

## RESEARCH ARTICLE

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## Magnetotelluric tensors, electromagnetic field scattering and distortion in three-dimensional environments

## Key Points:

- An apparent resistivity tensor allows visualization of MT data suited for geological interpretation
- A wave number tensor allows the computation of apparent horizontal and vertical wave numbers for MT
- A phase tensor and a distortion tensor, each with six elements, quantify static EM field distortions

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**Abstract** This paper describes how subsurface resistivity distributions can be estimated directly from the magnetotelluric (MT) tensor relationship between electric and magnetic fields observed on a three-dimensional (3-D) half-space. It presents an inhomogeneous plane wave analogy where relationships between horizontal electric and magnetic fields, and an apparent current density define an apparent resistivity tensor constructed from a quadratic function of the MT tensor. An extended-Born relationship allows the electric field to be normalized with respect to an apparent background current density. The model is generalized by including the vertical magnetic field in a 3 by 3 MT response tensor. A complex apparent wave number tensor, constructed from this tensor, has eigenvalues which, using the plane wave analogy, are the vertical wave numbers associated with the eigenpolarizations of propagating waves in the model half space. The elements associated with the vertical magnetic field transfer function define the horizontal wave numbers. An extended 3 by 3 phase tensor contains four elements of the conventional 2 by 2 phase tensor and two elements associated with the vertical magnetic transfer function. The extended phase tensor and a single real distortion tensor with six independent elements can be used to quantify static electric and magnetic field distortions. The approach provides a theoretical basis for visualization and migration of MT data, in comparison with results from other electrical and EM techniques, a starting point for constrained 3-D inversions, and an assessment of results with other geophysical and geological data.

## 1. Introduction

The frequency-dependent magnetotelluric (MT) response tensor, or MT apparent velocity tensor,  $\mathbf{Z}_h$  ( $\text{m s}^{-1}$ ) is the fundamental parameter estimated from the relationship  $\mathbf{E}_h = \mathbf{Z}_h \mathbf{B}_h$  between the horizontal electric field and magnetic induction,  $\mathbf{E}_h$  ( $\text{V m}^{-1}$ ) and  $\mathbf{B}_h$  (T), respectively, at the surface of the Earth. The tensor contains the subsurface conductivity or resistivity information that is sought in MT surveys. The representation of this information as an apparent resistivity began [e.g., Vozoff, 1986] when interpretation of the subsurface resistivity was restricted to one-dimensional (1-D) structures. In this situation, the Cagniard complex apparent resistivity,  $\rho_a$  ( $\Omega\text{-m}$ ) as a function of frequency,  $\omega$  is defined as  $\rho_a(\omega) = (\mu/\omega)|Z_h(\omega)|^2$ , where  $\pm Z_h$  are the two off-diagonal tensor elements of equal value and opposite sign, diagonal elements are zero, and  $\mu$  is the magnetic permeability, assumed to be that of free space. As the MT technique evolved to consider 3-D subsurface resistivity structures, there was no concomitant development of the concept of apparent resistivity; it was extended by computing the square of each of the four different elements,  $Z_{ij}$  of  $\mathbf{Z}_h$  so  $\rho_{a,ij} = (\mu/\omega)|Z_{ij}|^2$  and  $\varphi_{ij} = \tan^{-1}[\text{Im}(Z_{ij})/\text{Re}(Z_{ij})]$ ,  $i, j = 1, 2$ , where  $\rho_{a,ij}$  is the magnitude and  $\varphi_{ij}$  is the phase of the apparent resistivity associated with  $Z_{ij}$ .

These definitions for apparent resistivity are useful for quality control and presentation of MT data before inversions, helpful for semiquantitative discussion of subsurface resistivity structure beneath the MT site, and give first approximations to the resistivity depth distribution in special circumstances where the diagonal elements are close to zero. Their relationships with 3-D resistivity structures, where the diagonal elements are generally nonzero, are less intuitive.

The concept of apparent resistivity in other electrical and electromagnetic techniques evolved differently. In DC electrical resistivity and active source electromagnetic (EM) techniques, apparent resistivity attempts to account for the source strength and source receiver geometry that give rise to a nonuniform applied electric current density distribution [e.g., Spies and Frischknecht, 1991]. In a series of papers [Bibby, 1986; Caldwell and Bibby, 1998; Caldwell et al., 2002], apparent resistivity tensors were formulated to normalize electric field measurements with respect to a uniform background current density in DC electrical resistivity, and controlled source time domain and frequency domain techniques, respectively. The advantages of their approach are

that coordinate invariant apparent resistivities derived from these tensors are, to a good approximation, independent of source geometry and EM field polarization, are well behaved over a wide range of delay times and frequencies, and the invariants of the electric field apparent resistivity tensor provide good images of the 3-D resistivity distribution and the starting point for geophysical and geological interpretations.

This paper demonstrates how a complex apparent resistivity tensor, similar to tensors in DC and active source EM techniques, can be constructed from natural source MT magnetic (horizontal and vertical fields) and horizontal electric fields at the surface of a half-space containing a 3-D resistivity distribution. The apparent resistivity tensor allows the presentation of all data contained in the MT response tensor in a form suited for interpretations on maps and in pseudo sections.

The paper is organized into four subsequent sections. Section 2 uses an analogy for the propagation of plane EM waves through a homogeneous, dissipative, and anisotropic resistivity model half-space to give a solution for the governing equations that result in a frequency-dependent quadratic tensor function of  $\mathbf{Z}_h$ . The solution defines an apparent current density as a function of  $\mathbf{Z}_h$  and the observed magnetic field, and a complex apparent resistivity tensor relating the apparent current density with the observed electric field. Section 3 infers a spatially variable background apparent resistivity distribution and a linear scattering operator that provide an extended-Born approximation for the apparent background EM fields and the normalization of the observed electric field with the apparent background current density. Section 4 generalizes the model by developing a second-rank, three-dimensional (3 by 3) MT response tensor,  $\mathbf{Z}$  that incorporates the vertical magnetic transfer function. It introduces an apparent wave number tensor characterizing an obliquely propagating plane wave in a 3-D resistivity environment. Section 5 uses the relationships between the EM fields and the apparent current density to account for static electric and magnetic field distortions using a single real distortion tensor. It extends the conventional phase tensor analysis using the 3 by 3 MT tensor from section 4 and offers a strategy for interpreting "3-D/3-D" MT data acquired where distorting inhomogeneities are embedded within a 3-D resistivity distribution.

## 2. Apparent Resistivity Tensor

This paper recognizes that MT fields at typical frequencies are governed by the pre-Maxwell Equations within a quasi-uniform (nearly plane wave), near-field of ionospheric sources and are described by a parabolic diffusion equation rather than a hyperbolic wave equation [Weidelt and Chave, 2012]. Nevertheless, it presents a mathematical solution for MT fields in 3-D environments which can be described by terminology associated with propagating plane EM waves; the a posteriori justification is that it provides concepts that are useful for MT interpretation.

The solution is based on relationships between surface horizontal EM fields, which can be equated to apparent resistivity properties of an anisotropic Earth. Solutions for natural MT fields in anisotropic half-spaces are available for 1-D forward models [Mann, 1965; O'Brien and Morrison, 1967; Reddy and Rankin, 1971; Loewenthal and Landisman, 1973; Abramovici, 1974; Dekker and Hastie, 1980; Pek and Santos, 2002; Yin, 2006], 2-D forward models [Pek and Verner, 1997], and 3-D forward models [Weidelt, 1999]. Chew [1999] discusses the propagation of EM waves in anisotropic media.

