, prices and so on. They simply add to complications without changing the nature of the content of Theorem 2.*

One may wonder whether, using the vector field interpretation of Walras' Law, given an initial, arbitrary $p \in S$ and an arbitrary, continuous, $f(.)$, it would be possible to construct a continuous time dynamical system in such a way that it is algorithmically feasible to determine its basin of attraction to locate p^*. The idea is the following. Given Walras' Law, the excess demand function $X(p)$, at any p, is tangent to the price simplex, S. Then, given continuity it is clear that $X(p)$ defines a continuous vector-field on the price simplex. Clearly, therefore, the equilibrium price vector implied by **WET** must lie on this vector-field induced by $X(p)$. The question is, therefore: is it possible to construct a dynamical system consistent with the vector-field interpretation of $X(p)$ whose basin of attraction contains p^*? Put in the language of recursion theory, the question can *equivalently* be posed thus: is it possible to *construct* a Turing Machine equivalent of such a dynamical system such that it halts at a particular configuration? Obviously, theorem 2 precludes this possibility. Perhaps we can ask a slightly more structured question with hopes of an affirmative answer. Is it possible to identify a subclass of algorithms - a subclass of dynamical systems - with the specified property of halting at p^*- as dynamical systems, with the specified property of entering a particular basin of attraction - for computable initial conditions? The following result, stated in the form of a theorem, dashes any hope of an affirmative answer.

Theorem 5 *Let $M \sqsubset A$ be the proper subset of the set of (computable) algorithms that are candidate procedures for determining p^* for given, computable, initial conditions, p and $f(.)$. Then there is no algorithm to decide whether or not any given, arbitrary member $M_k \in M \sqsubset A$.*

Proof. A direct application of *Rice's Theorem.*

Remark 6 *A recent result of da Costa and Doria ([3], in particular Proposition 13 in §6) suggests that it is not even necessary to resort to the roundabout process of first constructing the algorithmic (i.e., Turing Machine) equivalent of a dynamical system so as to use the recursion theoretic version of Rice's Theorem. da Costa and Doria develop a version of Rice's Theorem for a fragment of real analysis in a way that makes it directly applicable to deriving undecidability results for continuous vector fields.*

■

3 Notes on Non-Constructivity in CGE Theory and Modelling

An algorithm, by definition, is a finite object, consisting of a finite sequence of instructions. However, such a finite object is perfectly compatible with 'an infinite sequence of successive refinements' ([12], p. 52), provided a stopping rule associated with a clearly specified and verifiable approximation value is part of the sequence of instructions that characterize the algorithm. Moreover, it is *not* 'the passage to the limit [that] is the nonconstructive aspect of Brouwer's [fix point] theorem' (ibid, p.52)[9]. Instead, the sources of non-constructivity are

[9] In [13], p. 1024, Scarf is more precise about the reasons for the failure of constructivity in the proof of Brouwer's fix point theorem:

> "In order to demonstrate Brouwer's theorem completely we must consider a sequence of subdivisions whose mesh tends to zero. Each such subdivision will yield a completely labeled simplex and, *as a consequence of the compactness* of the unit simplex, there is a convergent subsequence of completely labeled simplices all of whose vertices tend to a single point x^*. (This is, of course, the non-constructive step in demonstrating Brouwer's theorem, rather than providing an approximate fixed point)."

There are two points to be noted: first of all, even here Scarf does not pinpoint quite precisely to the main culprit for the cause of the non-constructivity in the proof of Brouwer's the undecidable disjunctions - i.e., appeal to the *law of the excluded middle* in infinitary instances - intrinsic to the choice of a convergent subsequence in the use of the Bolzano-Weierstrass theorem and an appeal to the *law of double negation* in an infinitary instance during a *retraction*. The latter reliance invalidates the proof in the eyes of the Brouwerian constructivists; the former makes it constructively invalid from the point of view of every school of constructivism, whether they accept or deny intuitionistic logic. Brief notes on these issues are discussed in this section.

