

# On the collection of topologies, generalised metrics and continuity spaces

PHD THESIS

by

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September 2013

*Dedicado a los Abus.  
Los extraño mucho!*

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# Acknowledgements

This work charts a four-year voyage of research investigation, discovery and collaboration resulting in intense fulfilment. Below is my thanks to a possibly incomplete set of people who have made this thesis possible.

As only a shadow of a reflection, my strongest acknowledgment and thanks go to my supervisor, Aisling McCluskey. *What a wonderful journey it has been!* That said, we must view this submission with delight for it signifies the beginning of a long and fruitful collaboration.

Ittay Weiss deserves my thanks for a rich supply of insightful ideas and for introducing me to the wonderful world of categories. The last two chapters of this thesis would have not been possible without him.

I must also thank Paul Bankston for invaluable conversations, mathematical guidance and his fantastic sense of humour. I have learned much from him.

A very special thanks goes to my family: although many leagues yonder you've managed to stay close. Ellen, for her love and support throughout all of these years (*gracias pequeña!*) and to the Murphys for making me feel right at home.

I wish also to thank Alexander Arhangel'skiĭ, Paul Gartside, Robin Knight and Chris Good, all of whom have, at one point or another, made a selfless contribution to this work.

Thanks also to all Riverside Terrapin postgrads; we've had great laughs, good times and a bit of maths! A special acknowledgement to my office-mate and long-time running partner, James McTigue. Many thanks to all members of staff and especially to Mary Kelly who has always been so attentive and helpful.

I must also mention Edwin O'Shea, Claas Röver, Tobias Rossmann and Javier Aramayona, four exceptional mathematicians to whom I am deeply indebted.

Finally, I would also like to thank the Irish Research Council for Science, Engineering and Technology for their financial support.

Jorge Bruno

# Introduction

This article-based thesis comprises a collection of four papers, each of which constitutes a chapter written in manuscript form. The elemental theme behind the first three chapters is the study of several collections of (topological) structures on an arbitrary set. A more holistic and unifying view of these structures is given in Chapter 4 where a categorical approach is taken and the individuality of fixing a set is abandoned.

For a fixed set  $X$ , the first collection of structures we investigate is that of all topologies on  $X$ . This collection, first introduced by Birkhoff ([Bir36]), is commonly denoted by  $Top(X)$  and has undergone exploration mainly as an order-theoretic structure ([LA75], [VL72]). The motivation behind Chapter 1 ([BM12]) is that of locating  $Top(X)$  as a subspace of the totally disconnected Hausdorff space  $2^{\mathcal{P}(X)}$ . Other well-known collections of structures on  $X$  (ultrafilters, filters, lattices, etc) are also investigated as subspaces of  $2^{\mathcal{P}(X)}$  with  $LatB(X)$  (the collection of all sublattices on  $\mathcal{P}(X)$  containing  $\emptyset$  and  $X$ ) shown to be a Hausdorff compactification of  $Top(X)$ . The last section of Chapter 1 illustrates a simple model-theoretic result that provides a connection between first-order definitions of subsets of  $2^{\mathcal{P}(X)}$  and a highly desired topological property: that of compactness. It is worth noting that the aforementioned first-order language will speak of the elements contained by each element of a given collection of structures on  $X$ . A simple corollary will then be that  $Top(X)$  is far from being first-order definable.

The following chapter ([BM13]) is a continuation and further refinement of results found in Chapter 1. This chapter begins by applying a well-known result by Frink ([Fri42]) in establishing an equivalence between the topological closure of sublattices

tices of  $2^{\mathcal{P}(X)}$  and their completely distributive completions (within the same ambient structure). Thereafter, the equivalence is exploited in searching for infinite compact subsets of  $Top(X)$  and in exploring the Borel complexity of  $Top(X)$  within  $2^{\mathcal{P}(X)}$ . In particular, the latter yields that (as far as  $G_\delta$  and  $F_\sigma$  sets can reach)  $Top(X)$  remains stubbornly elusive. The last section of this paper shows that as a topological space (with the subspace topology inherited from  $2^{\mathcal{P}(X)}$ )  $Top(X)$  is surprisingly versatile; copies of one-point and Stone-Čech compactifications of a large collection of spaces can be identified within  $Top(X)$ . As a simple example, it is shown that a certain collection of ultratopologies on  $X$  can be identified with Stone-Čech compactifications of  $X$  endowed with the discrete topology.

Another interesting collection of structures on a set  $X$  is what Chapter 3 refers to (and explores in detail) as *weight structures*. Simply put, these are elements of the compact Hausdorff space (and completely distributive lattice)  $[0, \infty]^{X \times X}$ . Since weight structures can be interpreted as distance assignments on pairs of points in  $X$ , it is natural to consider the following restrictions on the behaviour of any such structure.

- (i)  $\forall x \in X, d(x, x) = 0$
- (ii)  $\forall x, y \in X, d(x, y) < \infty$  (*finite-valued*)
- (iii)  $\forall x, y \in X, d(x, y) = 0 \Rightarrow x = y$  (*separation*)
- (iv)  $\forall x, y \in X, d(x, y) = d(y, x)$  (*symmetry*)
- (v)  $\forall x, y, z \in X, d(x, y) + d(y, z) \geq d(x, z)$  (*triangle inequality*).

By letting  $P$  denote any collection of these axioms and  $W_P(X) = \{d \in [0, \infty]^{X \times X} \mid d \text{ satisfies all axioms in } P\}$ , it is possible to view any such collection of axioms as a subset of the ambient set. In particular, the theory of metric spaces is the one concerned with the study (as objects) of weight structures that satisfy all of the above axioms. As noted above, the ambient set  $[0, \infty]^{X \times X}$  when ordered pointwise (i.e.  $d \leq m \in [0, \infty]^{X \times X} \Leftrightarrow \forall x, y \in X, d(x, y) \leq m(x, y)$ ) forms a complete lattice; every collection  $W_P(X)$  becomes a poset and the obvious inclusions  $W_P \hookrightarrow W_Q$

(for  $Q \subseteq P$ ) acquire more structure (that of order-preserving functions). The first section of Chapter 3 is concerned with the lattice-theoretic structure of the above defined collections. Moreover, as with Chapter 1, we explore a connection between certain first-order predicates and the order-theoretic collection of weight structures they define. However, in contrast with the previous chapters, this chapter introduces a categorical investigation of the inclusions  $W_P \hookrightarrow W_Q$ . For instance, given any weight structure that does not satisfy symmetry, it is then natural to ask:

*Is there a way to naturally symmetrize such an element? Is such a process unique? If not, is there an optimal one?*

Expressed in the language of categories, we provide answers to the above by means of adjunctions in much the same spirit as how the process of turning a base into a topology can be seen as an adjunction.

Another distinguishing feature of this chapter (in contrast to the previous two) can be interpreted as the study of connections between the structures found in Chapters 1 and 2 and those introduced in Chapter 3. For instance, one way to generate topologies given any weight structure  $d \in W_P$  is by mapping  $d$  to the covers its right and left  $\epsilon$ -balls generate. More precisely, for  $B_L(x)_\epsilon = \{y \in X \mid d(x, y) < \epsilon\}$  (resp.  $B_R(x)_\epsilon = \{y \in X \mid d(y, x) < \epsilon\}$ ) then

$$d \mapsto (\{B_L(x)_\epsilon \mid \epsilon > 0\}, \{B_R(x)_\epsilon \mid \epsilon > 0\}).$$

Since  $Top(X)$  is a complete lattice, the above assignment can be interpreted as an order-theoretic function  $W_P(X) \rightarrow Top(X) \times Top(X)$ . A large section in this chapter is then devoted to a detailed investigation of the above function with particular emphasis on meet/join preservation, structure of fibers and order-preservation.

In the literature ([KGM97]) one can find results involving lattice-embeddability within the complete lattice  $Top(X)$ . Along the same lines, the last section is concerned with the lattice-theoretic structure of the collection of all extended metrics  $Met(X) = W_{(i),(iv),(v)}(X)$  (i.e. pseudometrics), Menger convexity (and its dual) within  $Met(X)$  along with several lattice-embeddability properties of  $Met(X)$ .