The solution is based on Reilly [1979] which used a tensor notation for an arbitrary coordinate system. The solution presented in Appendix A of this paper uses a Cartesian tensor notation with the Einstein summation convention so it can be more easily translated into conventional MT matrix algebraic forms. The model for the MT fields assumes that a plane wave originating in an upper insulating half-space is vertically incident on the horizontal boundary of a uniformly anisotropic lower half-space. Starting from the Maxwell Equations, a second-order differential equation for the electric field vector  $\mathbf{E} = (E_x \ E_y \ E_z)^{\text{tr}}$  is obtained (A9) using exponential functions of spatial variables,  $\mathbf{r}$  of the form  $\mathbf{E} = \psi E e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ , typically used in time harmonic solutions for plane waves [e.g., Jackson, 1999, p. 296], where the superscript tr represents the transpose,  $\psi$  defines a set of orthogonal unit vectors,  $E$  is complex amplitude, and  $\mathbf{k}$  is a complex wave vector in a dissipative, conducting medium, whose imaginary part is a measure of the dissipation or attenuation in the medium. The solution contains a quadratic function of an apparent complex wave number tensor which can be equated to an apparent complex, conductivity tensor characterizing the electrical properties of the model half-space and approximating actual electrical properties in a volume beneath the measurement site. The quadratic function

is a second-rank, 3 by 3 tensor  $\gamma$  ( $\text{m}^{-2}$ ). In order to express  $\gamma$  in terms of the observed MT response tensor which operates in the  $x$ - $y$  plane on the surface of Earth, the derivation proceeds with constraints on  $\gamma$  and the assumption that  $E_z=0$ . This gives a second-order differential equation in only the vertical derivatives of the electric field vector (A13) which, by assuming that these can be expressed in terms of exponential functions of depth, results in a solution for  $\gamma$  with arbitrary electric fields (A17). The constraints on  $\gamma$  include  $\gamma_{zx} = \gamma_{zy} = 0$  implying that the vertical magnetic field is absent so the apparent propagation of the inhomogeneous plane wave is vertically downward through the anisotropic half-space. The properties of this half-space, by definition, do not vary in the horizontal direction so the horizontal EM fields do not vary as a function of horizontal variables. I relax the constraints on  $\gamma_{zx}$  and  $\gamma_{zy}$  in section 4 to allow for oblique downward propagation through the half-space. The constraint  $\gamma_{xz} = \gamma_{yz} = 0$  maintains the relationship between the horizontal electric and magnetic fields defined by the conventional 2 by 2 MT response tensor.

The tensor,  $\gamma$  therefore reduces to a 2 by 2 tensor,  $\gamma_h$  defining relationships between horizontal electric and magnetic fields with the upper left elements  $\gamma_{xx}$ ,  $\gamma_{xy}$ ,  $\gamma_{yx}$ , and  $\gamma_{yy}$ . The tensor is a quadratic function (A22) of the observed apparent slowness tensor  $\mathbf{Y}_h = \mathbf{Z}_h^{-1}$  ( $\text{s m}^{-1}$ ) and becomes:

$$\gamma_h = -\omega^2 \det(\mathbf{Y}_h) \mathbf{Z}_h^{\text{tr}} \mathbf{Y}_h = -\omega^2 \begin{pmatrix} Y_{yy}Y_{xx} - Y_{yx}Y_{yx} & Y_{yy}(Y_{xy} - Y_{yx}) \\ Y_{xx}(Y_{yx} - Y_{xy}) & Y_{xx}Y_{yy} - Y_{xy}Y_{xy} \end{pmatrix} \quad (1)$$

where  $\det$  represents the determinant of a matrix.

The apparent complex conductivity tensor  $\sigma_h$  ( $\text{S m}^{-1}$ ) can be defined from  $\gamma_h = -i\omega\mu \sigma_h$ , a generalization of the well-known relationship,  $k^2 = -i\omega\mu\sigma$  between wave number and conductivity in a uniform isotropic medium. Unlike a symmetric conductivity tensor representing a physical property at a point in a real material, the apparent conductivity tensor is nonsymmetric; it represents the frequency-dependent, complex electrical properties of a model half-space below the observation point that will give the observed MT response tensor. A nonsymmetric apparent conductivity tensor is required to represent the MT response in a situation where the subsurface conductivity is not symmetrically distributed around the measurement location. The real part of the apparent conductivity tensor represents the gross dissipative behavior of the model half-space. The imaginary part represents the effects on the phases from conductivity heterogeneities (e.g., due to electric fields from charges on boundaries); the imaginary part is *not* a consequence of the dielectric properties of the subsurface medium.

The apparent complex resistivity tensor,  $\rho_h$  ( $\Omega\text{-m}$ ) is defined as

$$\rho_h = \sigma_h^{-1} = (i\mu/\omega) \det(\mathbf{Z}_h) \mathbf{Z}_h \mathbf{Y}_h^{\text{tr}} \quad (2)$$

Using Ohm's Law and defining  $\mathbf{J}_h$  as the apparent current density vector, at the surface of the uniform anisotropic model half-space, which results in the observed electric and magnetic fields at the same surface point, then

$$\mathbf{E}_h = (i\mu/\omega) \det(\mathbf{Z}_h) \mathbf{Z}_h \mathbf{Y}_h^{\text{tr}} \mathbf{J}_h \quad (3)$$

The magnetic field is therefore

$$\mathbf{B}_h = (i\mu/\omega) \det(\mathbf{Z}_h) \mathbf{Y}_h^{\text{tr}} \mathbf{J}_h \quad (4)$$

so providing an equivalent definition of the apparent current density:

$$\mathbf{J}_h = \left(-i\omega/\mu\right) \det(\mathbf{Y}_h) \mathbf{Z}_h^{\text{tr}} \mathbf{B}_h \quad (5)$$

The two sets of four real and four imaginary parts of the apparent resistivity tensor can be treated as independent, asymmetric tensors, and each may be represented graphically as an ellipse with axes representing the principal axes of the tensor. Each tensor is associated with three coordinate invariants, which correspond to the lengths of the major and minor axes of the ellipse and a skew angle which measures the amount by which the principal axes have been rotated with respect to a corresponding symmetric tensor. The principal axes of the real and imaginary tensors are parallel only in situations with symmetry, e.g., where the resistivity distribution is 2-D. In 3-D environments, the seven coordinate invariants that characterize the MT response [Szarka and Menville, 1997] include a "mixed" invariant which is most easily

interpreted as the angle between the major axes of the two ellipses. The remaining coordinate-dependent parameter is the orientation of the major axis of one of the ellipses.

The expressions for the representation and invariants of a single ellipse have been developed in a series of papers [Bibby, 1986; Caldwell and Bibby, 1998; Caldwell et al., 2002; Caldwell et al., 2004; Bibby et al., 2005]. Booker [2014] provides algorithms for the display of phase tensor ellipse geometry that avoid ambiguities in the original formulations and summarizes an approach [Efron, 1982] to the estimation of uncertainties in the phase tensor ellipse based on the full covariance matrix of the observed MT response tensor. A similar approach will be needed for the propagation of errors into the MT apparent resistivity tensor and how these can be visualized along with the parameters defining the geometry of the real and imaginary ellipses.

There are several examples of ellipse representations of apparent resistivity tensors computed from theoretical responses of 3-D subsurface resistivity models for (a) bipole-dipole direct current electrical resistivity [Bibby and Hohmann, 1993], (b) long-offset time domain electric field data [Caldwell and Bibby, 1998], (c) controlled-source MT data [Caldwell et al., 2002], and (d) natural source MT data [Caldwell and Bibby, 1996]. Weckmann et al. [2003] also present an example for the real part of the apparent resistivity tensor from observed MT data. The ellipses calculated from the 3-D models show that the apparent resistivity tensors from the four techniques all behave in a similar manner. They provide a compact presentation of all the data, are independent of the coordinate system chosen to measure the EM fields, are physically intuitive, and facilitate geological interpretations in simple 3-D resistivity environments. They may also be the best way to present all data from field experiments where a combination of two or more electrical and EM techniques is deployed to guarantee the resolution of subsurface structures across many spatial scales.

