



## The gluing problem for some block fusion systems

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Author(s)	Park, Sejong
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# THE GLUING PROBLEM FOR SOME BLOCK FUSION SYSTEMS

SEJONG PARK

ABSTRACT. We answer the gluing problem of blocks of finite groups [7, 4.2] for tame blocks and the principal  $p$ -block of  $\mathrm{PSL}_3(p)$  for  $p$  odd. In particular, we show that the gluing problem for the principal  $p$ -block of  $\mathrm{PSL}_3(p)$  does not have a unique solution when  $p \equiv 1 \pmod{3}$ .

## 1. INTRODUCTION

Let  $P$  be a finite  $p$ -group for some prime  $p$ . A fusion system  $\mathcal{F}$  on  $P$  is a category whose objects are the subgroups of  $P$  and whose morphisms are injective group homomorphisms satisfying some axioms formulated by Puig in the early 1990s.(cf. [10]) Axioms of fusion systems are modeled on common features of conjugation maps in finite groups having  $P$  as a Sylow  $p$ -subgroup and conjugation maps of Brauer pairs of blocks of finite groups having  $P$  as a defect group. As such, fusion systems provide a uniform framework for studying local structures of finite groups and blocks of finite groups. For further details and terminology, we refer the reader to [3]. All fusion systems appearing in this paper are saturated, and hence we drop the adjective ‘saturated’ and call them simply fusion systems.

One of the main themes of modular representation theory is the global-local principle, which is exemplified by a celebrated conjecture of Alperin [1]. Alperin’s weight conjecture, as it is usually called, predicts that a global invariant, the number of isomorphism classes of simple modules, of a block is equal to a local invariant, the number of conjugacy classes of weights, of the block, which in turn is determined by the fusion system of the block on its defect group plus some extra data. See [4, §5] for more details. The gluing problem of blocks [7, 4.2] asks if these extra data can be encoded into a single cohomology class of a certain category related to the fusion system of the block. If so, one obtains a reformulation of Alperin’s weight conjecture [7, 4.5, 4.7][8, 4.3] which provides a structural viewpoint.

In this paper, we solve the gluing problem for some blocks. Let us first explain the gluing problem more precisely. Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $k$  be an algebraically closed field. Let  $[S(\mathcal{F}^c)]$  be the poset of  $\mathcal{F}$ -conjugacy classes  $[\sigma]$  of chains

$$\sigma = (R_0 < R_1 < \cdots < R_n), \quad n \geq 0,$$

of  $\mathcal{F}$ -centric subgroups  $R_i$  of  $P$ , with partial order induced by taking subchains. Here  $\mathcal{F}^c$  denotes the full subcategory of  $\mathcal{F}$  consisting of the  $\mathcal{F}$ -centric subgroups of  $P$ , and  $S(\mathcal{F}^c)$  denotes the subdivision of the EI-category  $\mathcal{F}^c$ . For further details and precise definitions, we refer the reader to [8]. Also, let

$$\mathrm{Aut}_{\mathcal{F}}(\sigma) = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(R_n) \mid \alpha(R_i) = R_i \text{ for all } i\}.$$

For any positive integer  $i$ , there is a covariant functor

$$\mathcal{A}_{\mathcal{F}}^i: [S(\mathcal{F}^c)] \rightarrow \mathbf{Ab}$$

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sending  $[\sigma] \in [S(\mathcal{F}^c)]$  to  $H^i(\text{Aut}_{\mathcal{F}}(\sigma), k^\times)$ , where the poset  $[S(\mathcal{F}^c)]$  is viewed as a category with the morphisms given by the partial order and  $\mathbf{Ab}$  denotes the category of abelian groups.