As noted above, it is possible to abandon the individuality of fixing a set by con-

sidering the class of all topological spaces in one fell swoop. Augmenting this class by adding continuous functions as their respective morphisms, the category **Top** is then constructed. Similarly, for any weight structure,  $d$ , that satisfies (i) there exists a standard topology,  $\tau$ , generated in terms of  $\epsilon$ -balls. That is,  $A \in \tau$  iff for all  $x \in A$  there exists  $\epsilon > 0$  so that  $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\} \subseteq A$ . These  $\epsilon$ -balls are only guaranteed to be open (and thus form a base for  $\tau$ ) provided  $d$  satisfies (v). In the literature ([ML98] and [Law02]) one can find an ample selection of candidates for morphisms between weight structures: continuous maps, contractions, uniformly continuous maps, Lipschitz maps, etc. We adopt continuous functions as morphisms between weight structures. More precisely, we adopt the  $\epsilon$ - $\delta$  definition of continuity between weight structures. It is important to notice that topological continuity and  $\epsilon$ - $\delta$  continuity are equivalent iff the weight structure satisfies the triangle inequality. Otherwise we can only assume that  $\epsilon$ - $\delta$  continuity implies topological continuity. It is then natural to construct two collections (categories) of weight structures: those that demand the triangle inequality and those that do not. Let  $\mathbf{W}^\Delta$  (resp.  $\mathbf{W}$ ) denote the category of weight structures that satisfy (i) and (v) (resp. (i)) with  $\epsilon$ - $\delta$  continuous functions. Let  $\mathcal{M}$  and  $\mathcal{P}$  denote the functors that take a weight structure from  $\mathbf{W}^\Delta$  and  $\mathbf{W}$ , respectively, and send it to the topology it generates in **Top**. For reasons outlined above,  $\mathcal{P}$  is a faithful extension of  $\mathcal{M}$  and  $\mathcal{M}$  is fully faithful. Moreover, since all objects from  $\mathbf{W}$  generate sequential spaces then  $\mathcal{P}[\mathbf{W}]$  is a proper subcategory of **Top**. It is natural then to ask the following.

*Is there a way to naturally extend either or both categories so as to capture the abstract concept of topology in terms of the intuitive notion of ‘distance assignment’ between points? More precisely*

- (a) Locate an extension  $\mathbf{W}^\Delta \rightarrow \mathbf{M}$  of  $\mathbf{W}^\Delta$  such that the functor  $\mathbf{W}^\Delta \rightarrow \mathbf{Top}$  extends to an equivalence  $\mathcal{M} : \mathbf{M} \rightarrow \mathbf{Top}$ .
- (b) Locate an extension  $\mathbf{W} \rightarrow \mathbf{P}$  of  $\mathbf{W}$  such that the functor  $\mathbf{W} \rightarrow \mathbf{Top}$  extends to faithful surjective functor  $\mathcal{P} : \mathbf{P} \rightarrow \mathbf{Top}$ .

The last chapter of this thesis focuses on possible solutions to the above by naturally extending  $\mathbf{W}^\Delta$  and  $\mathbf{W}$ . The objects for such extensions are based on R. C. Flagg's *continuity spaces* where the concept of positivity is an intrinsic one ([Fla97]). In a nutshell, continuity spaces are a generalization of weight structures where distances on a set are allowed to be *valued* on a very special collection of completely distributive lattices. In particular, since Flagg's construction essentially gives rise to  $\text{Ob}(\mathbf{M})$  (we only define the morphisms in  $\mathbf{M}$  and note the equivalence of categories) our aim within  $\mathbf{M}$  is to provide a topological characterization of an obvious collection of full subcategories of  $\mathbf{M}$  (that of symmetric, disjoint and separated continuity spaces). On the other hand,  $\mathbf{P}$  will be constructed based on  $\mathbf{M}$  so as to naturally extend  $\mathbf{W}$ . We will investigate  $\mathbf{P}$  in greater detail than  $\mathbf{M}$ : we provide constructions for all topological limits and some colimits within  $\mathbf{P}$ . Even though  $\mathcal{P}$  will turn out not to be full,  $\mathbf{Top}$  and  $\mathbf{P}$  will be shown to be surprisingly similar; we show that  $\mathcal{P}$  is continuous and lifts all small limits and that  $\mathcal{P}$  preserves and lifts all small coproducts and direct limits.



# Chapter 1

## Topologies as points within $2^{\mathcal{P}(X)}$

Jorge L. Bruno and Aisling E. McCluskey

### Abstract

A topology on a nonempty set  $X$  specifies a natural subset of  $\mathcal{P}(X)$ . By identifying  $\mathcal{P}(\mathcal{P}(X))$  with the totally disconnected compact Hausdorff space  $2^{\mathcal{P}(X)}$ , the lattice  $Top(X)$  of all topologies on  $X$  is a natural subspace therein. We investigate topological properties of  $Top(X)$  and give sufficient model-theoretic conditions for a general subspace of  $2^{\mathcal{P}(X)}$  to be compact.

### 1.1 Introduction

Families of subsets of a nonempty set  $X$  occur naturally as elements of the product space  $2^{\mathcal{P}(X)}$  (where  $2$  carries the discrete topology) when one identifies  $\mathcal{P}(\mathcal{P}(X))$  with  $2^{\mathcal{P}(X)}$ . Thus for example  $Ult(X)$ ,  $Fil(X)$ ,  $Top(X)$  and  $Lat(X)$ , denoting the families of all ultrafilters, filters and topologies on  $X$  and sublattices of  $\mathcal{P}(X)$  respectively, occur as natural subspaces of  $2^{\mathcal{P}(X)}$ . Our investigation focuses principally on the topological nature of the subspace  $Top(X)$  while also establishing sufficient model-theoretic conditions for a general subspace of  $2^{\mathcal{P}(X)}$  to be compact.

$Top(X)$  has been an object of study since its introduction by Birkhoff in [Bir36]. It

is a complete atomic and complemented lattice under inclusion and it has undergone exploration mainly as an order-theoretic structure ([LA75], [VL72]). More recent work ([GMW02] and [MW04]) has concerned the order-theoretic nature of intervals of the form  $[\sigma, \rho] = \{\tau \in Top(X) : \sigma \leq \tau \leq \rho\}$ . Our interest is motivated by the location of  $Top(X)$  within the product space  $2^{\mathcal{P}(X)}$  and of the subsequent topological structure it assumes from this location, about which little appears to be known. For Tychonoff  $X$ , the recognition of  $Ult(X)$  when regarded as a subspace of  $2^{\mathcal{P}(X)}$  as a homeomorphic copy of  $\beta(X)$  suggests potential for this study.

## 1.2 Preliminaries and Examples

We shall assume that  $X$  is an infinite set throughout. We begin by recalling the subbasic open sets of the product space  $2^{\mathcal{P}(X)}$ :

**Definition 1.2.1.** Given  $A \subseteq X$ , we define  $A^+ := \{\mathcal{F} \in 2^{\mathcal{P}(X)} : A \in \mathcal{F}\}$  and  $A^- := \{\mathcal{F} \in 2^{\mathcal{P}(X)} : A \notin \mathcal{F}\}$  and observe that these are the subbasic open subsets of the product space  $2^{\mathcal{P}(X)}$ .

*Remark 1.2.2.* Note that the subbasic open sets are also closed since  $A^+ = 2^{\mathcal{P}(X)} \setminus A^-$ . Adopting standard order-theoretic notation, we observe that  $A^+ = \{A\}^\uparrow$  where  $\theta^\uparrow = \{\phi \in \mathbb{P} : \theta \leq \phi\} = [\theta, \infty)$  for a partially ordered set (poset)  $(\mathbb{P}, \leq)$ . Given  $\phi, \theta$  and  $\psi$  in the poset  $2^{\mathcal{P}(X)}$ , we note the more general result that  $\phi^\downarrow = \{\psi \mid \psi \subseteq \phi\}$  and  $\phi^\uparrow = \{\psi \mid \phi \subseteq \psi\}$  are each closed sets. Simply observe, for example, that for any  $\theta \in 2^{\mathcal{P}(X)} \setminus \phi^\downarrow$ , there is some  $B \in \theta$  such that  $B \not\subseteq \phi$ . Then  $B^+$  is an open neighbourhood of  $\theta$  that is disjoint from  $\phi^\downarrow$ , whence  $\phi^\downarrow$  is closed.