The underlying reason for the similarity in the behavior of MT apparent resistivity tensors with the behavior of apparent resistivity tensors from the other techniques is explored in the following section.

### 3. Background and Scattered Fields

The computation of EM fields at a point on the surface or within a 3-D conductivity distribution may be derived from the Maxwell Equations using the volume integral equation method [e.g., Wannamaker, 1991; Avdeev, 2005]. The solution, a Fredholm integral equation (scattering equation) of the second kind for the total electric field, is equal to the sum of the background (normal) field in a background conductivity distribution and the anomalous electric field scattered from 3-D conductivity anomalies. The background conductivity is typically assumed to be a 1-D conductivity variation with depth and, in practice, is estimated from Niblett-Bostick transformations [Zhdanov et al., 1996] or 1-D inversions [Zhdanov et al., 2000] using the frequency-dependent MT response tensor. A numerical solution can then be obtained in operator form using a Liouville-Neumann series expansion [e.g., Chew, 1999] representing multiple scattering from the anomalous conductivity distribution. If the initial approximation of the scattered electric field is zero, the truncation of this series after the first term leads to the first-order Born approximation. Habashy et al. [1993] introduced an extended-Born approximation using a scattering tensor that allows the anomalous electric field at a point within heterogeneous conductivity to be linearly related to the background electric field at that point, formally equivalent to a second-order Neumann series expansion with truncation after the second term [Zhdanov and Fang, 1996]. These concepts are used below to construct an extended-Born MT scattering tensor and apparent background electric and magnetic fields without any restrictions on the magnitude of anomalous apparent conductivity deviations from the apparent background conductivity.

The apparent background and anomalous conductivities at a frequency  $\omega$  may be inferred from the apparent conductivity tensor. The apparent background conductivity must be invariant to rotations about the z axis if it is to function in a manner similar to a 1-D background conductivity inferred from observed MT data. In this paper, we make use of the rotationally invariant trace to decompose the apparent conductivity tensor as follows:

$$-i\omega\mu\sigma_h = -i\omega\mu\sigma_0(\mathbf{I}_2 + \mathbf{S}) = \gamma_0(\mathbf{I}_2 + \mathbf{S}) = \gamma_h \quad (6)$$

$$\sigma_0(\omega) = [\frac{1}{2}\text{trace}(\sigma_h)] = (i/\omega\mu)k_0^2 = \rho_0^{-1} \quad (7)$$

where  $\sigma_0(\omega)$  is the frequency-dependent apparent background conductivity,  $k_0$  is a wave number (see section 4) associated with the apparent, background, horizontal electric and magnetic fields,  $\mathbf{E}_0$  and  $\mathbf{B}_0$ , respectively (i.e., the electric and magnetic fields excited in the absence of conductivity inhomogeneities),

$\rho_0$  is the apparent background resistivity,  $\mathbf{I}_2$  is the 2 by 2 identity matrix and  $\mathbf{S}$  is a scattering tensor, representing the effects of scattering from the anomalous conductivity structure. The scattering tensor is easily estimated from the observed MT response tensor via (1) and (6), and the background conductivity tensor as a function of depth may be obtained via 1-D inversions of  $\sigma_0(\omega)$ .

The Neumann series for the scattered electric fields may be written as

$$\mathbf{E}_s^{(N)} = \sum_{n=0}^N \mathbf{S}^n \mathbf{E}_0 \quad (8)$$

where subscript  $s$  refers to scattered fields, and superscript  $n$  refers to the  $n$ th power of the scattering operator. The classical (first-order) Born approximation assumes that the scattered fields in the anomalous conductivity distribution are zero, i.e.,  $\mathbf{E}_s^{(0)} = 0$ , and truncates the Neumann series at  $n = 1$ :

$$\mathbf{E}_s^{(1)} = \mathbf{S} \mathbf{E}_0 \quad (9)$$

The  $N$ th iteration of the Neumann series is the sum of the terms in (8). The matrix representing  $\mathbf{S}$  has the property that  $\mathbf{S}^2 = -\det(\mathbf{S})\mathbf{I}_2$  ( $\mathbf{S}$  is involutory to within a scale factor). Substituting for  $\mathbf{S}^2$  gives the total electric field:

$$\begin{aligned} \mathbf{E}^{(N)} &= \mathbf{E}_0 + \mathbf{S} \mathbf{E}_0 + (-\det(\mathbf{S})) \mathbf{E}_0 + (-\det(\mathbf{S})) \mathbf{S} \mathbf{E}_0 + (-\det(\mathbf{S}))^2 \mathbf{E}_0 + \dots \\ &= \left\{ \sum_{n=0}^N (-\det(\mathbf{S}))^n + \mathbf{S} \sum_{n=0}^N (-\det(\mathbf{S}))^n \right\} \mathbf{E}_0 \end{aligned} \quad (10)$$

In the limit  $N \rightarrow \infty$

$$\sum_{n=0}^N (-\det(\mathbf{S}))^n = (1 + \det(\mathbf{S}))^{-1} \quad (11)$$

resulting in an infinite Born series approximation:

$$\mathbf{E}^{(\infty)} = \left\{ \frac{\mathbf{I}_2 + \mathbf{S}}{1 + \det(\mathbf{S})} \right\} \mathbf{E}_0 \quad (12)$$

which can be compared with the second-order Born approximation from (10):

$$\mathbf{E}^{(2)} = \mathbf{E}_0 + \mathbf{E}_s^{(2)} = (\mathbf{I}_2 + \mathbf{S} - \det(\mathbf{S})) \mathbf{E}_0 \quad (13)$$

The observed electric field  $\mathbf{E}_h = \mathbf{E}^{(\infty)}$ , so (12) demonstrates that  $\mathbf{E}_h$  is linearly related to the apparent background electric field  $\mathbf{E}_0$  and can therefore be regarded as an extended-Born approximation for  $\mathbf{E}_0$ . The apparent background current density is given by  $\mathbf{E}_0 = \rho_0 \mathbf{J}_0$  and can be regarded as the apparent current density in the absence of scattering from resistivity inhomogeneities embedded in an isotropic half-space. The background magnetic field is  $\mathbf{B}_0 = \mathbf{Y}_0 \mathbf{E}_0$  where the background slowness tensor is

$$\mathbf{Y}_0 = (l/2\omega) k_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (14)$$

The relationships between the observed electric field and apparent current density distributions are given by

$$\mathbf{J}_h = \left\{ \frac{(\mathbf{I}_2 + \mathbf{S})^2}{1 + \det(\mathbf{S})} \right\} \mathbf{J}_0 = \frac{(1 - \det(\mathbf{S}))}{(1 + \det(\mathbf{S}))} \mathbf{J}_0 + \frac{2\mathbf{S}}{(1 + \det(\mathbf{S}))} \mathbf{J}_0 \quad (15)$$

$$\mathbf{E}_h = \rho_0 \left\{ \frac{\mathbf{I}_2 + \mathbf{S}}{1 + \det(\mathbf{S})} \right\} \mathbf{J}_0 \quad (16)$$

Equation (16) is therefore an extended-Born normalization of the electric field with respect to a uniform background current density in an isotropic half-space similar to the practice in DC electrical resistivity and active source EM techniques for the definition of apparent resistivity [Caldwell *et al.*, 2002, and references therein]. It summarizes the theoretical basis for the similarity in the behavior of MT apparent resistivity tensors with that from the other techniques. All parameters in (16) can be estimated directly from the observed MT response tensor.