To the best of my knowledge, every textbook proof of the Brouwer fix point theorem that proceeds via some variant of Sperner's Lemma, however roundabout, invokes the Bolzano-Weierstrass Theorem[10]. This theorem relies on an undecidable disjunction which is, often, submerged and unconsciously unrecognised as such. The reason, almost invariably, is the wording used to describe the undecidable disjunction. For example, Starr's clear and detailed presentation of the proof of Brouwer's fix point theorem is based on the excellent and almost elementary exposition in [16] (particularly, pp.424-7). There, in turn, the appeal to the Bolzano-Weierstrass theorem is made almost as with a magician's wand[11]:

> "Making [the] assumption [that given any simplex S, there are subdivisions that are arbitrarily fine] we can now finish the proof of Browuer's fixed-point theorem. We take an infinite sequence of subdivisions of S with *mesh*, that is, length of the longest one-dimensional edge, approaching 0. From each subdivision, we *choose* one simplex that carries all labels, and in this simplex we *choose* a single point. We thus have an infinite sequence of points in the original simplex S, and *we can choose a subsequence that converges to a single point*. This point .. is the limit point of the sequence of all

theorem; secondly, nothing in the construction of the algorithm provides a justification to call the value generated by it to be an approximation to x^*. In fact the value determined by Scarf's algorithm has no theoretically meaningful connection with x^* (i.e., to p^*) for it to be referred to as an approximate equilibrium.

[10] Just for ease of reading the discussion in this section I state, here, the simplest possible statement of this theorem:

> **Bolzano-Weierstrass Theorem:** Every bounded sequence contains a convergent subsequence

[11] In the clear and elementary proof of the Brouwer fix point therem given in Starr's textbook (op.cit), the appeal to the Bolzano-Weierstrass theorem is made when proving the KKM theorem (p. 62). In Scarf's own elegant text (op.cit) invoking of this theorem occurs, during the proof of Brouwer's theorem, on p. 51:

> "As the vectors are increasingly refined, a convergent subsequence of subsimplices *may be found*, which tend in the limit to a single vector x^*." (italics added)

Scarf is careful to claim that the required subsequence '*may be found*', but does not claim that it can be found algorithmically. One may wonder: if not found algorithmically, then how?

> vertices of all the simplexes from which the points of the convergent subsequence were originally *chosen.*" ([16], p.427; all italics, except the first one, are added)

The deceptive use of the word '*choose*' in the above description of mathematical processes conveys the impression that the '*choices*', in each case, are *algorithmically* implementable. However, it is only the first use of the word '*choose*' and the implied *choice* - i.e., choosing simplexes from increasingly fine subdivisions - that can be algorithmized constructively. The part that invokes the Bolzano-Weierstrass theorem, i.e., '*Choosing a subsequence that converges to a single point*' - incidentally, this point is the sought after fixed-point of the Brouwer theorem - entails undecidable disjunctions and as long as any proof relies on this aspect of the theorem, it will remain unconstructifiable[12].

An alternative approach to demonstrating the Brouwer fix point theorem is by way of the *non-retraction* theorem (cf. [13], §4)[13]. To the best of my knowledge, every demonstration of the Brouwer fix point theorem via a non-retraction theorem proceeds by way of proof by contradiction and an appeal to the law of double negation in infinitary instances. Moreover, unless the retractions are on highly structured spaces, there is no hope whatsoever of devising computable methods to locate the fix points that are non-constructively shown to exist. Since Brouwer's original proof was along these lines, just before he saw the 'light', so to speak, and began repudiating his theorem for its non-intuitionistic and non-constructive nature, it may be useful to make it known to an economic readership[14].

Brouwer's proof of his celebrated fix point theorem was indirect in two ways: he proved, first, the following:

Theorem 7 *Given a continuous map of the disk onto itself with no fixed points,* $\exists$ *a continuous retraction of the disk to its boundary.*

Having proved this, he then took its *contrapositive:*

[12] Over fifty years ago, when Brouwer returned to the topic of his famous theorem with an *Intuitionist* version of it, he made the trenchant observation that seems to have escaped the attention of mathematical economists:

> "[T]he validity of the Bolzano-Weierstrass theorem [in intuitionism] would make the classical and the intuitionist form of fixed-point therems equivalent." ([2], p.1).

The invalidity of the Bolzano-Weierstrass theorem in any form of constructivism is due to its reliance on the law of the excluded middle in an infinitary context of choices (cf. also [5], pp. 10-12)

[13] Scarf's discussion of the version of this theorem is based on results in [6]. However, Hirsch's proof of the Brouwer fix point theorem in [6] is incorrect (cf. [7]) and, therefore, I shall not discuss the non-constructive aspects *in the connection* between Scarf's combinatorial lemma on labelling a restricted simplex (op.cit, Lemma 3.43) and the non-retraction theorem. But I shall have something to say about the general logical strategy intrinsic to proofs of the existence of fixed points of continuous maps using retractions.