Given a functor from a small category to an abelian category, one can define the cohomology of the small category with coefficients in the functor much the same way as one defines the cohomology of a group with coefficients in a module. See [12] for further details. Using the contractibility of  $[S(\mathcal{F}^c)]$  proved in [9, 1.1], Linckelmann finds in [6] that for every fusion system  $\mathcal{F}$  there exists an exact sequence in cohomology as follows:

**Theorem 1.1** ([6, 1.1]). *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $k$  be an algebraically closed field. Then there exists an exact sequence of abelian groups*

$$0 \rightarrow H^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) \rightarrow H^2(\mathcal{F}^c, k^\times) \rightarrow H^0([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^2) \rightarrow H^2([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) \rightarrow H^3(\mathcal{F}^c, k^\times).$$

*In particular, the group  $H^2(\mathcal{F}^c, k^\times)$  is finite, of order coprime to  $\text{char}(k)$  if  $\text{char}(k)$  is positive.*

If  $\mathcal{F}$  is the fusion system of a block of a finite group, then the block determines an element of  $H^0([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^2)$  by the work of Külshammer and Puig [5, 1.8, 1.12]. The gluing problem asks whether this element is the image of the map

$$H^2(\mathcal{F}^c, k^\times) \rightarrow H^0([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^2).$$

In particular, if  $H^2([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) = 0$ , then the gluing problem will have solutions, the number of which is equal to the order of the group  $H^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1)$ .

In this paper, we compute  $H^i([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1)$  ( $i = 1, 2$ ) for fusion systems  $\mathcal{F}$  of tame blocks and the principal  $p$ -block of  $\text{PSL}_3(p)$  for  $p$  odd. Recall that a tame block is a 2-block whose defect groups are dihedral, semidihedral, or (generalized) quaternion 2-groups. The gluing problem for tame blocks has a unique solution as the next theorem, proved in Section 2, shows.

**Theorem 1.2.** *Let  $P$  be a dihedral, semidihedral, or (generalized) quaternion 2-group, and let  $\mathcal{F}$  be a fusion system on  $P$ . Let  $k$  be an algebraically closed field of characteristic 2. Then we have*

$$H^2(\mathcal{F}^c, k^\times) = H^0([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^2) = 0.$$

*In particular, the gluing problem for tame blocks has the zero class as a unique solution.*

In the case of the principal block of  $\text{PSL}_3(p)$  we also obtain a unique solution for the gluing problem when  $p \not\equiv 1 \pmod{3}$  and, unexpectedly, multiple solutions when  $p \equiv 1 \pmod{3}$ . This is the main result of this paper and is proved in Section 3.

**Theorem 1.3.** *Let  $p$  be an odd prime number and let  $P$  be an extraspecial group of order  $p^3$  and exponent  $p$ . Then for any fusion system  $\mathcal{F}$  on  $P$  we have*

$$H^2([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) = 0.$$

*If  $\mathcal{F} = \mathcal{F}_P(\text{PSL}_3(p))$ , then we have*

$$H^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) = \begin{cases} 0, & \text{if } p \not\equiv 1 \pmod{3} \\ \mathbb{Z}/3, & \text{if } p \equiv 1 \pmod{3}. \end{cases}$$

*In particular, the gluing problem for the principal  $p$ -block of  $\text{PSL}_3(p)$  has a unique solution if  $p \not\equiv 1 \pmod{3}$ , and three solutions if  $p \equiv 1 \pmod{3}$ .*

## 2. TAME FUSION SYSTEMS

Let  $P$  be either a dihedral group  $D_{2^n}$  ( $n \geq 2$ ), a semidihedral group  $SD_{2^n}$  ( $n \geq 4$ ), or a (generalized) quaternion group  $Q_{2^n}$  ( $n \geq 3$ ) of order  $2^n$ . It is well known that the subgroups  $R$  of  $P$  are cyclic, dihedral, semidihedral, or quaternion, and their automorphism groups are 2-groups except when  $R \cong D_4$  or  $Q_8$ , in which cases we have  $\text{Aut}(D_4) \cong S_3$ ,  $\text{Aut}(Q_8) \cong S_4$ . From this, one can easily deduce the following proposition.