An extended-Born (or quasi-analytical) approximation has also been used to reduce computation time in volume integral equation-based inversions of 3-D frequency domain EM data [Zhdanov *et al.*, 1996;

Zhdanov et al., 2000; Tseng et al., 2003; Song and Liu, 2004; Gribenko and Zhdanov, 2007]. One advantage of these methods compared with other 3-D forward modeling methods is that they require discretization only in the anomalous domain so the selection of an appropriate background resistivity distribution is important. A variable background and small anomalous resistivity distribution improves the accuracy of the forward model, the speed of calculation of Fréchet derivatives and the efficiency of an inversion procedure [Gribenko and Zhdanov, 2007]. In this paper, the calculation in (6) of an apparent variable background resistivity  $\rho_0(\mathbf{r}_h, \omega)$ , where  $\mathbf{r}_h$  is the horizontal location of the sounding on the surface, can be used for any 3-D resistivity problem, ranging from discrete conductive heterogeneities in a 1-D half-space often used in mining geophysics to lithospheric scale MT tectonic studies.

#### 4. Three-Dimensional MT Tensors

If a plane wave is incident from a nonconducting atmosphere onto the surface of a uniform, arbitrarily anisotropic, i.e., finite strike, dip, and slant [Pek and Santos, 2002], conductive half-space, the induced current flow is horizontal at all points on the surface, the vertical magnetic field is zero but the vertical electric field is not necessarily zero [Yin, 2006; Heise et al., 2006]. Therefore, the restrictions of the model (section 2 and Appendix A), are only strictly true for a 1-D isotropic or an azimuthally anisotropic (dip and slant are zero) half-space, and in the E-polarization strike direction of a 2-D resistivity distribution. The theory leading to the apparent resistivity tensor in (2) already reproduces the conventional Cagniard apparent resistivities in situations where the subsurface structure is 1-D or 2-D, and provides a simple generalization to the characterization of apparent resistivities over 3-D subsurface structures (rather than merely squaring the elements of the impedance tensor). This section relaxes the constraint of apparent vertical propagation to account for all magnetic field components observed at Earth's surface. Its practical application is evident in section 5.

Consider a second-rank, three-dimensional MT tensor  $\mathbf{Z}$  constructed to relate  $\mathbf{E} = (E_x \ E_y \ 0)^{tr}$  and  $\mathbf{B} = (B_x \ B_y \ B_z)^{tr}$ ; i.e.,  $E_z$  is assumed to be zero but  $B_z$  is not necessarily so:

$$\mathbf{Z} = \begin{pmatrix} Z_{xx} & Z_{xy} & Z_{xz} \\ Z_{yx} & Z_{yy} & Z_{yz} \\ -T_x Z_{zz} & -T_y Z_{zz} & Z_{zz} \end{pmatrix} \quad (17)$$

As a consequence of assuming that  $E_z = 0$  at the surface of the Earth,  $B_z$  can be predicted from the horizontal magnetic field using the observed vertical magnetic transfer function  $\mathbf{T} = (T_x \ T_y)$ . Away from lateral resistivity boundaries,  $E_z \ll E_h$  so  $E_z \sim 0$  at the surface is a reasonable constraint. The plane wave analogy becomes one of oblique downward propagation with all three magnetic field components and two horizontal electric field components.

As there is a linear relationship between  $B_z$  and the horizontal magnetic field components,  $Z_{xz}$  and  $Z_{yz}$  may be set to zero, resulting in  $\gamma_{xz} = \gamma_{yz} = 0$  and the preservation of relationships among  $E_x$ ,  $E_y$ ,  $B_x$ , and  $B_y$  in the MT response tensor.  $Z_{zz}$  is indeterminate so, for reasons discussed below, is chosen to be  $\det(\mathbf{Z}_h)^{1/2}$  ( $\text{m s}^{-1}$ ) in (19) and (24). The 3-D tensors,  $\mathbf{Z}$  and  $\mathbf{Y}$ , and the resulting apparent resistivity tensor (18) all have upper left submatrices identical to the equivalent 2 by 2 matrices.

$$\rho = (i\mu/\omega) \det(\mathbf{Z}_h) \mathbf{Z} \mathbf{Y}^{tr} \quad (18)$$

This is not the case for the 3 by 3 tensor:

$$\begin{aligned} \gamma &= -\omega^2 \det(\mathbf{Y}_h) \mathbf{Z}^{tr} \mathbf{Y} \\ &= \begin{pmatrix} \gamma_h(1,1) + \delta\gamma_{11} & \gamma_h(1,2) + \delta\gamma_{12} & \omega^2 T_x \det(\mathbf{Y}_h) \\ \gamma_h(2,1) + \delta\gamma_{21} & \gamma_h(2,2) + \delta\gamma_{22} & \omega^2 T_y \det(\mathbf{Y}_h) \\ \delta\gamma_{31} & \delta\gamma_{32} & -\omega^2 \det(\mathbf{Y}_h) \end{pmatrix} \end{aligned} \quad (19)$$

The  $\delta\gamma_{ij}$  terms are functions of  $Z_{zz}$ ,  $\mathbf{T}$ , and  $\mathbf{Y}_h$  and can introduce significant differences between the corresponding elements of the upper left submatrix of  $\gamma$  and  $\gamma_h$ . The upper left submatrix of  $\sigma$  is therefore also different from  $\sigma_h$ .

The apparent current density vector,  $\mathbf{J}$  has the same horizontal components as  $\mathbf{J}_h$  and a vertical component responsible for the vertical magnetic field:

$$\mathbf{J} = \left( -i\omega / \mu \right) \det(\mathbf{Y}_h) \mathbf{Z}^{tr} \mathbf{B} \quad (20)$$

#### 4.1. Apparent Wave Number Tensor

For a quasi-uniform MT source field, subject to approximations of spatial structure explained in *Weaver* [1994, p. 68] and *Weidelt and Chave* [2012, pp. 123–124], the spatial variations in surface EM fields, and therefore the tensor relationship between them, result from lateral variations in subsurface resistivity. The MT response (and apparent slowness) tensor can be estimated only if this source field has two linearly independent polarizations which excite subsurface resistivity structures differently [e.g., *Weidelt and Chave*, 2012]. The magnetic field for one source polarization is not necessarily orthogonal to its associated electric field. In the presence of 3-D resistivity distributions, the surface electric field is generally elliptically polarized if the magnetic field is linearly polarized. Where a linearly polarized magnetic field results in a linearly polarized electric field, the diagonal terms of the MT tensor are zero, and the orientation of the magnetic field defines either the strike direction of a 2-D resistivity structure or of an anisotropic 1-D structure where  $B_z = 0$ .

However, it is possible for two linearly polarized waves, each with orthogonal electric and magnetic fields, to propagate through the subsurface and retain their original polarization (eigenstates). The MT tensor  $\mathbf{Z}_h$  at the surface of a 3-D resistivity distribution contains these two eigenstates [*Eggers*, 1982]. For each state, the horizontal electric field eigenvector represents the same polarization ellipse as that of the horizontal magnetic field, scaled and phase shifted by its complex, scalar eigenvalue and rotated about the vertical axis through 90°; the orientations of the two polarization ellipses are approximately but not necessarily perpendicular. The apparent wave number tensor, defined in (A19) as  $\mathbf{K}_h = i\omega \mathbf{Y}_h$  ( $\text{m}^{-1}$ ), contains these two eigenstates. The elements of  $\mathbf{K}_h$  contain information about the subsurface apparent wave number components excited by the two polarization sources. I interpret the eigenvalues of  $\mathbf{K}_h$  as the vertical wave numbers,  $k_{z1}$  and  $k_{z2}$ , invariant to rotations about the vertical axis and consistent with apparent vertical wave propagation in a 2 by 2 MT response tensor when we ignore the existence of a finite  $B_z$ . Then

$$k_{z1} = \frac{1}{2} (K_{xy} - K_{yx}) + \frac{1}{2} \left[ (K_{xy} - K_{yx})^2 - 4 \det(\mathbf{K}_h) \right]^{1/2} \quad (21a)$$

$$k_{z2} = \frac{1}{2} (K_{xy} - K_{yx}) - \frac{1}{2} \left[ (K_{xy} - K_{yx})^2 - 4 \det(\mathbf{K}_h) \right]^{1/2} \quad (21b)$$

$$k_{z1} + k_{z2} = K_{xy} - K_{yx} = \text{skew}(\mathbf{K}_h) \quad (22a)$$

$$k_{z1} k_{z2} = K_{xx} K_{yy} - K_{xy} K_{yx} = \det(\mathbf{K}_h) \quad (22b)$$