Given elements  $\phi, \theta$  in an arbitrary poset  $\mathbb{P}$ ,  $\phi^\uparrow$  and  $\theta^\downarrow$  define the subbasic closed sets for the *interval topology* on  $\mathbb{P}$ . Thus  $2^{\mathcal{P}(X)}$ 's product topology contains the associated interval topology. Note further that the product space  $2^{\mathcal{P}(X)}$  is both minimal Hausdorff and maximal compact. Thus a lattice that can be realized as a closed subset of  $2^{\mathcal{P}(X)}$  has a Hausdorff interval topology precisely when the lattice-morphism is also a homeomorphism.

## Examples in $Top(X)$

**Example 1.2.3.** The set of  $T_1$  topologies on  $X$  is closed in  $Top(X)$ .

*Proof.* Any  $T_1$  topology on  $X$  must contain the cofinite topology  $\mathcal{C} = \{A \subseteq X : A \text{ is cofinite}\} = \{A_i : i \in I\}$ . Notice that  $\mathcal{C}^\uparrow = \bigcap_i A_i^+$ . Then  $\mathcal{C}^\uparrow \cap Top(X)$  is closed in  $Top(X)$  and it is precisely the set of  $T_1$  topologies on  $X$ . □

As an easy consequence of the above remark, we have that if  $Q$  is an expansive (contractive) topological property for which the set  $Top_Q(X)$  of all  $Q$  topologies on  $X$  has finitely many minimal (maximal) elements, then  $Top_Q(X)$  must be closed in  $Top(X)$ . Moreover, if  $\mathbb{Q} = \{Q_i : i \in I\}$  is a collection of expansive (contractive) topological properties for which  $Top_{Q_i}(X)$  has finitely many minimal (maximal) elements for each  $i$ , then  $Top_{\mathbb{Q}}(X)$  is closed in  $Top(X)$ .

**Example 1.2.4.** Let  $f : X \rightarrow X$  be a function and  $\sigma$  any topology on  $X$ . Then

$$\begin{aligned}\sigma_f &= \{\rho \in Top(X) \mid f : (X, \sigma) \rightarrow (X, \rho) \text{ is continuous}\} \text{ and} \\ \sigma^f &= \{\rho \in Top(X) \mid f : (X, \rho) \rightarrow (X, \sigma) \text{ is continuous}\}\end{aligned}$$

are closed in  $Top(X)$ .

*Proof.* Let  $\tau = \{X\} \cup \{A \mid f^{-1}(A) \in \sigma\}$ . In a similar spirit to the quotient topology, this topology is the largest topology which makes  $f$  continuous and thus we have  $\sigma_f = \tau^\downarrow \cap Top(X)$ . As shown above, this set is closed in  $Top(X)$ . The proof is similar for  $\sigma^f$ . □

Note that if  $f : X \rightarrow X$  is injective and if  $\sigma$  is the cofinite topology  $\mathcal{C}$ , then  $\sigma^f$  is precisely the set of all  $T_1$  topologies on  $X$ .

**Example 1.2.5.** Let  $f : X \rightarrow X$  be a function. Then

$$\begin{aligned} Cns(f) &= \{\rho \in Top(X) \mid f : (X, \rho) \rightarrow (X, \rho) \text{ is continuous}\}, \\ Open(f) &= \{\rho \in Top(X) \mid f : (X, \rho) \rightarrow (X, \rho) \text{ is open}\}, \text{ and} \\ Closed(f) &= \{\rho \in Top(X) \mid f : (X, \rho) \rightarrow (X, \rho) \text{ is closed}\} \end{aligned}$$

are closed in  $Top(X)$ .

*Proof.* Observe that if  $\rho \in Top(X) \setminus Cns(f)$  we must have  $A \in \rho$  so that  $f^{-1}(A) \notin \rho$ . Then  $A^+ \cap [f^{-1}(A)]^- \cap Top(X)$  is open in  $Top(X)$ , contains  $\rho$  and is disjoint from  $Cns(f)$ , whence  $Cns(f)$  is closed in  $Top(X)$ . The proof is similar for  $Open(f)$  and  $Closed(f)$ .  $\square$

In view of the above, if  $f$  is a bijection then  $Hom(f) = \{\sigma \in Top(X) \mid f : (X, \sigma) \rightarrow (X, \sigma) \text{ a homeomorphism}\}$  is also closed in  $Top(X)$  (since  $Hom(f) = Cns(f) \cap Open(f)$ ). Note also that for any function  $f : X \rightarrow X$ ,  $Cns(f)$  is a complete sublattice of  $Top(X)$ .

## Examples of function spaces

Let  $Y$  be an infinite set with  $A, B \subseteq Y$ , and denote by  $\mathbb{F}_B^A$  the collection of all partial functions from  $A$  to  $B$ ; that is,  $\mathbb{F}_B^A = \{f : C \rightarrow B \mid C \subseteq A\}$ . Then  $[\mathbb{F}_B^A]^{<\omega} = \{f \in \mathbb{F}_B^A \mid |f| \in \omega\}$  is the set of all finite such partial functions. Clearly  $[\mathbb{F}_B^A]^{<\omega} = \mathbb{F}_B^A$  when  $A$  is finite. Consider next an example where  $X$  is a special type of set derived from  $Y$ :

**Example 1.2.6.** Let  $X = \mathcal{P}(Y)$  where  $Y$  is infinite and let  $A, B \subseteq Y$ . Then

- (i)  $\mathbb{F}_B^A$  is a compact subspace of  $2^{\mathcal{P}(X)}$ ,
- (ii)  $[\mathbb{F}_B^A]^{<\omega}$  is a proper dense subset of  $\mathbb{F}_B^A$  if and only if  $A$  is infinite, and
- (iii)  $[\mathbb{F}_B^A]^{<\omega}$  is not locally compact if and only if  $A$  is infinite.

*Proof.* (i) Note first that  $\langle a, b \rangle \in \mathcal{P}(X)$  for  $a, b \in Y$  (that is,  $\{\{a\}, \{a, b\}\} \in \mathcal{P}(X)$ ). Thus,  $\mathbb{F}_B^A \subset 2^{\mathcal{P}(X)}$ . If a point  $p$  in  $2^{\mathcal{P}(X)}$  is not a function from a subset of  $A$  to  $B$  then either

- (1)  $p \not\subseteq A \times B$  so that  $p$  contains an element  $C$  where  $C$  is not an element of  $A \times B$ . Then  $C^+ \cap \mathbb{F}_B^A = \emptyset$ , or
- (2)  $p \subseteq A \times B$  and  $p$  contains at least two sets are of the form:  $\langle a, b \rangle$  and  $\langle a, b' \rangle$  where  $b \neq b'$ . In this case,  $\langle a, b \rangle^+ \cap \langle a, b' \rangle^+$  is an open neighbourhood of  $p$  disjoint from  $\mathbb{F}_B^A$ .

Thus  $\mathbb{F}_B^A$  is closed in  $2^{\mathcal{P}(X)}$  and thus compact.

(ii) It is easy to see that any basic open set about any  $f \in \mathbb{F}_B^A$  defines a finite partial function from  $A$  to  $B$ .

(iii) Observe from (i) and (ii) that  $\mathbb{F}_B^A$  is a Tychonoff compactification of  $[\mathbb{F}_B^A]^{<\omega}$  and recall that a Tychonoff space is locally compact if and only if its remainder is closed in each of its compactifications. Let  $p \in [\mathbb{F}_B^A]^{<\omega}$  and let  $p \in \bigcap C_i^+ \cap \bigcap D_j^- = U$ . Each  $C_i$  is of the form  $\langle a, b \rangle$  since  $p$  is a function, while in the worst case scenario each  $D_j$  has also the form  $\langle a^*, b^* \rangle$ . Thus any function in  $U$  is compelled to include finitely many ordered pairs (as specified by each  $C_i$ ) and at worst to exclude finitely many certain other ordered pairs within  $A \times B$  (as potentially specified by  $D_j$ ). Since  $A$  is infinite, infinitely many infinite extensions of  $p$  must exist in  $U$ . Thus  $[\mathbb{F}_B^A]^{<\omega}$  has empty interior in  $\mathbb{F}_B^A$ . In particular,  $\mathbb{F}_B^A \setminus [\mathbb{F}_B^A]^{<\omega}$  is not closed and so  $[\mathbb{F}_B^A]^{<\omega}$  is not locally compact. □

Notice that item (1) from the previous example shows that the set of all subsets from  $A \times B$  is also compact. We can define in a similar way the set  $\mathbb{I}_B^A$  of injective partial functions from  $A$  to  $B$ .