**Proposition 2.1.** *Let  $\mathcal{F}$  be any fusion system on  $P$  and let  $R$  be an  $\mathcal{F}$ -centric subgroup of  $P$ .*

- (1) *If  $R \not\cong D_4, Q_8$ , then  $\text{Out}_{\mathcal{F}}(R)$  is a 2-group.*
- (2) *If  $R \cong D_4, Q_8$  and  $R < P$ , then  $\text{Out}_{\mathcal{F}}(R) \cong C_2$  or  $S_3$ .*
- (3) *If  $R \cong D_4, Q_8$  and  $R = P$ , then  $\text{Out}_{\mathcal{F}}(R) = 1$  or  $C_3$ .*

**Corollary 2.2.** *Let  $\mathcal{F}$  be any fusion system on  $P$  and let  $\sigma$  be a chain of  $\mathcal{F}$ -centric subgroups of  $P$ . Let  $k$  be an algebraically closed field of characteristic 2. We have*

$$H^1(\text{Aut}_{\mathcal{F}}(\sigma), k^\times) \cong \begin{cases} \mathbb{Z}/3, & \text{if } \sigma = (P), P \cong D_4 \text{ or } Q_8, \text{Out}_{\mathcal{F}}(P) \cong C_3 \\ 0, & \text{otherwise} \end{cases}$$

and

$$H^2(\text{Aut}_{\mathcal{F}}(\sigma), k^\times) = 0.$$

*Proof.* Since  $k$  is an algebraically closed field of characteristic 2, we have

$$H^1(A, k^\times) = \text{Hom}(A, k^\times) \cong \text{Hom}(A/([A, A]O^{2'}(A)), k^\times)$$

for any finite group  $A$ . Thus we have

$$H^1(C_2, k^\times) = H^1(S_3, k^\times) = 0, \quad H^1(C_3, k^\times) \cong \mathbb{Z}/3.$$

On the other hand, we have  $H^2(A, k^\times) = 0$  for any finite 2-group  $A$ , because any central extension of  $A$  by  $k^\times$  splits. ([2, 3.7.5]) Also, it is well known that

$$H^2(C_3, k^\times) = H^2(S_3, k^\times) = 0.$$

Now the result follows from Proposition 2.1. □

*Proof of Theorem 1.2.* For simplicity denote  $\mathcal{C} = [S(\mathcal{F}^c)]$  and  $\mathcal{A}^i = \mathcal{A}_{\mathcal{F}}^i$ . By Corollary 2.2, we have  $\mathcal{A}^2 = 0$ , and hence  $H^0(\mathcal{C}, \mathcal{A}^2) = 0$ . By Theorem 1.1, it remains to show  $H^1(\mathcal{C}, \mathcal{A}^1) = 0$ .

Case 1:  $\text{Out}_{\mathcal{F}}(P) = 1$ . Then  $\mathcal{A}^1 = 0$  and so  $H^1(\mathcal{C}, \mathcal{A}^1) = 0$ .

Case 2:  $P \cong D_4$ ,  $\text{Out}_{\mathcal{F}}(P) \cong C_3$ . Then  $\mathcal{F}^c$ , and hence  $\mathcal{C}$ , has one object. Thus  $H^1(\mathcal{C}, \mathcal{A}^1) = 0$ .

Case 3:  $P \cong Q_8$ ,  $\text{Out}_{\mathcal{F}}(P) \cong C_3$ . Then  $P$  has a unique (up to  $\mathcal{F}$ -conjugacy)  $\mathcal{F}$ -centric proper subgroup  $R \cong C_4$ . Thus the poset  $\mathcal{C}$  and the functor  $\mathcal{A}^1$  are as follows:

$$\begin{array}{ccc} & [P] & \\ & \swarrow \quad \searrow & \\ [R < P] & & 0 \\ & \swarrow \quad \searrow & \\ & [R] & \end{array} \quad \begin{array}{ccc} & \mathbb{Z}/3 & \\ & \swarrow \quad \searrow & \\ 0 & & 0 \\ & \swarrow \quad \searrow & \end{array}$$

Thus we have  $H^1(\mathcal{C}, \mathcal{A}^1) = 0$ . □

### 3. THE EXTRASPECIAL GROUP OF ORDER $p^3$ AND EXPONENT $p$ , $p$ ODD

Let  $p$  be an odd prime and let  $P$  be the extraspecial group of order  $p^3$  and exponent  $p$ . Ruiz and Viruel [11] classified all fusion systems  $\mathcal{F}$  on  $P$ . First let us recall some basic facts from [11]. Explicitly, one can view  $P$  as a Sylow  $p$ -subgroup of  $\text{SL}_3(p)$  as follows:

$$P = \left\{ \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{F}_p \right) \right\}.$$