From (1), the second-order tensor  $\gamma_h$  can be written:

$$\gamma_h = \det(\mathbf{K}_h) (\mathbf{K}_h^{-1})^{tr} \mathbf{K}_h = \begin{pmatrix} K_{xx} K_{yy} - K_{yx}^2 & K_{yy} (K_{xy} - K_{yx}) \\ K_{xx} (K_{yx} - K_{xy}) & K_{xx} K_{yy} - K_{xy}^2 \end{pmatrix} \quad (23)$$

The (Lanczos) eigenvalues of  $\gamma_h$  are  $-k_{z1}^2$  and  $-k_{z2}^2$ ,  $\text{trace}(\gamma_h) = -(k_{z1}^2 + k_{z2}^2)$ ,  $\det(\gamma_h) = k_{z1}^2 k_{z2}^2$  and  $k_0^2 = \gamma_0 = -1/2 (k_{z1}^2 + k_{z2}^2)$ .

The extension to a 3 by 3 apparent wave number tensor which includes a finite  $B_z$  is compatible with this eigenstate analysis. The relationships between  $E_x, E_y, B_x,$  and  $B_y$  in  $\mathbf{Z}_h$  are preserved in the upper left submatrix of  $\mathbf{Z}$  when  $Z_{xz}$  and  $Z_{yz}$  are zero in (17). Noting that if  $E_z = 0$ , the horizontal electric and magnetic fields still satisfy the formal definition of the two eigenstates in the linear MT response tensor [*Eggers*, 1982] obtained by imposing the constraint that  $\mathbf{E} \cdot \mathbf{B} = 0$ . This constraint is true within 1-D isotropic resistivity distributions, or 1-D azimuthally anisotropic and 2-D isotropic resistivity distributions, where a linearly polarized magnetic field is parallel or perpendicular to the resistivity strike direction; it is only an approximation within 3-D resistivity environments. By choosing  $Z_{zz}$  to be a rotational invariant of  $\mathbf{Z}_h$ ,  $Z_{zz} = \det(\mathbf{Z}_h)^{1/2} = i\omega \det(\mathbf{K}_h)^{-1/2}$  then

$$\mathbf{K} = \begin{pmatrix} K_{xx} & K_{xy} & 0 \\ K_{yx} & K_{yy} & 0 \\ T_x K_{xx} + T_y K_{yx} & T_y K_{yy} + T_x K_{xy} & \det(\mathbf{K}_h)^{1/2} \end{pmatrix} \quad (24)$$

This has two eigenvalues equal to the eigenvalues of  $\mathbf{K}_h$ , and the third eigenvalue equal to  $\det(\mathbf{K}_h)^{1/2}$  offers no additional information. These eigenvalues are the apparent vertical wave numbers,  $k_{z1}$  and  $k_{z2}$ , and the elements  $K_{zx}$  and  $K_{zy}$  are horizontal wave numbers  $-k_y$  and  $k_x$ . The apparent total wave numbers,  $k_1$ ,  $k_2$  associated with the eigenstates can be obtained [e.g., *Claerbout*, 1985] by assuming a plane wave dispersion relationship, e.g.,  $k_1^2 = k_x^2 + k_y^2 + k_{z1}^2$ . Defining  $k_z = \sqrt{k_{z1}k_{z2}}$ , allows the estimation of a total wave number  $\mathbf{k}$  and the condition  $\mathbf{k} \cdot \mathbf{B} = 0$  results in what appears to be a single inhomogeneous plane wave propagating obliquely into the lower half-space; its surfaces of constant amplitude and constant phase are planes but they are not parallel [*Jackson*, 1999]. This is a plane wave analogy for MT where the EM fields diffuse in the dissipative conducting subsurface with attenuation and phase relationships that are frequency and directionally dependent. A single plane wave is consistent with the requirement that no energy is reflected at subsurface boundaries [*Mandelis et al.*, 2001].

An apparent vertical wave number is a minimum requirement for migration, analytic continuation, or imaging of MT data [e.g., *Zhdanov et al.*, 1996].

## 5. EM Field Distortions

The most complete description of the physical mechanism of EM field distortion was presented by *Chave and Smith* [1994] who developed an integral equation approach, based on the extended-Born approximation of *Habashy et al.* [1993], to the scattering of electric and magnetic fields from resistivity inhomogeneities near an observation site. The theory assumes that (1) the regional electric field is constant across the distorting inhomogeneity and (2) it can be approximated within the inhomogeneity by its value at the site. The first condition implies that induction is negligible in local inhomogeneities where their spatial scale is smaller than the regional inductive scale or the source field scale, usually where the inhomogeneities are not significantly less resistive than the background resistivity and the frequencies employed are low [*Groom and Bailey*, 1989]. In these circumstances, the distorting effects on the electric and magnetic fields is due to electric charges on the boundaries of the inhomogeneities (galvanic or static distortion). The second condition is easily achieved where the regional resistivity structure is 1-D but may be problematic for 3-D, or even 2-D, regional resistivity distributions where the observation point is close to a boundary in the regional structure. As it is impossible to test these conditions at each site prior to distortion analyses, most analysts take them as axiomatic and proceed by assuming that the horizontal components of the observed, distorted EM fields,  $\mathbf{E}_h$  and  $\mathbf{B}_h$  can be related to regional, undistorted counterparts,  $\mathbf{E}_h^r$  and  $\mathbf{B}_h^r$  by

$$\mathbf{E}_h = \mathbf{C}_h \mathbf{E}_h^r \quad (24a)$$

$$\mathbf{B}_h = \mathbf{B}_h^r + \mathbf{D}_h \mathbf{E}_h^r = (\mathbf{I}_2 + \mathbf{D}_h \mathbf{Z}_h^r) \mathbf{B}_h^r \quad (25b)$$

where superscript  $r$  refers to regional fields (i.e., fields in the absence of the distorting inhomogeneities),  $\mathbf{C}_h$  and  $\mathbf{D}_h$  are second-rank electric and magnetic distortion tensors. The distorted MT response tensor  $\mathbf{Z}_h$  is

$$\mathbf{E}_h = \mathbf{C}_h \mathbf{Z}_h^r (\mathbf{I}_2 + \mathbf{D}_h \mathbf{Z}_h^r)^{-1} \mathbf{B}_h = \mathbf{Z}_h \mathbf{B}_h \quad (26)$$

Some investigators have considered the effects of magnetic distortion where  $\mathbf{D}_h \neq 0$  [*Chave and Smith*, 1994; *Smith*, 1997; *Agarwal and Weaver*, 2000]. If induction is small in the local inhomogeneity,  $\mathbf{C}_h$  and  $\mathbf{D}_h$  are real and frequency independent. Moreover, the effects of magnetic galvanic distortion decrease with decreasing frequency so the most common practice is to assume that  $\mathbf{C}_h$  is real and frequency independent, and  $\mathbf{D}_h = 0$  [e.g., *Jones*, 2012].