**Example 1.2.7.**  $\mathbb{I}_B^A$  is compact in  $2^{\mathcal{P}(X)}$  where  $X = Y \cup \mathcal{P}(Y)$  with  $A, B \subseteq Y$ .

*Proof.* The above example illustrates how to separate all functions in  $\mathbb{F}_B^A$  from other objects in  $2^{\mathcal{P}(X)}$ . Thus we must only focus on separating all injective functions from

all other functions in  $\mathbb{F}_B^A$ . Let  $f : C \rightarrow B$  be a function in  $\mathbb{F}_B^A$  which is not injective. Then there exists distinct elements  $a, a'$  in  $C$  so that  $f(a) = f(a')$  from which it follows that  $\langle a, f(a) \rangle^+ \cap \langle a', f(a) \rangle^+ \cap \mathbb{I}_B^A = \emptyset$ .  $\square$

Let  $\mathbb{O}_B^A = \{f \in \mathbb{F}_B^A \mid f \text{ is onto}\}$ . The following example assumes that  $|A| \geq |B|$  and the preceding notation. Of course, when  $A$  is finite, then  $\mathbb{O}_B^A$  is finite and thus compact.

**Example 1.2.8.**  $\mathbb{O}_B^A$  is a dense proper subset of  $\mathbb{F}_B^A$  precisely when  $A$  is infinite.

*Proof.* Let  $f \in \mathbb{F}_B^A$  and let  $\bigcap C_i^+ \cap \bigcap D_j^-$  be a basic open neighbourhood of  $f$ . As described in the proof of Example 2.6 (iii) since  $A$  is infinite, it is straightforward to find an onto function from  $A$  to  $B$  in this neighbourhood.  $\square$

## Almost disjoint families

**Definition 1.2.9.** Let  $\kappa$  be an infinite cardinal. If  $x, y \subset \kappa$  and  $|x \cap y| < \kappa$ , then  $x$  and  $y$  are said to be **almost disjoint** (*a.d.*). An *a.d.* **family** is a collection  $\mathcal{A} \subseteq \mathcal{P}(\kappa)$  so that for all  $x \in \mathcal{A}$ ,  $|x| = \kappa$  and any two distinct elements of  $\mathcal{A}$  are *a.d.* Such a family is **maximal** (*m.a.d.f.*) whenever it is not contained in any other *a.d.* family.

The following lemma shows that given a cardinal  $\kappa$  the collection of all *a.d.* families of  $\kappa$  is a compact subset of  $2^{\mathcal{P}(\kappa)}$ .

**Lemma 1.2.10.** *Let  $\kappa$  be any cardinal. For any cardinal  $\lambda \leq \kappa$  the set  $\mathcal{A}_\lambda = \{\mathcal{A} \in 2^{\mathcal{P}(\kappa)} \mid \forall x, y \in \mathcal{A}, |x| = |y| = \kappa \text{ and } |x \cap y| < \lambda\}$  is compact in  $2^{\mathcal{P}(\kappa)}$ . In particular, the collection of all almost disjoint families (i.e.  $\mathcal{A}_\kappa$ ) is compact.*

*Proof.* Note that if  $\alpha \in p \in 2^{\mathcal{P}(\kappa)}$  with  $|\alpha| < \kappa$ , then  $\alpha^+ \cap \mathcal{A}_\lambda = \emptyset$ . Suppose that  $p \in 2^{\mathcal{P}(\kappa)} \setminus \mathcal{A}_\lambda$  and that for all  $\alpha \in p$ ,  $|\alpha| = \kappa$ . Then  $p$  must contain at least two subsets of  $\kappa$ ,  $\alpha$  and  $\beta$  say, so that  $|\alpha \cap \beta| \geq \lambda$ . In turn,  $\alpha^+ \cap \beta^+ \cap \mathcal{A}_\lambda = \emptyset$ .  $\square$

In light of the above, for a given cardinal  $\kappa$ , in ZFC we must have that the cardinality of any maximal chain in  $\mathcal{A}_\kappa$  is at least  $\kappa^+$ . To see this, take any chain  $C$  with

cardinality  $\kappa$  and notice that  $\bigcup C \in \overline{C} \subset \mathcal{A}_\kappa$ . If  $|\bigcup C| = \kappa^+$  then we can always augment  $C$  to a chain of size  $\kappa^+$ . If not, then a simple diagonal argument shows that  $\bigcup C$  is not a *m.a.d.f.* family and in turn  $C$  is not maximal.

### 1.3 Main results

We begin with some preliminary definitions:

**Definition 1.3.1.** For  $X$  an infinite set,  $Lat(X)$  denotes the set of all sublattices of  $\mathcal{P}(X)$ ,  $Lat(X, \vee)$  is the set of all join complete sublattices of  $\mathcal{P}(X)$  ( $Lat(X, \wedge)$  is defined dually) and  $LatB(X)$  is the set of all sublattices of  $\mathcal{P}(X)$  that contain  $\emptyset$  and  $X$ .

**Lemma 1.3.2.** *Lat(X) is closed in  $2^{\mathcal{P}(X)}$  and each of  $Lat(X, \vee)$  and  $Lat(X, \wedge)$  is dense in  $Lat(X)$ .*

*Proof.* It is routine to show that an element  $\theta$  of  $2^{\mathcal{P}(X)}$  that is not a sublattice of  $\mathcal{P}(X)$  has a basic open neighbourhood that is disjoint from  $Lat(X)$ . The failure of  $\theta$  to be a sublattice provides sets, according to how the failure is manifested, with which to assemble an appropriate open neighbourhood. Next, let  $L$  be a sublattice of  $\mathcal{P}(X)$  and let  $\bigcap A_i^+ \cap \bigcap B_j^-$  be a basic open neighbourhood of  $L$ . Since  $L$  is a lattice, the join of any family of  $A_i$ s can not yield any  $B_j$ . Thus we can generate from the  $A_i$ s (by closing under intersection and union) a finite and therefore complete lattice .

□

**Proposition 1.3.3.** *LatB(X) is a (Hausdorff) compactification of Top(X).*

*Proof.* To see that  $LatB(X)$  is closed, and therefore compact, in  $2^{\mathcal{P}(X)}$ , it suffices by Lemma 3.2 to consider those sublattices of  $\mathcal{P}(X)$  that omit either  $X$  or  $\emptyset$ . We can then use  $X^-$  or  $\emptyset^-$ , as appropriate, as a neighbourhood of the given sublattice that is disjoint from  $LatB(X)$ . Finally, if  $L \in LatB(X)$  with  $\bigcap A_i^+ \cap \bigcap B_j^-$  an open neighbourhood of  $L$ , then no  $B_j$  can be the meet or join of any collection of  $A_i$ s. Thus we can define a (finite) topology with subbase  $\{A_i\}$  which clearly belongs to the given neighbourhood. That is,  $Top(X)$  is dense in  $LatB(X)$ .

□

The previous theorem indicates that the concept of generalized topologies on a fixed set  $X$  is that of sublattices of  $\mathcal{P}(X)$  containing  $X$  and  $\emptyset$ . That is, limits of topologies are just elements from  $LatB(X)$ .