In particular, we have

$$Z(P) = [P, P] = \Phi(P) = \left\{ \left( \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid z \in \mathbb{F}_p \right) \right\}.$$

The sequence of groups

$$1 \rightarrow \text{Inn}(P) \xrightarrow{\iota} \text{Aut}(P) \xrightarrow{\pi} \text{Aut}(P/Z(P)) \rightarrow 1,$$

where  $\iota$  is the inclusion and  $\pi$  sends each  $\alpha \in \text{Aut}(P)$  to the induced automorphism  $uZ(P) \mapsto \alpha(u)Z(P)$  of  $P/Z(P)$ , is split exact. More precisely,  $\text{Out}(P) \cong \text{Aut}(P/Z(P)) \cong \text{GL}_2(p)$  and, through the splitting map, one can view  $\text{Out}(P)$  as a subgroup of  $\text{Aut}(P)$ . Moreover, the inclusion of  $\text{GL}_2(p) \cong \text{Out}(P)$  in  $\text{Aut}(P)$ , compatible with the splitting, can be given by sending each  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(p)$  to the automorphism

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & \frac{1}{2}ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & b & \frac{1}{2}bd \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & ad - bc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote the image of  $A$  under this inclusion by  $A_P$ .

Also we have  $\text{Inn}(P) \cong C_p \times C_p$ , and an isomorphism can be given by sending each  $(a, b) \in C_p \times C_p$  to the inner automorphism

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & z + ay - bx \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

There are exactly  $p + 1$  proper centric subgroups of  $P$ :

$$V_i = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & ix \\ 0 & 0 & 1 \end{pmatrix} \mid x, z \in \mathbb{F}_p \right\} \quad (0 \leq i < p), \quad V_p = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{F}_p \right\}.$$

All  $V_i$  ( $0 \leq i \leq p$ ) are elementary abelian normal subgroups of  $P$  of order  $p^2$ . Hence  $\text{Aut}(V_i) \cong \text{GL}_2(p)$  and an isomorphism can be given by sending  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(p)$  to the automorphism

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & ix \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ax + bz & cx + dz \\ 0 & 1 & i(ax + bz) \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & cy + dz \\ 0 & 1 & ay + bz \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote the image of  $A$  under this isomorphism by  $A_{V_i}$ .

Now let  $\mathcal{F}$  be an arbitrary fusion system on  $P$ , and let  $k$  be an algebraically closed field of characteristic  $p$ . The chains in  $[S(\mathcal{F}^c)]$  have length at most 1, and hence

$$(1) \quad H^2([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) = 0.$$

We use the following lemma for computing  $H^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1)$ . It enables us to work with a cochain complex smaller than the one induced from the standard projective resolution of the constant covariant functor  $\underline{\mathbb{Z}}: [S(\mathcal{F}^c)] \rightarrow \mathbf{Ab}$ .

**Lemma 3.1** ([8, 3.2]). *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $\mathcal{A}: [S(\mathcal{F}^c)] \rightarrow \mathbf{Ab}$  be a covariant functor. Let  $C(\mathcal{A})$  be the cochain complex of abelian groups whose component in degree  $n \geq 0$  is equal to*

$$C(\mathcal{A})^n = \bigoplus_{[\sigma]} \mathcal{A}([\sigma])$$

where the direct sum is taken over the set of  $\mathcal{F}$ -conjugacy classes  $[\sigma]$  of chains  $\sigma$  of  $\mathcal{F}$ -centric subgroups of  $P$  of length  $n$ , and whose coboundary maps  $\delta^n: C(\mathcal{A})^{n-1} \rightarrow C(\mathcal{A})^n$  are given by

$$\delta^n(\alpha)([\sigma]) = \sum_{i=0}^n (-1)^i \mathcal{A}([\sigma(i) \rightarrow \sigma])(\alpha([\sigma(i)]))$$

where  $\alpha \in C(\mathcal{A})^{n-1}$ ,  $\sigma = (R_0 < \cdots < R_n)$ , and  $\sigma(i) = (R_0 < \cdots < R_{i-1} < R_{i+1} < \cdots < R_n)$ . Then we have

$$H^n([S(\mathcal{F}^c)], \mathcal{A}) \cong H^n(C(\mathcal{A}))$$

for any integer  $n \geq 0$ .