This section uses the theory developed in the paper to reformulate the effects of distortion and extend the phase tensor analysis of *Caldwell et al.* [2004].

### 5.1. Extended Phase Tensor With Magnetic Distortion

The phase tensor analysis assumes that  $\mathbf{C}_h$  is real and frequency independent and  $\mathbf{D}_h = 0$ , so that  $\mathbf{B}_h = \mathbf{B}_h^r$ . The latter assumption implies a decoupling between the distortion effects on the electric and magnetic fields, and is inconsistent, in a strict sense, with the Maxwell Equations. If the assumption is accepted, the MT response tensor can be decomposed into real and imaginary parts,  $\mathbf{U}_h^r$  and  $\mathbf{V}_h^r$ , respectively, so that  $\mathbf{Z}_h = \mathbf{C}_h \mathbf{Z}_h^r = \mathbf{C}_h (\mathbf{U}_h^r + i \mathbf{V}_h^r) = \mathbf{C}_h \mathbf{U}_h^r (\mathbf{I}_2 + i (\mathbf{U}_h^r)^{-1} \mathbf{V}_h^r) = \mathbf{C}_h \mathbf{U}_h^r (\mathbf{I}_2 + i \varphi_h)$ . The phase tensor  $\varphi_h$  is free from distortion.

The relationships in sections 2 and 4 between the regional magnetic and electric fields and regional current density are summarized (in new notation):

$$\mathbf{B}^r = (i\mu/\omega) \det(\mathbf{Z}_h^r) (\mathbf{Y}^r)^{\text{tr}} \mathbf{J}^r \quad (27a)$$

$$\mathbf{E}^r = (i\mu/\omega) \det(\mathbf{Z}_h^r) \mathbf{Z}^r (\mathbf{Y}^r)^{\text{tr}} \mathbf{J}^r \quad (27b)$$

It follows that the regional electric and magnetic fields are not in phase with the regional apparent current density. If local distorting inhomogeneities are then introduced in the vicinity of the observation point, the distorted apparent current density is changed across the inhomogeneities according to  $\mathbf{J} = \mathbf{P}\mathbf{J}^r$ . If the effect of the inhomogeneities on the regional electric field at the observation point is to be described by a real distortion tensor  $\mathbf{C}$  to be consistent with the theory of *Chave and Smith* [1994], the premultiplication in (27a) of  $(\mathbf{Y}^r)^{\text{tr}}$  by  $\mathbf{C}^{-\text{tr}}$  (the transpose of the inverse of  $\mathbf{C}$ ) requires  $\mathbf{C}^{-\text{tr}}(\mathbf{Y}^r)^{\text{tr}} = (\mathbf{Y}^r)^{\text{tr}}\mathbf{P}$  so  $\mathbf{P}$  is complex. The distorted apparent current density is therefore different in amplitude and phase from the regional apparent current density. The premultiplication of  $(\mathbf{Y}^r)^{\text{tr}}$  by  $\mathbf{C}^{-\text{tr}}$  is equivalent to the premultiplication of  $\mathbf{Z}^r$  by  $\mathbf{C}$  so (27) becomes

$$\mathbf{B} = (i\mu/\omega) \det(\mathbf{Z}_h) \mathbf{C}^{-\text{tr}} (\mathbf{Y}^r)^{\text{tr}} \mathbf{J}^r = \mathbf{C}^{-\text{tr}} \mathbf{B}^r \quad (28a)$$

$$\mathbf{E} = \mathbf{C}\mathbf{Z}^r \mathbf{B} = \mathbf{C}\mathbf{Z}^r \mathbf{C}^{-\text{tr}} \mathbf{B}^r \quad (28b)$$

where the replacement of the term  $\det(\mathbf{Z}_h^r)$  in (27a) by  $\det(\mathbf{Z}_h)$  represents a site gain which is a function of  $\det(\mathbf{C})$ .

The distorted electric field is in phase with the regional electric field and the distorted magnetic field is constrained to be in phase with the regional magnetic field. The usual assumption in distortion analysis is  $\mathbf{B} = \mathbf{B}^r$ , an extreme condition of the distorted field in phase with the regional field. None of these electric and magnetic fields is in phase with the apparent current density.

The phase tensor in this model is therefore shown to be valid in the presence of a distorted magnetic field in phase with the regional magnetic field. Its importance can be recognized by rewriting the distorted fields in (28) as

$$\mathbf{B} = \left( i\mu/\omega \right) \det(\mathbf{Z}_h) \mathbf{C}^{-\text{tr}} \mathbf{U}_Y^{\text{tr}} (\mathbf{I}_3 - i\varphi^{\text{tr}}) \mathbf{J}^r \quad (29a)$$

$$\mathbf{E} = \mathbf{C}\mathbf{U}^r (\mathbf{I}_3 + i\varphi) \mathbf{B} \quad (29b)$$

where  $\mathbf{U}^r$  is the real part of the regional 3 by 3 MT response tensor,  $\mathbf{U}_Y^{\text{tr}}$  is the real part of the transpose of the regional 3 by 3 apparent slowness tensor and it is easily demonstrated that the “phase tensor” of  $(\mathbf{Y}^r)^{\text{tr}}$  is  $-\varphi^{\text{tr}}$ . Equation (29a) provides the link to the modeling results of *Caldwell et al.* [2004] which suggested that the phase tensor major axis indicates the preferred flow direction of the regional induction current. Equation (29b) demonstrates explicitly that the geometry of the phase tensor is meaningful where magnetic field distortion is present [Booker, 2014].

By construction,  $Z_{xz} = Z_{yz} = 0$ , so  $\varphi_{xz} = \varphi_{yz} = 0$ , regardless of (real)  $\mathbf{C}$ , and the extended phase tensor consists of the four parameters of the conventional phase tensor in the upper left submatrix and two distortion-free parameters  $\varphi_{zx}$  and  $\varphi_{zy}$  associated with the vertical magnetic transfer function. The parameters,  $\varphi_{zx}$  and  $\varphi_{zy}$ , equal to the phases of  $k_y$  and  $k_x$ , are similar to those in the distortion-free vertical field phase vector [Booker, 2014], based on the enhanced admittance phase tensor [Pankratov and Kuvshinov, 2010]. The three eigenvalues of  $\varphi$  represent the semimajor axes of a phase tensor ellipsoid. Two eigenvalues are equal to those of  $\varphi_h$ , the third is equal to  $\varphi_{zz}$ , the phase of  $Z_{zz}$ , and contains no additional information.

If  $\mathbf{C}$  is constrained so that  $C_{xz} = C_{yz} = 0$ ,  $\mathbf{Z}$  has upper left submatrix elements identical to its 2 by 2 distorted equivalent, so the two eigenstates of  $\mathbf{Z}$  satisfy the orthogonality constraint  $\mathbf{E} \cdot \mathbf{B} = 0$  for the horizontal electric and magnetic fields. The two independent eigenvalues of  $\mathbf{K} = i\omega(\mathbf{Z})^{-1}$  become distorted vertical wave numbers and may be used in the computation of a distorted background apparent resistivity. Finite values of  $C_{zx}$  and  $C_{zy}$  result in distortion of each component of the vertical magnetic transfer function and introduce a finite vertical electric field, a useful property in marine MT where strong bathymetric variations of the seafloor can distort  $E_z$  [Chave and Smith, 1994], and in terrestrial MT where inhomogeneities can distort  $\mathbf{E}_h$  and generate  $E_z$  near Earth’s surface, even at spatial scales ( $\sim 20$  m to 100 m) commensurate with the practical measurement of electric fields. If  $C_{zx}$  and  $C_{zy}$  are constrained to be zero, the vertical magnetic transfer function is unaffected. The phase tensor is unaffected by the choice of  $C_{zz}$ .