**Corollary 1.3.4.**  *$Top(X)$  is not closed, and therefore not compact, in  $2^{\mathcal{P}(X)}$ .*

*Proof.*  $LatB(X)$  properly contains  $Top(X)$  as a dense subset. Since  $LatB(X)$  is closed in  $2^{\mathcal{P}(X)}$ , then  $\overline{Top(X)} = LatB(X)$ . Thus  $Top(X)$  can be neither closed nor compact in  $2^{\mathcal{P}(X)}$ . □

**Lemma 1.3.5.**  *$Top(X)$  has empty interior in  $LatB(X)$ .*

*Proof.* Take any topology  $\tau$  on  $X$  and a basic open neighbourhood  $\bigcap A_i^+ \cap \bigcap B_j^-$  of  $\tau$ . We seek an infinite subset  $S$  of  $X$  and a partition  $\mathcal{P} = \{P_k\}_{k \in \kappa}$  of  $S$  into infinitely many subsets, whose finite joins and meets cannot be any  $B_j$ .  $\mathcal{P}$  together with the  $A_i$ s will generate an element of  $LatB(X)$  that is in the above-mentioned neighbourhood and that is not join complete (and thus not a topology). There are two possibilities: either the complement of  $\bigcup A_i \cup \bigcup B_j$  is finite or it is not. In the latter case, just take  $S$  to be the (infinite) complement, in which case any partition of  $S$  into infinitely many subsets will suffice. For the former case, rename all infinite  $A_i$ s and  $B_j$ s as  $C_k$ s. Let  $\mathcal{C}_1 = C_1$ ,

$$\mathcal{C}_2 = \begin{cases} \mathcal{C}_1 \cap C_2, & \text{if } |\mathcal{C}_1 \cap C_2| \geq \aleph_0 \\ \mathcal{C}_1, & \text{otherwise.} \end{cases}$$

In general,

$$\mathcal{C}_k = \begin{cases} \mathcal{C}_{k-1} \cap C_k, & \text{if } |\mathcal{C}_{k-1} \cap C_k| \geq \aleph_0 \\ \mathcal{C}_{k-1}, & \text{otherwise.} \end{cases}$$

The process terminates with an infinite set  $\mathcal{C}_n$ . Let  $S$  be that subset of  $\mathcal{C}_n$  obtained

by removing any elements that occur in intersections of finite cardinality with  $A_i$ s and  $B_j$ s (that is, those that did not partake in constructing  $\mathcal{C}_n$ ). That is,

$$S = \mathcal{C}_n \setminus \left[ \bigcup_{|\mathcal{C}_n \cap A_i| \in \omega} (\mathcal{C}_n \cap A_i) \cup \bigcup_{|\mathcal{C}_n \cap B_j| \in \omega} (\mathcal{C}_n \cap B_j) \right].$$

Then any partition  $\mathcal{P} = \{P_k\}_{k \in \kappa}$  of  $S$  into infinitely many subsets together with the  $A_i$ s will generate the required family  $L$ , say, in  $LatB(X)$ . Indeed, note that by design  $A_i \cap S = \emptyset$  or  $A_i \cap S = S$  for all  $i$  (the same is true for all  $B_j$ ). It remains to show that no  $B_j$  belongs to  $L$ . Let  $k$  be any number so that  $B_k \cap S = \emptyset$ . Then for  $B_k \in L$  it must be that  $B_k$  is generated solely by the use of  $A_i$ s (contradiction). Otherwise,  $B_k \cap S = S$ . To this end, notice that no finite collection of elements from  $\mathcal{P}$  can yield  $S$ . Consequently, we find again that  $B_k$  must then be generated by taking a finite union of  $A_i$ s (contradiction). □

**Corollary 1.3.6.** *Top(X) is not locally compact.*

*Proof.* By Lemma 1.3.5,  $LatB(X) \setminus Top(X)$  is not closed and the result follows. □

**Corollary 1.3.7.** *Top(X) is dense and co-dense in LatB(X).*

*Proof.* The proof is immediate from Proposition 1.3.3 and Lemma 1.3.5. □

## Some Model Theory

Let  $\mathcal{L}$  be the first order language of Boolean algebras and  $P$  a unary predicate symbol ([Man99]). Expand  $\mathcal{L}$  by adding  $P$  and denote the expanded language by  $\mathcal{L}(P)$  ([Man96]).

**Definition 1.3.8.** Let  $\Phi$  be a set of sentences of  $\mathcal{L}(P)$ . We say that  $\Phi$  is a definition for a set  $\mathcal{F} \subset 2^{\mathcal{P}(X)}$  provided that  $\mathcal{F}$  is the collection of all  $F \subset \mathcal{P}(X)$  so that  $(\mathcal{P}(X), F) \models \Phi$ . Similarly, a set of sentences  $\{\Phi_i\}$  defines a family  $\mathcal{F}$  provided that  $\mathcal{F}$  contains all  $F$  so that  $(\mathcal{P}(X), F) \models \Phi_i$  for all  $i$ .

**Theorem 1.3.9.** *A subspace of  $2^{\mathcal{P}(X)}$  is compact if it has a definition expressible as a universal sentence of  $\mathcal{L}(P)$ .*

*Proof.* Let  $\Phi(x_1, x_2, \dots, x_n, P)$  be a well formed formula. We can assume that  $\Phi$  is in conjunctive normal form; that is,  $\Phi$  is a conjunction of atomic formulas and negation of atomic formulas. Thus  $\Phi(x_1, x_2, \dots, x_n, P) = \alpha_1(x_1, x_2, \dots, x_n, P) \wedge \dots \wedge \alpha_k(x_1, x_2, \dots, x_n, P)$  where each  $\alpha_i$  is atomic or a negation of an atomic formula. Note that there is no loss of generality here since universal quantifiers distribute over conjunctions. We need only prove the theorem for an atomic formula since intersections of closed sets are closed. Suppose then that  $\Phi(x_1, x_2, \dots, x_n, P)$  is atomic. Define  $\Phi^*(P) = \forall x_1, x_2, \dots, x_n \Phi(x_1, x_2, \dots, x_n, P)$  and let  $\mathcal{F} = \{S \subseteq \mathcal{P}(X) \mid (\mathcal{P}(X), S) \models \Phi^*(P)\}$ . If  $S' \notin \mathcal{F}$  then  $(\mathcal{P}(X), S') \not\models \Phi^*(P)$ , thus  $\exists A_1, A_2, \dots, A_n \in \mathcal{P}(X)$  so that  $(\mathcal{P}(X), S')[A_1/x_1, A_2/x_2, \dots, A_n/x_n]$  does not satisfy  $\Phi(x_1, x_2, \dots, x_n, P)$ . In particular, some of the  $A_i$  might belong to  $S'$  while the rest might not; denote them by  $B_k$  ( $k \leq r$ ) and  $C_l$  ( $l \leq s$ ), respectively, with  $r + s = n$ . Hence, no element of  $\mathcal{F}$  can contain all of the  $B_k$ s and miss all of the  $C_j$ s. It follows that  $C_1^- \cap \dots \cap C_s^- \cap B_1^+ \cap \dots \cap B_r^+ \cap \mathcal{F} = \emptyset$ .

□

**Corollary 1.3.10.** *A subspace of  $2^{\mathcal{P}(X)}$  is compact if it has a definition expressible as a collection of universal sentences of  $\mathcal{L}(P)$ .*

*Proof.* Let  $\{\Phi_i\}$  be a collection of universal sentences in  $\mathcal{L}(P)$ . Theorem 1.3.9 proves that each sentence defines a compact subspace  $\mathcal{F}_i \subset 2^{\mathcal{P}(X)}$ . Lastly,  $\{\Phi_i\}$  is then a definition for  $\bigcap_i \mathcal{F}_i$ , which is closed and thus compact. □

**Corollary 1.3.11.** *Top(X) is not first-order definable via universal sentences.*

*Remark 1.3.12.* Clearly, not all compact sets can be defined in terms of universal sentences for there are not enough sentences in our countable language to capture them all. What is not as obvious is that there are instances where the use of a unary predicate symbol is essential. Take for instance  $\mathcal{A}_\kappa$  as defined in Lemma 1.2.10. Add to  $\mathcal{L}$  a unary predicate symbol  $P$  with the following intended interpretation:  $\forall x$

$P(x) \iff |x| = \kappa$ . Then we can easily define  $\mathcal{A}_\kappa$ . However, the same is not true with the use of  $\mathcal{L}$  alone.

Lastly, we can expand  $\mathcal{L}$  to an uncountable language by adding for all  $Z \subset X$  a unary predicate symbol  $P_Z$ . Given  $Y \subset X$ , we want  $P_Z(Y)$  to be true if and only if  $Z = Y$ . It is then possible to define all subbasic closed sets of the form  $A^-$  for if we let  $\Phi_A = \forall x \neg P_A(x)$ , then  $A^-$  is the unique subset of  $2^{\mathcal{P}(X)}$  for which  $(\mathcal{P}(X), A^-) \models \Phi_A$ . Similarly,  $A^+$  is the unique set so that  $(\mathcal{P}(X), A^+) \models \neg\Phi_A$ . In turn, any basic closed set satisfies a sentence expressed as a finite disjunction of the aforementioned ‘subbasic’ sentences. Finally, a closed subset in  $2^{\mathcal{P}(X)}$  satisfies an arbitrary collection of ‘basic’ sentences as described above.