3.1.  $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}_3(p))$ ,  $3 \nmid (p-1)$ . From [11], we have that

- (1)  $\mathrm{Out}_{\mathcal{F}}(P) \cong \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}_P \right\rangle \times \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}_P \right\rangle$  ( $\mathbb{F}_p^\times = \langle \zeta \rangle$ );
- (2) the  $\mathcal{F}$ -conjugacy classes among the  $V_i$  are  $\{V_0\}$ ,  $\{V_p\}$ ,  $\{V_i \mid 1 \leq i \leq p-1\}$ ;
- (3)  $V_i$  is  $\mathcal{F}$ -radical if and only if  $i = 0, p$ ;
- (4)  $\mathrm{Aut}_{\mathcal{F}}(V_0) \cong \mathrm{Aut}_{\mathcal{F}}(V_p) \cong \mathrm{GL}_2(p)$ .

Then we have

$$\mathrm{Aut}_{\mathcal{F}}(V_1 < P) \cong (C_p \times C_p) \rtimes \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_P \right\rangle,$$

and the restriction map

$$\begin{aligned} \mathrm{Aut}_{\mathcal{F}}(V_1 < P) &\rightarrow \mathrm{Aut}_{\mathcal{F}}(V_1) \\ \alpha &\mapsto \alpha|_{V_1} \end{aligned}$$

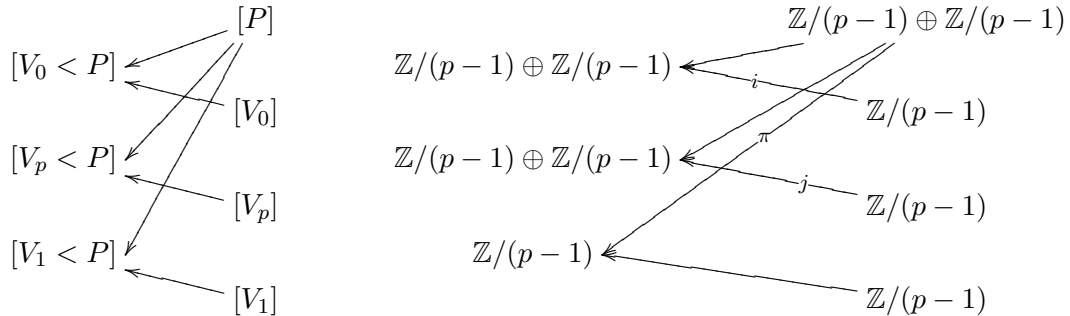
is surjective by Alperin's fusion theorem. The above map has kernel  $\cong C_p$  contained in  $\mathrm{Inn}(P)$  and it sends  $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_P$  to  $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}_{V_1}$ . It follows that

$$\mathrm{Aut}_{\mathcal{F}}(V_1) \cong C_p \rtimes \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}_{V_1} \right\rangle.$$

Also we have  $\mathrm{Aut}_{\mathcal{F}}(V_0 < P) = \mathrm{Aut}_{\mathcal{F}}(V_p < P) = \mathrm{Aut}_{\mathcal{F}}(P)$ , and the restriction maps to  $V_0$  and  $V_p$  are given as follows:

$$\begin{array}{ccc} \mathrm{Aut}_{\mathcal{F}}(P) &\rightarrow & \mathrm{Aut}_{\mathcal{F}}(V_0) & & \mathrm{Aut}_{\mathcal{F}}(P) &\rightarrow & \mathrm{Aut}_{\mathcal{F}}(V_p) \\ \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}_P &\mapsto & \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_{V_0} & & \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}_P &\mapsto & \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}_{V_p} \\ \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}_P &\mapsto & \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}_{V_0} & & \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}_P &\mapsto & \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_{V_p} \end{array}$$