[14] I am, essentially, summarizing ultra-concisely the lucid discussion in [9], pp. 120-7.

Theorem 8 *If there is no continuous retraction of the disk to its boundary then there is no continuous map of the disk to itself without a fixed point.*

Using the logical principle of equivalence between a proposition and its contrapositive (i.e., logical equivalence between theorems 7 & 8) and the law of double negation ($\nexists$ a continuous map with **no** fixed point = $\exists$ a continuous map with a fixed point) Brouwer demonstrated the existence of a fixed point for a continuous map of the disk to itself. This latter principle is what makes the proof of the Brouwer fix point theorem via retractions (or the non-retraction theorem) essentially unconstructifiable. Scarf's attempt to discuss the 'relationship between these two theorems [i.e., between the non-retraction and Brouwer fix point theorems] and to interpret [his] combinatorial lemma [on effectively labelling a restricted simplex] as an example of the non-retraction theorem is incongruous. This is because Scarf, too, like the Brouwer at the time of the original proof of his fix-point theorem, uses the full paraphernalia of non-constructive logical principles to link the Brouwer and non-retraction theorems and his combinatorial lemma[15]

I shall conclude this section with two conjectures:

Conjecture 9 *The non-retraction theorem is unconstructifiable*

To put the next conjecture within the context of Scarf's (constructive) combinatorial lemma (op.cit) some elementary terminological and conceptual preparation is necessary. The unit simplex is given a (restricted) simplical subdivision with t vertices v^j $(j = 1, 2...., n, ..., t)$ such that all but the first n vertices are strictly interior to the simplex and they are labelled with integers, say $\ell(v^j)$, without restriction, except for $\ell(v^j) = j$, $\forall j = 1, 2, ..., n$.

Conjecture 10 $\nexists$ *computable solutions for $g(x) = 0$, for the continuous, piecewise, linear mapping $g(x)$ of the unit simplex into itself*

Where:

$$x = \alpha_1 v^{j_1} + \alpha_2 v^{j_2} + ... + \alpha_n v^{j_n} \tag{5}$$

and: $\alpha_i \geq 0$, $\sum \alpha_i = 1$

These two conjectures summarize the aim of the discussion in this section in the following senses. Non-constructivity pervades the basic underpinnings of **CGE** Theory and hence any modelling based on this kind of theory cannot escape being non-constructive. Moreover, since the mathematical foundations of **CGE** Theory, relying as it does on topological fix point theorems, are essentially unconstructifiable, any numerical exercise or application of **CGE** modelling will have inevitable and unavoidable *ad hoc* aspects, entirely divorced from theory. I am not however led to believe that the impressive applied work based on **CGE**

[15] Scarf uses, in addition, proof by contradiction where, implicitly, LEM (*tertium non datur*) is also invoked in the context of an infinitary instance (cf. [13], pp. 1026-7).

modelling is entirely worthless. I subscribe, in fact, to the enlightened view, eloquently expressed by Sydney Afriat in another, not unrelated context[16]:

> "Practice can stand without theory; put another way, it constitutes its own theory. All that is accomplished by expressing anything in practice in terms of ..... theory is to show it does not conflict with that theory. Practice is consistent with a poor specialization of that theory, so poor that nothing is aided by bringing it in." ([1], p.3)

4 Concluding Notes

In his characteristically perceptive review of the important papers by Pour-El and Richards ([10]), Kreisel ([8], p.900) observed:

> "The [papers by Pour-El and Richards] add to the long lists of operations μ in analysis with some recursive 'input' I for which no output in $\mu(I)$ is recursive. ... Familiar examples are provided by (i) Brouwer's fixed point theorem in dimension >1 .... where I ranges over (i) continuous maps of the unit cicle into itself ... and where $\mu(I)$ is the set of (i) fixed points.... ."