## 1.4 Conclusion

This paper offers results from a preliminary investigation into the topological nature of  $Top(X)$  as a subspace of  $2^{\mathcal{P}(X)}$ . The outcomes thus far suggest that  $Top(X)$ ’s topological character may well be difficult to gauge. Knowing whether  $Top(X)$  is a  $G_\delta$  or an  $F_\sigma$ , for example, will shed further light on its topological complexity and crystallize the extent to which it may yield to further analysis. In this paper, we have exhibited examples of sets that are closed in  $Top(X)$  but not in  $2^{\mathcal{P}(X)}$ . It would be nice to find (infinite) subsets of  $Top(X)$  that are closed in  $2^{\mathcal{P}(X)}$ , thus giving a source of infinite compact subsets of  $Top(X)$ . For example, is it possible to find a copy of  $\beta\mathbb{N}$  or a Cantor set in  $Top(X)$ ? Answers to these questions would provide further interesting aspects of  $Top(X)$ .

## 1.5 Acknowledgement

It is a pleasure to record our indebtedness to Paul Bankston, Marquette University, Milwaukee for valuable conversations and insights in pursuit of these results.

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# Chapter 2

## Topologies as points within a Stone space: lattice theory meets topology

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### Abstract

For a non-empty set  $X$ , the collection  $Top(X)$  of all topologies on  $X$  sits inside the Boolean lattice  $\mathcal{P}(\mathcal{P}(X))$  (when ordered by set-theoretic inclusion) which in turn can be naturally identified with the Stone space  $2^{\mathcal{P}(X)}$ . Via this identification then,  $Top(X)$  naturally inherits the subspace topology from  $2^{\mathcal{P}(X)}$ . Extending ideas of Frink (1942), we apply lattice-theoretic methods to establish an equivalence between the topological closures of sublattices of  $2^{\mathcal{P}(X)}$  and their (completely distributive) completions. We exploit this equivalence when searching for countably infinite compact subsets within  $Top(X)$  and in crystalizing the Borel complexity of  $Top(X)$ . We exhibit infinite compact subsets of  $Top(X)$  including, in particular, copies of the Stone-Čech and one-point compactifications of discrete spaces.

### 2.1 Introduction

For a non-empty set  $X$ , the collection  $Top(X)$  of all topologies on  $X$  sits inside the Boolean lattice  $\mathcal{P}(\mathcal{P}(X))$  (when ordered by set-theoretic inclusion) which in turn can

be naturally identified with the Stone space  $2^{\mathcal{P}(X)}$  (zero-dimensional compact  $T_2$ ). Via this identification then,  $Top(X)$  naturally inherits the subspace topology from  $2^{\mathcal{P}(X)}$  (see [BM12]), a subspace about which little is known. Frink [Fri42] showed that endowing the lattice  $\mathcal{P}(\mathcal{P}(X))$  with either the *interval* or the *order* topology yields the same space as  $2^{\mathcal{P}(X)}$ . In the same paper, Frink also proved that a lattice is complete if and only if it is compact in its interval topology.

These ideas enable us to apply lattice-theoretic techniques in the investigation of an object, whose individual elements provide a rich source of topological inquiry and knowledge. Just as the Stone-Čech compactification  $\beta\mathbb{N}$  for discrete space  $\mathbb{N}$  is homeomorphic to the subspace of all ultrafilters on  $\mathbb{N}$ , so the question of how all topologies on a fixed infinite set  $X$  behave collectively as a natural subspace of  $2^{\mathcal{P}(X)}$  is interesting and yet is little explored. We establish that complete sublattices of  $\mathcal{P}(\mathcal{P}(X))$  provide a rich supply of compact subsets within  $2^{\mathcal{P}(X)}$ . It is then possible to find infinite compact subsets of  $Top(X)$  by purely lattice-theoretic means and to gain further insight into the topological complexity of  $Top(X)$ .

The first section of this paper focuses on extending the aforementioned results by describing the equivalence between the topological closures of sublattices of  $2^{\mathcal{P}(X)}$  and their (completely distributive) completions. We exploit such an equivalence when searching for countably infinite compact subsets within  $Top(X)$  and in crystalizing the Borel complexity of  $Top(X)$ . The last section is devoted to describing other infinite compact subsets of  $Top(X)$  including, in particular, copies of the Stone-Čech and one-point compactifications of discrete spaces.

## 2.2 Preliminaries

For convenience and unless otherwise indicated,  $2^{\mathcal{P}(X)}$  shall denote the usual Boolean algebra for an infinite set  $X$  equipped with the subset inclusion order ( $\subseteq$ ) in addition to the usual product space, where  $2$  is the discrete space. We reserve the use of the symbol  $\subset$  for cases of proper or strict containment only. The topology of any subset  $P$  of  $2^{\mathcal{P}(X)}$  then is simply the usual subspace topology on  $P$  (which we shall denote where necessary by  $P_{\text{II}}$ ) while  $\overline{P}$  will denote the topological closure of  $P$  in the space

$2^{\mathcal{P}(X)}$ .

For a sublattice  $P$  of  $2^{\mathcal{P}(X)}$  we denote by  $\hat{P}$  its lattice-theoretic completion. It is simple to show that  $\hat{P} = \bigcap \{L \subseteq 2^{\mathcal{P}(X)} \mid P \subseteq L, L \text{ a sublattice of } 2^{\mathcal{P}(X)} \text{ and } L = \hat{L}\}$ . Of course, finite sublattices are trivially complete and, in general, the *free completion* of a lattice does not exist (see [CD59], [Whi41], [Whi42]). That said, the *completely distributive completion* of any partial order exists and is unique up to isomorphism [Mor04]. Moreover, for a partial order  $P$ :

$$x \in \hat{P} \text{ if and only if } x = \bigwedge \bigvee S \text{ for all } S \subseteq P \text{ so that } x \leq \bigvee S.$$

Since  $2^{\mathcal{P}(X)}$  (as well as any sublattice thereof) is completely distributive we have an explicit description of each element in the lattice-theoretic completion of any sublattice of  $2^{\mathcal{P}(X)}$ .

**Definition 2.2.1.** Let  $(P, \leq)$  be a poset with  $p \in P$ . We define  $p^\downarrow = \{x \in P \mid x \leq p\}$ ,  $p^\uparrow = \{x \in P \mid p \leq x\}$ ,  $p_\downarrow = p^\downarrow \setminus \{p\}$  and  $p_\uparrow = p^\uparrow \setminus \{p\}$ .

**Definition 2.2.2.** We shall adopt the following notation:

- (i) If  $P$  is a lattice, and  $S \subseteq P$ , then we denote by  $\langle S \rangle_L$  the sublattice of  $P$  generated by closing off  $S$  under finite meets and finite joins.
- (ii) If  $\mathcal{S} \subseteq \mathcal{P}(X)$ , then  $\langle \mathcal{S} \rangle_T$  denotes the topology  $\sigma$  on  $X$  generated by closing off  $\mathcal{S}$  under finite intersections and arbitrary unions. Thus  $\mathcal{S} \cup \{\emptyset, X\}$  is a subbase for  $\sigma$ .

**Definition 2.2.3.** Given any  $S \subseteq 2^{\mathcal{P}(X)}$  we let  $R_S = \{a \in 2^{\mathcal{P}(X)} \mid \forall b \in S, \text{ either } b \subseteq a \text{ or } a \subseteq b\}$  and refer to it as the *set of relations of  $S$* .

## 2.3 Completeness and compactness of sublattices

**Lemma 2.3.1.** *Let  $P$  be a sublattice of  $2^{\mathcal{P}(X)}$ , let  $S \subseteq P$  and let  $x = \bigvee S$ . If  $x \notin P$ , then  $x$  is a limit point of  $P$ .*

*Proof.* Let  $\bigcap A_i^+ \cap \bigcap B_j^-$  be an arbitrary open neighbourhood of  $x$ . Then for each of the finitely many  $i$ , there is  $s_i \in S$  such that  $A_i \in s_i$ ; furthermore  $B_j \notin s_i$  for each  $j$  and each  $i$ . Thus  $\bigvee_i s_i \in \bigcap A_i^+ \cap \bigcap B_j^- \cap P$  and clearly  $\bigvee_i s_i \neq x$ . □

Recall that the *interval topology* on a poset  $P$  is the one generated by  $\{x^\uparrow \mid x \in P\} \cup \{x^\downarrow \mid x \in P\} \cup \{P, \emptyset\}$  as a subbase for the closed sets; we denote it by  $P_<$ . The *order topology*  $P_O$  on a lattice  $P$  is defined in terms of Moore-Smith convergence. A filter  $\mathcal{F}$  of subsets from  $P$  is said to Moore-Smith-converge to a point  $l \in P$  whenever

$$\bigwedge_{F \in \mathcal{F}} \bigvee F = l = \bigvee_{F \in \mathcal{F}} \bigwedge F.$$

We then take  $F \subseteq P$  to be closed if and only if any convergent filter that contains  $F$  converges to a point in  $F$ . For a lattice  $P$ ,  $P_< \subseteq P_{\Pi} \subseteq P_O$  [Fri42].