Thus the poset  $[S(\mathcal{F}^c)]$  and the functor  $\mathcal{A}_{\mathcal{F}}^1$  are as follows:



where  $\pi(1, 0) = \pi(0, 1) = 1$ ,  $i(1) = (2, 1)$ ,  $j(1) = (1, 2)$ , and all other maps on the right-hand side are the identity maps. Thus the cochain complex  $C(\mathcal{A}_{\mathcal{F}}^1)$  of Lemma 3.1, after splicing off the identity map on  $\mathbb{Z}/(p-1)$  induced from the inclusion  $[V_1] \rightarrow [V_1 < P]$ , is

$$4(\mathbb{Z}/(p-1)) \xrightarrow{\delta^1} 4(\mathbb{Z}/(p-1)) \rightarrow 0 \rightarrow \cdots$$

where the image of  $\delta^1$  is generated by  $(1, 0, 1, 0)$ ,  $(0, 1, 0, 1)$ ,  $(2, 1, 0, 0)$ , and  $(0, 0, 1, 2)$ . Thus  $\delta^1$  is a bijection, and hence

$$(2) \quad H^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) = 0.$$

3.2.  $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}_3(p))$ ,  $3 \mid (p-1)$ . From [11], we have that

- (1)  $\mathrm{Out}_{\mathcal{F}}(P) = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_P \right\rangle \times \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta^3 \end{pmatrix}_P \right\rangle (\mathbb{F}_p^\times = \langle \zeta \rangle)$ ;
- (2) the  $\mathcal{F}$ -conjugacy classes among the  $V_i$  are  $\{V_0\}$ ,  $\{V_p\}$ ,  $\{V_{\zeta^{3i}} \mid 0 \leq i < \frac{p-1}{3}\}$ ,  $\{V_{\zeta^{3i+1}} \mid 0 \leq i < \frac{p-1}{3}\}$ ,  $\{V_{\zeta^{3i+2}} \mid 0 \leq i < \frac{p-1}{3}\}$ ;
- (3)  $V_i$  is  $\mathcal{F}$ -radical if and only if  $i = 0, p$ ;
- (4)  $\mathrm{Aut}_{\mathcal{F}}(V_0) \cong \mathrm{Aut}_{\mathcal{F}}(V_p) \cong \mathrm{SL}_2(p) \rtimes C_{(p-1)/3}$ .

Then we have

$$\mathrm{Aut}_{\mathcal{F}}(V_i < P) \cong (C_p \times C_p) \rtimes \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_P \right\rangle (0 < i < p),$$

and the restriction map

$$\begin{aligned} \mathrm{Aut}_{\mathcal{F}}(V_i < P) &\rightarrow \mathrm{Aut}_{\mathcal{F}}(V_i) \\ \alpha &\mapsto \alpha|_{V_i} \end{aligned}$$