The distorted magnetic field in (25b), based on the theory of *Chave and Smith* [1994], can be extended to the 3 by 3 MT tensor and related to the formalism developed in this paper:

$$\mathbf{B} = (\mathbf{I}_3 + \mathbf{DZ}^T)\mathbf{B}' = ((\mathbf{I}_3 + \mathbf{DU}^T) + i\mathbf{DU}^T\varphi)\mathbf{B}' \quad (30a)$$

$$\mathbf{B} = (\mathbf{C}^{-tr} + i(\mathbf{C}^{-tr} - \mathbf{I}_3)\varphi)\mathbf{B}' \quad (30b)$$

where  $\mathbf{D} = (\mathbf{C}^{-tr} - \mathbf{I}_3)(\mathbf{U}')^{-1}$ .

Equations (28a), (28b), (30a), and (30b) demonstrate that static distortion of the horizontal electric fields and all magnetic field components can be quantified using a single EM distortion tensor with six independent elements along with the six distortion-free elements of the extended phase tensor. Equations (30a) and (30b) state explicitly which of the components of  $\mathbf{B}$  are in phase and out of phase with  $\mathbf{B}'$ , respectively. The omission of the out-of-phase term is likely to be significant only in a restricted high-frequency band.

## 5.2. Implications for 3-D Inversion

As 3-D inversion codes generally incorporate the vertical magnetic transfer functions as well as the MT response tensor, it makes sense to include the six distortion-free, frequency-independent parameters of the three-dimensional phase tensor as data in an inversion. Rather than simply modeling the MT response tensor, 3-D inversion strategies incorporating the vertical magnetic transfer function, phase tensor, rotationally invariant quantities and/or interstation transfer functions may provide a fruitful approach [*Miensepust et al.*, 2013; *Patro et al.*, 2013; *Tietze et al.*, 2015]. The real and imaginary apparent resistivity tensor ellipses in section 2 each provide three coordinate invariants, as does the angle between their major axes. All of these invariants, and the coordinate dependent angle expressing the orientation of the major axis of one ellipse, contain the diagonal elements of the MT response tensor, and therefore may help to deal with some of the issues associated with incorporating diagonal elements in 3-D inversions [*Miensepust et al.*, 2013]. Their efficacy may be assessed using the visualization advantage associated with the apparent resistivity ellipses to demonstrate behavior across neighboring sites that may (or not) be consistent with regional structures.

Regularized 3-D inversion strategies are usually implemented to obtain the smallest possible misfit between the model response and data, traded against the roughness of the resistivity model, and it is also possible to incorporate distortion constraints by including terms in the objective function. As local distortion varies from site to site, it will introduce structure into the (1-D) apparent background resistivity  $\rho_0(\mathbf{r}_h, \omega)$ . The choice of a background resistivity as a prior model is important in 3-D inversion, particularly when inverting with vertical magnetic transfer functions and the phase tensor [e.g., *Patro et al.*, 2013]. The background apparent resistivity is constructed from the apparent vertical wave numbers (section 4) and represents a simple migration of MT data at a single site. It should be smoother in the horizontal direction over all sites than the apparent resistivity, so it may provide the basis for a minimum-structure constraint in the objective function. Inversions could proceed by incorporating distortion constraints on the six independent elements of  $\mathbf{C}$ . In circumstances where the model misfit is poor at high frequencies but good at low frequencies, a distorted magnetic field out-of-phase with the regional magnetic field is indicated. A correction using (30b) may be introduced at those frequencies along the lines described by *Chave and Smith* [1994].

## 6. Conclusions

The paper has presented an analogy for the MT method using the propagation of an inhomogeneous EM plane wave through a dissipative, conducting half-space with an arbitrary anisotropy. It derives a quadratic function of an apparent wave number tensor whose components contain the complex wave numbers of the two linearly independent polarizations necessary for a full description of the relationship between the horizontal electrical and magnetic fields. The eigenvalues of the 2 by 2 apparent wave number tensor are associated with vertically propagating inhomogeneous waves in a 3-D Earth, so providing a simple method for the migration of the EM fields. The formulation allows for the construction of an apparent resistivity tensor. An apparent current density may be estimated from the observed magnetic field.

The model results in an extended-Born approximation for the relationship between the observed electric field and an apparent background electric field. It allows MT electric field measurements to be normalized with respect to a current density in a uniform isotropic half-space, similar to formulations for DC electrical resistivity, and controlled-source time domain and frequency domain apparent resistivity tensors. In doing

so, it provides an explanation for the similarity in the behavior of these tensors reported in the literature. The extended- Born approximation provides an estimate of an apparent variable background resistivity that may be used to improve the accuracy of 3-D MT forward models using integral equations, the speed of the calculation of Frechét derivatives, and the starting model in 3-D inversions.

The formulation is used to construct a 3 by 3 MT tensor that encapsulates relationships between all magnetic field components and the horizontal electric fields, and their relationships with an apparent current density vector. The 3 by 3 apparent wave number tensor contains the two vertical wave number components associated with its independent eigenvalues, and apparent horizontal wave numbers associated with the vertical magnetic transfer functions. The plane wave MT analogy is represented by a single inhomogeneous plane wave, propagating obliquely into the subsurface, with a total apparent wave number that can be estimated by assuming a plane wave dispersion relationship.

An extended 3 by 3 phase tensor may be constructed from the 3 by 3 MT tensor. It contains the conventional 2 by 2 phase tensor and two distortion-free elements related to the apparent horizontal wave number components. The model allows the definition of a single real EM distortion tensor which accounts for the distortion of electric and magnetic fields. It provides an explicit demonstration that the phase tensor is useful where there is static electric and magnetic field distortion. An approach to 3-D inversion is sketched where the phase tensor, the EM distortion tensor, and a constraint on the rotationally invariant apparent background resistivity may be incorporated into a regularized multisite, multifrequency strategy. The modeled apparent resistivity tensor ellipses are particularly suited to assess how closely their behavior across neighboring MT sites is consistent with known geological structures and other geophysical interpretations.

A forthcoming paper will explore the theory using simulated and observed 3-D/3-D MT data.

### Appendix A: Derivation of the Governing Equations

The derivation is based on *Reilly* [1979] which used a notation of tensor calculus independent of any particular coordinate system. In this Appendix A, the original derivation has been reformulated by using tensor notation for a Cartesian coordinate system. The following symbols are defined:

1.  $\varepsilon_{ij}, \varepsilon_{ijk}$  are the Levi-Civita tensor densities (alternating tensors) in two and three dimensions, respectively:

$$\varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon_{12} = 1, \varepsilon_{21} = -1$$

$$\begin{aligned} \varepsilon_{ijk} &= 1 && \text{if } ijk \text{ is any cyclic permutation of } (123). \\ \varepsilon_{ijk} &= -1 && \text{if } ijk \text{ is any cyclic permutation of } (321). \\ \varepsilon_{ijk} &= 0 && \text{if any two of } (i,j,k) \text{ are equal.} \end{aligned}$$

2.  $\delta_{ij}$  is the Kronecker delta in two and three dimensions. It is also the metric tensor for the Cartesian coordinate system.

$$\begin{aligned} \delta_{ij} &= 1 && \text{if } i = j \\ \delta_{ij} &= 0 && \text{if } i \neq j \end{aligned}$$