**Lemma 2.3.2.** *Let  $P$  be a sublattice of  $2^{\mathcal{P}(X)}$ . Then  $P_< \subseteq P_{\Pi} \subseteq P_O$  and all three topologies coincide when  $P$  is a complete sublattice of  $2^{\mathcal{P}(X)}$ . Moreover, all three topologies on  $P$  are compact if and only if  $P$  is complete.*

*Proof.* The first inequality is true since for any  $x \in P$ , we have that

$$x^\uparrow \cap P = \bigcap_{A \in x} (A^+ \cap P).$$

A similar argument holds for  $x^\downarrow \cap P$ . For the second inequality and without loss of generality, take any subbasic closed set  $A^+$  and let  $\mathcal{F}$  be a convergent filter in  $P$  containing  $A^+ \cap P$ ; this forces  $\bigwedge (A^+ \cap P)$  to exist in  $P$ . Since  $P$  is a sublattice of  $2^{\mathcal{P}(X)}$  then

$$\bigvee_{F \in \mathcal{F}} \bigwedge F = \bigcup_{F \in \mathcal{F}} \bigcap F$$

and  $A \in \bigcup_{F \in \mathcal{F}} \bigcap F$ . Consequently,  $\bigcup_{F \in \mathcal{F}} \bigcap F \in A^+ \cap P$ .

Next, if  $P$  is complete then for any  $A \subseteq X$ , it follows that  $\bigcap(A^+ \cap P) \in P$  and  $\bigcup(A^- \cap P) \in P$ . In turn,  $(A^+ \cap P) = (\bigcap(A^+ \cap P))^\uparrow$ ,  $(A^- \cap P) = (\bigcup(A^- \cap P))^\downarrow$  and  $P_< = P_\Pi$ . Since  $P_<$  is  $T_2$  then  $P_O$  must be compact Hausdorff [Ins64] and  $P_O = P_<$ . The last claim is true since Frink [Fri42] shows that a complete lattice is compact in its interval topology if and only if it is complete. □

Not only is a sublattice  $P$  compact in  $2^{\mathcal{P}(X)}$  precisely when  $P$  is complete but also the closure of  $P$  within  $2^{\mathcal{P}(X)}$  is indeed its lattice theoretic completion:

**Theorem 2.3.3.** *Given an infinite sublattice  $P$  of  $2^{\mathcal{P}(X)}$  and  $x \in 2^{\mathcal{P}(X)}$ ,*

- (i)  *$x$  is a limit point of  $P$  only if  $x$  can be expressed in the form  $\bigwedge_{j \in J} \bigvee_{i \in I_j} x_{i,j}$ , where  $x \neq x_{i,j}$  for each  $i, j$  and  $\{x_{i,j} \mid i \in I_j\}_{j \in J}$  are infinite subsets of  $P$ .*
- (ii)  *$\bar{P} = \hat{P}$ ; that is, the topological closure of  $P$  in  $2^{\mathcal{P}(X)}$  coincides with its lattice-theoretic completion.*

*Proof.* (i) Let  $x$  be a limit point of  $P$  and assume, without loss of generality, that  $x$  is infinite. If  $x = \mathcal{P}(X)$  then since all neighbourhoods of  $x$  meet  $P$ , we must have  $x = \bigvee(x_\downarrow \cap P)$  and we are done. Otherwise fix any  $B \notin x$  and notice that  $\forall A \in x$  we have that  $A^+ \cap B^- \cap P \neq \emptyset$ . Furthermore  $|A^+ \cap B^- \cap P| \geq \aleph_0$  since otherwise we can find (in Hausdorff  $2^{\mathcal{P}(X)}$ ) a neighbourhood of  $x$  which is disjoint from  $P$ . Holding  $B$  fixed, it is simple to see that  $x \subseteq \bigvee_{A \in x} (\bigvee(A^+ \cap B^- \cap P))$  while  $B \notin \bigvee_{A \in x} (\bigvee(A^+ \cap B^- \cap P))$ . To this end, we must only intersect all such suprema for each  $B \notin x$  and we have the required form. In symbols:

$$x = \bigwedge_{B \notin x} \bigvee_{A \in x} \left( \bigvee(A^+ \cap B^- \cap P) \right).$$

For (ii) we must only notice that (i)  $\Rightarrow$  (ii). Indeed, take any  $x \in \hat{P}$ . If  $x \in P$ , we are done. Otherwise, if  $x = \bigvee S$ , for  $S \subseteq P$  then Lemma 3.1 applies. The last possibility is for  $x = \bigwedge_{k \in K} \bigvee S_k$  where  $S_k \subseteq P$  and  $x \subset \bigvee S_k$ . Take a basic open set  $\bigcap A_i^+ \cap \bigcap B_j^-$  about  $x$  and observe that for all  $i$  and for all  $k \in K$ ,  $A_i^+ \cap S_k \neq \emptyset$ . As for the  $B_j^-$ , we know that for any  $j$  we can find a  $k_j$  for which  $B_j \not\subseteq \bigvee S_{k_j}$ . Hence, for each  $j$  take a finite collection of elements from its corresponding  $\langle S_{k_j} \rangle_L$  (i.e. the one for which  $B_j \not\subseteq \bigvee S_{k_j}$ ) so that the join of such a collection contains all  $A_i$ . Taking the intersection of all such collections for each  $j$  we have an element of  $P$  that is contained in the aforementioned basic open set and thus  $x$  is a limit point of  $P$ .

□

Thus for example, a chain  $\Omega$  in  $2^{\mathcal{P}(X)}$  is by default a sublattice of  $2^{\mathcal{P}(X)}$  and so its closure  $\bar{\Omega}$  in  $2^{\mathcal{P}(X)}$  is its lattice-theoretic completion, which is again a chain. In fact, observe that

$$\hat{\Omega}(=\bar{\Omega}) = \left\{ \bigcap \Omega \right\} \cup \left\{ \bigcup \Omega \right\} \cup \Omega \cup \left\{ \bigcup (b_{\downarrow} \cap \Omega) \mid b \in R_{\Omega} \right\} \cup \left\{ \bigcap (b_{\uparrow} \cap \Omega) \mid b \in R_{\Omega} \right\}.$$

*Remark 2.3.4.* Let  $(X, \sigma)$  be any topological space containing a convergent sequence  $(x_n)_{n \in \omega}$  where  $x_n \rightarrow x_{\omega}$ . That  $x_n \rightarrow x_{\omega}$  is equivalent to demanding that any open set containing  $x_{\omega}$  must contain all but finitely many points from  $(x_n)$ . Notice that the same is true for  $\omega$  in the ordinal space  $\omega + 1$  (with the order topology). Moreover, any natural number in  $\omega + 1$  is isolated and hence  $\omega$  is a discrete subspace of  $\omega + 1$ . Thus  $\omega + 1$ , as an indexing set for any convergent sequence with its limit  $\{x_n, x_{\omega}\}$ , sets up a natural and continuous mapping  $\phi : \omega + 1 \rightarrow \{x_n, x_{\omega}\}_{n \in \omega}$  (where  $n \rightarrow x_n$ ) whereupon compactness naturally transfers. With that in mind, let  $\Omega = \{a_1, a_2, \dots\}$  be a well-ordered chain in  $2^{\mathcal{P}(X)}$  with  $\alpha$  as its indexing ordinal. If  $\beta \in \alpha$  is a limit ordinal, then any open set about  $\beta$  contains infinitely many ordinals below  $\beta$ . Notice that this might not be the case with  $a_{\beta}$ , for if  $a_{\beta} \neq \bigcup (a_{\beta})_{\downarrow}$  then  $a_{\beta}$  can be separated from  $(a_{\beta})_{\downarrow}$  by means of open sets. Thus, the bijection  $h : \alpha \rightarrow \Omega$  for which  $\beta \mapsto a_{\beta}$  is clearly open:  $h$  is a homeomorphism if and only if for any limit ordinal  $\beta \in \alpha$  we have  $a_{\beta} = \bigcup (a_{\beta})_{\downarrow}$ .