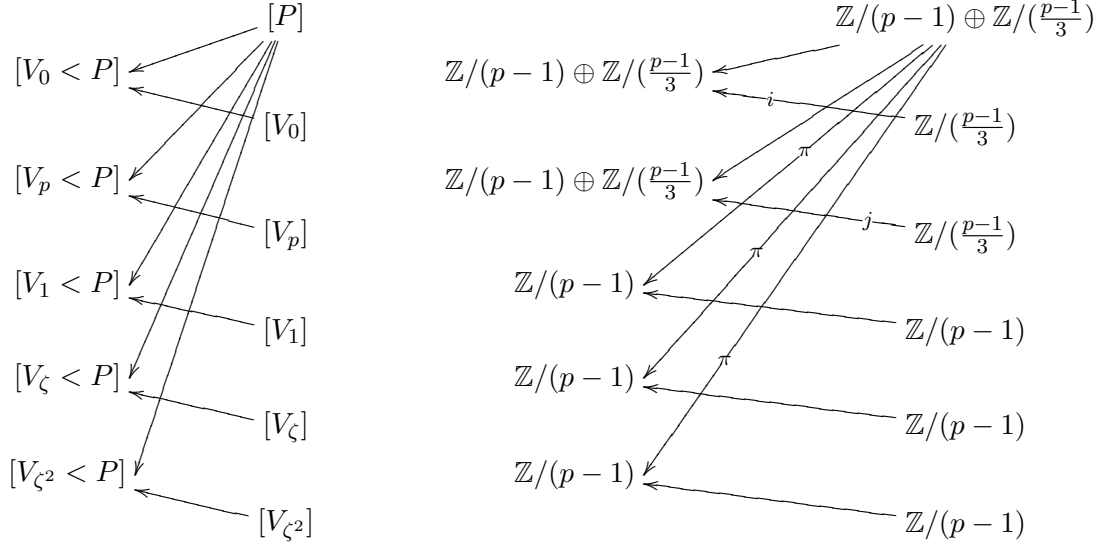
is surjective by Alperin's fusion theorem for  $0 < i < p$ . The above map has kernel  $\cong C_p$  contained in  $\mathrm{Inn}(P)$  and it sends  $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_P$  to  $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}_{V_i}$ . It follows that

$$\mathrm{Aut}_{\mathcal{F}}(V_i) \cong C_p \rtimes \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}_{V_i} \right\rangle (0 < i < p).$$

Also we have  $\mathrm{Aut}_{\mathcal{F}}(V_0 < P) = \mathrm{Aut}_{\mathcal{F}}(V_p < P) = \mathrm{Aut}_{\mathcal{F}}(P)$ , and the restriction maps to  $V_0$  and  $V_p$  are given as follows:

$$\begin{array}{ccc} \mathrm{Aut}_{\mathcal{F}}(P) \rightarrow \mathrm{Aut}_{\mathcal{F}}(V_0) & & \mathrm{Aut}_{\mathcal{F}}(P) \rightarrow \mathrm{Aut}_{\mathcal{F}}(V_p) \\ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_P \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}_{V_0} & & \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}_P \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}_{V_p} \\ \begin{pmatrix} 1 & 0 \\ 0 & \zeta^3 \end{pmatrix}_P \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta^3 \end{pmatrix}_{V_0} & & \begin{pmatrix} 1 & 0 \\ 0 & \zeta^3 \end{pmatrix}_P \mapsto \begin{pmatrix} \zeta^3 & 0 \\ 0 & \zeta^3 \end{pmatrix}_{V_p} \end{array}$$

Thus the poset  $[S(\mathcal{F}^c)]$  and the functor  $\mathcal{A}_{\mathcal{F}}^1$  are as follows:



where  $\pi(1, 0) = 1$ ,  $\pi(0, 1) = 0$ ,  $i(1) = (3, 1)$ ,  $j(1) = (3, 2)$ , and all other maps on the right-hand side are the identity maps. Thus the cochain complex  $C(\mathcal{A}_{\mathcal{F}}^1)$  of Lemma 3.1, after splicing off the three identity maps on  $\mathbb{Z}/(p-1)$  induced from the inclusions  $[V_{\zeta^i}] \rightarrow [V_{\zeta^i} < P]$  ( $i = 0, 1, 2$ ) is

$$\mathbb{Z}/(p-1) \oplus 3(\mathbb{Z}/(\frac{p-1}{3})) \xrightarrow{\delta^1} 2(\mathbb{Z}/(p-1) \oplus \mathbb{Z}/(\frac{p-1}{3})) \rightarrow 0 \rightarrow \dots$$

where the image of  $\delta^1$  is generated by  $(1, 0, 1, 0)$ ,  $(0, 1, 0, 1)$ ,  $(3, 1, 0, 0)$ , and  $(0, 0, 3, 2)$ . Thus  $\delta^1$  is injective, and comparing the order of the groups, we get

$$(3) \quad H^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) \cong \mathbb{Z}/3.$$

Now Theorem 1.3 follows from (1), (2), and (3).

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, AB24 3UE, SCOTLAND, UNITED KINGDOM  
*E-mail address:* `s.park@abdn.ac.uk`