The third and fourth Maxwell Equations (Maxwell-Faraday and Maxwell-Ampère Equations, respectively) in tensor notation are

$$\varepsilon_{ijk} E_{k,j} = -\frac{\partial B_i}{\partial t} \tag{A1}$$

$$\varepsilon_{ijk} B_{k,j} = \mu \left( J_i + \frac{\partial D_i}{\partial t} \right) \tag{A2}$$

where  $E_{k,j}$  is the covariant derivative of the (covariant) electric field vector,  $E_k$ ,  $B_i$  is the magnetic flux density,  $J_i$  is the current density,  $D_i$  is the displacement current, and  $\mu$  is the scalar magnetic permeability, assumed to be that of free space. For time variations of the form  $e^{-i\omega t}$ , the differential equations in the frequency domain become

$$\varepsilon_{ijk} E_{k,j} = i\omega B_i \tag{A3}$$

$$\varepsilon_{ijk} B_{k,j} = \mu (J_i - i\omega D_i) \tag{A4}$$

The constitutive relations for an anisotropic half-space are

$$\begin{aligned} J_i &= \sigma_{ij} E_j \\ D_i &= \zeta_{ij} E_j \end{aligned} \quad (A5)$$

where  $\sigma_{ij}$  and  $\zeta_{ij}$  are three-dimensional conductivity and permittivity tensors defined at the observation point. Substitution from (A5) into (A4) leads to

$$\varepsilon_{ijk} B_{kj} = \mu (\sigma_{ij} - i\omega \zeta_{ij}) E_j \quad (A6)$$

The magnetic flux density  $B_i$  is eliminated between (A3) and (A6) by introducing

$$\gamma_{ij} = i\omega \mu (\sigma_{ij} - i\omega \zeta_{ij}) \quad (A7)$$

Taking the curl of (A3) gives

$$\varepsilon_{pqr} \varepsilon_{rjk} E_{k,jq} = i\omega \varepsilon_{pqr} B_{r,q} \quad (A8)$$

Multiplying (A6) by  $i\omega$  allows the elimination of the magnetic flux density between (A6) and (A8) obtaining a second-order differential equation in the electric field vector:

$$\varepsilon_{pqr} \varepsilon_{rjk} E_{k,jq} - \gamma_{ps} E_s = 0 \quad (A9)$$

At this point in the derivation  $\gamma_{ps}$  is a tensor operating in the three-dimensional space in which MT electric and magnetic fields are measured. The aim is to express  $\gamma_{ps}$  in terms of the observed slowness tensor which operates in a two-dimensional space ( $x$ - $y$  plane). Therefore, it is necessary to reformulate (A9) so that it only contains the horizontal components of  $\gamma_{ps}$ .

This can be achieved by imposing two restrictions that (1) a plane EM wave of angular frequency  $\omega$  is vertically incident at the horizontal surface of the half-space and (2) its direction of propagation within the half-space is vertically downward. By introducing  $n_i$  as a unit vector in the vertical directions and writing the specific Cartesian components of vectors and tensors using numerical indices 1, 2, and 3 to represent east, north, and vertical, respectively, the two assumptions imply among other things that

1. the vertical direction is a principal axis of anisotropy so four of the nine components of the second-rank tensor,  $\gamma_{ps}$  are zero:

$$\gamma_{13} = \gamma_{23} = \gamma_{31} = \gamma_{32} = 0 \quad (A10)$$

2. the vertical component of  $E_k$  is zero:

$$E_k n_k = E_3 = 0 \quad (A11)$$

which implies that all components of the form  $\gamma_{p3}$  in (A9) are indeterminate. Since two of these are already zero in (A10), then  $\gamma_{33}$  is indeterminate.

By introducing these constraints all horizontal derivatives of  $E_k$  vanish so (A9) reduces to

$$\varepsilon_{rpk} \varepsilon_{kr} E_{k,jq} n_j n_q - \gamma_{ps} E_s = 0 \quad (A12)$$

which can be written simply as

$$E_{p,33} - \gamma_{ps} E_s = 0 \quad (A13)$$

This is a second-order differential equation in only the vertical derivatives of the electric field vector. A general solution to this equation can be obtained by assuming that the vertical derivative of  $E_k$  can be expressed in terms of exponential functions of the spatial variables which, with the constraints above, is simply the depth. The vertical derivative of  $E_k$  becomes

$$E_{p,j} n_j = E_{p,3} = U_{tp} E_t \quad (A14)$$

in which the two-dimensional tensor  $U_{tp}$  is constant under differentiation and is a function only of the properties of the half-space. For the second derivative,

$$E_{p,33} = E_{p,jq} n_j n_q = (E_{p,j} n_j)_q n_q = (U_{tp} E_t)_q n_q = U_{tp} (E_{t,q} n_q) = U_{tp} (U_{st} E_s) \quad (A15)$$

so (A13) becomes

$$U_{tp}U_{st}E_s - \gamma_{ps}E_s = 0 \quad (A16)$$

If this is to hold for arbitrary  $E_s$ , then

$$\gamma_{ps} = U_{tp}U_{st} \quad (A17)$$

The two-dimensional form of the third Maxwell Equation in (A3) can now be expressed, using (A14) as

$$i\omega B_j = \varepsilon_{ijk}E_{k,j} = \varepsilon_{ki}U_{tk}E_t \quad (A18)$$

and using the definition of the slowness tensor  $B_j = Y_{ij}E_j$  and the wavenumber tensor  $K_{ij} = i\omega Y_{ij}$ , then

$$K_{it} = i\omega Y_{it} = \varepsilon_{ki}U_{tk} \quad (A19)$$

$$U_{tk} = -i\omega \varepsilon_{ki}Y_{it} \quad (A20)$$

The substitution for  $U_{tp}$  and  $U_{st}$  into (A17) leads to

$$\begin{aligned} \gamma_{ps} &= -\omega^2 \varepsilon_{pi}Y_{it}\varepsilon_{tj}Y_{js} \\ &= -\omega^2 (\delta_{pj}\delta_{it} - \delta_{pt}\delta_{ij})Y_{it}Y_{js} \\ &= -\omega^2 (Y_{ii}Y_{ps} - Y_{ip}Y_{is}) \end{aligned} \quad (A21)$$

The four Cartesian components are then

$$\begin{aligned} \gamma_{11} &= -\omega^2(Y_{22}Y_{11} - Y_{21}Y_{21}) & \gamma_{12} &= -\omega^2Y_{22}(Y_{12} - Y_{21}) \\ \gamma_{21} &= -\omega^2Y_{11}(Y_{21} - Y_{12}) & \gamma_{22} &= -\omega^2(Y_{11}Y_{22} - Y_{12}Y_{12}) \end{aligned} \quad (A22)$$

The equations in (A22) can be written in matrix form:

$$\mathbf{Y}_h = -\omega^2(\text{trace}(\mathbf{Y}_h)\mathbf{Y}_h - \mathbf{Y}_h^{\text{tr}}\mathbf{Y}_h) \quad (A23)$$

$$= -\omega^2 \det(\mathbf{Y}_h)\mathbf{Z}_h^{\text{tr}}\mathbf{Y}_h \quad (A24)$$

where “trace” and “det” are trace and determinant,  $\mathbf{Y}_h^{\text{tr}}$  is the transpose of  $\mathbf{Y}_h$  and  $\mathbf{Z}_h = \mathbf{Y}_h^{-1}$  is the MT response tensor, and subscript  $h$  indicates relationships among horizontal EM fields at Earth’s surface.

The quadratic form of the slowness tensor in (A23) is similar to the form of the apparent resistivity tensor derived from an algebraic argument for horizontal magnetic fields produced by grounded sources [Caldwell *et al.*, 2002, Appendix A]. Both quadratic functions are contractions of the fourth-rank tensor (outer) product of the slowness tensor with itself. Five independent contractions of the fourth-rank tensor are possible and the most general, second-rank tensor quadratic form is a linear combination of these contractions. Caldwell *et al.* [2002] develop an argument that reduces the combination to symmetric and nonsymmetric terms similar to  $\mathbf{Y}_h^{\text{tr}}\mathbf{Y}_h$ , and  $\text{trace}(\mathbf{Y}_h)\mathbf{Y}_h$  in (A23).

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