Somehow this kind of important observation seems to have escaped the attention of mathematical economists and, more importantly, it is not part of the basic knowledge of CGE modelers. Had the latter been aware of this important but elementary fact - that Brouwer's fixed point theorem entails, for recursive inputs, only non-recursive reals as outcomes - they may have attempted to adapt the underlying theory of the CGE model to computability assumptions *ab initio.*

Although the substantive content of the results and discussion in this paper may have a pessimistic flavour the aim is, in fact, positive. The implicit aim has been to dissect the computable and constructive content in the foundations of **CGE** theory and modelling. The problem in such an exercise has been most appropriately pointed out by Richman:

> "Even those who like algorithms have remarkably little appreciation of the thoroughgoing algorithmic thinking that is required for a constructive proof. ...... I would guess that most realist mathematicians are unable even to recognize when a proof is constructive in the intuitionist's sense.
>
> It is a lot harder than one might think to recognize when a theorem depends on a nonconstructive argument. One reason is that proofs are rarely self-contained, but depend on other theorems whose proofs depend on still other theorems. These other theorems have often been internalized to such an extent that we are not aware whether or not nonconstructive arguments have been used, or must be used, in their proofs. Another reason is that the law of excluded

[16] In the context of index number theory and its foundations in utility analysis.

> middle [LEM] is so ingrained in our thinking that we do not distinguish between different formulations of a theorem that are trivially equivalent given LEM, although one formulation may have a constructive proof and the other not." ([11], p.125)

In a sense, this paper substantiates the perceptive points made by Fred Richman in the above quote. Very few economists, even the very competent mathematical economists, pause to wonder whether standard theorems that are routinely invoked are numerically meaningful. Thus the computable content of the Heine-Borel or the Hahn-Banach theorem or the constructive content of the Bolzano-Weierstrass theorem are never raised as issues even though far reaching policy conclusions depend on applied models that invoke them, albeit implicitly.

The positive aim in the task I undertook in this paper is the following. I have no doubt that fix point theorems are important because economic analysis needs equilibrium benchmarks. Such benchmarks are best formalized and derived as fix points, particularly in dynamic contexts. The question - and this is where the pessimistic flavour with which this paper seems to have been lined may camouflage the positive aim - is whether it is necessary or advisable to model economic fundamentals in terms of the mathematics of real analysis and its handmaiden, topological analysis - especially topological fix point theorems. Is there no alternative, in view of the admirable success of **CGE** modelling in many applied contexts? I think there is and it is to formalize and model economic fundamentals, *ab initio*, in terms of constructive or computable mathematics. I do not think any aspect of the proverbial baby will have to be thrown out with the bathwater that is real analysis, particularly that part of the bathwater that contains the non-constructive and non-computable elements. By the latter I mean, of course, any part of economic theory that depends on topological fix point theorems. The reason is my final conjecture:

Conjecture 11 *every result in economic theory that depends on topological - non-constructive and uncomputable - fix point theorems can be derived, with imperceptible change in content, with recursion theoretic fix point theorems that are also constructive.*

I choose to end with this unnecessarily vague and obviously bold conjecture to pose it as a counterpoint to an equally bold one with which Starr ends his pedagogically lucid discussion of Uzawa's Equivalence Theorem (but cf. also [4], pp. 919-720):

> "What are we to make of the Uzawa Equivalence Theorem? It says that use of the Brouwer Fixed-Point Theorem is not merely one way to prove the existence of equilibrium. In a fundamental sense, it is the only way. Any alternative proof of existence will include, inter alia, an implicit proof of the Brouwer Theorem. Hence this mathematical method is essential; one cannot pursue this branch of economics without the Brouwer Theorem." ([15], p.138)

If Starr is correct about *his* conjecture, it would appear that economic theory is condemned to non-constructive and uncomputable existence and, hence, to numerical and applied work that will always be divorced from theory in uncomfortable ways.

My conjecture suggests that there are non-topological fix-point theorems whose constructive and computable content patch the wedge that divides theory and practice. The implementation of a research program to realize this conjecture is not a simple task. That it is realizable is, on the other hand, quite clear - at least to me. There are, after all, many different kinds of mathematics at the disposal of the mathematically inclined economist and to have had the fate of economic theory 'locked into' one, non-constructive, uncomputable, kind may, after all, have been entirely due to an accident of history.

The main reason for my conviction about the realizability of such a research program is economic: it appears to me that almost all the more important and fundamental theorems of economic theory were first derived on the basis of unalloyed economic intuition, before being given formal respectability in terms of mathematical dressing. Not least of these was the question of equilibrium existence, for which the powerful intuition of Adam Smith, Léon Walras, Alfred Marshall, Friedrich Hayek and a host of other distinguished economists are a testimony.

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