## 2.4 For $X$ infinite, $Top(X)$ is neither a $G_\delta$ nor an $F_\sigma$ set

**Lemma 2.4.1.** *For any  $\{A_i \mid i \in \omega\} \subseteq \mathcal{P}(X)$ ,  $\bigcap_{i \in \omega} A_i^+$  contains a sublattice of  $\mathcal{P}(X)$  that is not join complete; that is,  $(\bigcap_{i \in \omega} A_i^+) \cap (LatB(X) \setminus Top(X)) \neq \emptyset$ .*

*Proof.* Consider  $\langle \{A_i \mid i \in \omega\} \rangle_L$  and suppose that it is join complete (otherwise, we are done). Notice that its countable cardinality demands that only finitely many of the  $A_i$ s can be singletons. Since  $X$  is infinite, we may choose a countable infinite collection of singletons  $\mathcal{S} = \{\{p\} \mid p \in X\}$  from  $\mathcal{P}(X)$  and generate a lattice  $K = \langle \{A_i\} \cup \mathcal{S} \rangle_L$ . Then  $K$  cannot be join complete for there are uncountably many subsets of  $\bigcup \mathcal{S}$  (i.e. joins of  $\mathcal{S}$ ) and only  $\aleph_0$  many elements in  $K$ . □

**Theorem 2.4.2.**  *$Top(X)$  is not a  $G_\delta$  set.*

*Proof.* Suppose that  $Top(X) = \bigcap_{k \in \omega} \mathcal{O}_k$ , where

$$\mathcal{O}_k = \bigcup_{\alpha \in \beta_k} \left( \left( \bigcap_{i_\alpha \leq n_\alpha} A_{i_\alpha}^+ \right) \cap \left( \bigcap_{j_\alpha \leq m_\alpha} B_{j_\alpha}^- \right) \right).$$

Now, the discrete topology  $\mathcal{D}$  on  $X$  must be in this intersection of open sets. Thus for each  $k \in \omega$ , it must belong to at least one basic open set of the form  $(\bigcap_{i_\alpha \leq n_\alpha} A_{i_\alpha}^+) \cap (\bigcap_{j_\alpha \leq m_\alpha} B_{j_\alpha}^-)$  and since  $\mathcal{D}$  contains all sets, then no subbasic open set can be of the form  $B^-$ . That is,  $\mathcal{D} \in \bigcap_{k \in \omega} A_k^+$  after some renumeration of the  $A$ s. Applying Lemma 2.4.1 to  $\bigcap_{k \in \omega} A_k^+$  we can find a sublattice of  $\mathcal{P}(X)$  that belongs to  $\bigcap_{k \in \omega} A_k^+$  and that is not join complete - a contradiction. □

In fact, Lemma 2.4.1 proves something much stronger. Define recursively:

$$\begin{aligned}
G_\delta^0 &:= \{\text{all } G_\delta \text{ sets}\} \\
G_{\delta\sigma}^0 &:= \{\text{all countable unions of } G_\delta \text{ sets}\} \\
G_\delta^\beta &:= \{\text{all countable intersections of } G_{\delta\sigma}^{\beta-1} \text{ sets}\} \quad (\text{for } \beta \text{ a successor ordinal}) \\
G_{\delta\sigma}^\beta &:= \{\text{all countable unions of } G_\delta^\beta \text{ sets}\} \quad (\text{for } \beta \text{ a successor ordinal}) \\
G_\delta^\gamma &:= \bigcup_{\beta \in \gamma} G_\delta^\beta \quad (\text{for } \gamma \text{ limit ordinal}) \\
G_{\delta\sigma}^\gamma &:= \bigcup_{\beta \in \gamma} G_{\delta\sigma}^\beta \quad (\text{for } \gamma \text{ limit ordinal})
\end{aligned}$$

Take for any  $n \in \mathbb{N}$  a set  $G_n$ , say, from  $G_\delta^n$  and assume that  $Top(X) = \bigcap_{n \in \omega} G_n$ . Since  $\mathcal{D} \in \bigcap_n G_n$  then for each  $n \in \mathbb{N}$  we can find a countable collection  $\{A_{i_n}\}_{i \in \omega}$  of subsets of  $X$  corresponding to each  $G_n$  so that

$$\mathcal{D} \in \bigcap_{i_n} A_{i_n}^+ \subset G_n$$

and consequently

$$\mathcal{D} \in \bigcap_{n \in \omega} \bigcap_{i_n} A_{i_n}^+ \subset \bigcap_{n \in \omega} G_n.$$

By Lemma 2.4.1, there is a lattice that is not join complete belonging to  $\bigcap_{n \in \omega} \bigcap_{i_n} A_{i_n}^+$ . Hence,  $Top(X) \neq \bigcap_{n \in \omega} G_n$ . Notice that the same is true for any countable limit ordinal. That is, for any  $\beta \in \omega_1$  so that  $\mathcal{D} \in G \in G_\delta^\beta$  it is possible to extract a countable collection of open sets  $A_i$  ( $i \in \omega$ ) so that  $\mathcal{D} \in \bigcap_{i \in \omega} A_i^+ \subseteq G$ , in which case Lemma 2.4.1 completes the proof.

**Corollary 2.4.3.**  $Top(X) \notin G_\delta^\beta$  for  $\beta \in \omega_1$ .

In other words, it is not possible to generate (in the sense of Borel sets)  $Top(X)$  by means of open sets. If  $2^{P(X)}$  was metrizable (which it is not) then the above corollary would suffice to show that  $Top(X)$  is not a Borel set.

**Corollary 2.4.4.**  $Top(X)$  is not Čech complete.

*Proof.* We showed above that any countable intersection of open sets from  $2^{\mathcal{P}(X)}$  containing the discrete topology on  $X$  contains an element of  $LatB(X) \setminus Top(X)$ . Hence,  $Top(X)$  is not a  $G_\delta$  set in  $LatB(X)$ . □

**Theorem 2.4.5.**  $Top(X)$  is not an  $F_\sigma$  set.

*Proof.* If  $Top(X)$  is an  $F_\sigma$  set, then it must be of the form

$$Top(X) = \bigcup_{k \in \omega} \mathcal{C}_k$$

where each  $\mathcal{C}_k$  is a closed set. We will show by contradiction that at least one such closed set must contain a sequence of topologies whose limit is not a topology. Since the limit of any sequence must be present in the closure of the sequence, then the aforementioned closed set will contain an element that is not a topology. We prove the above for  $|X| = \aleph_0$  and note that the same is true for any  $X$  with  $|X| \geq \aleph_0$ .

Let  $k : [0, 1] \rightarrow \mathcal{P}(\mathbb{N})$  be an injective order morphism so that  $\forall a \in [0, 1], \bigcup_{b < a} k(b) = k(a)$  and  $k(1) \neq \mathbb{N}$ . That is,  $k([0, 1])$  is a dense and uncountable linear order in  $\mathcal{P}(\mathbb{N})$  where  $a < b \Rightarrow k(a) \subset k(b)$ . Next, for any  $a$  define  $\tau_a = \mathcal{P}(k(a)) \cup \{\mathbb{N}\}$ . Notice that  $\forall a \in [0, 1], \tau_a \in Top(\mathbb{N})$  and  $\bigcup_{b < a} \tau_b \notin Top(\mathbb{N})$  (since  $k(a) \notin \bigcup_{b < a} \tau_b$ ), and  $\{\tau_a\}_{a \in [0, 1]}$  is an uncountable dense linear order in  $Top(\mathbb{N})$ . If  $Top(\mathbb{N}) = \bigcup_{k \in \omega} \mathcal{C}_k$  where each  $\mathcal{C}_k$  is closed then there must exist one set  $\mathcal{C}$  from  $\{\mathcal{C}_k\}_{k \in \omega}$  which contains an uncountable set  $D \subset \{\tau_a\}_{a \in [0, 1]}$  for which  $\mu(D) > 0$  (non-zero measure). We immediately get that  $D$  must contain a densely ordered subset that in turn contains a strictly increasing sequence, call it  $S$ . By Theorem 2.3.3,  $\bigcup S \in \hat{S}$  but  $\bigcup S \notin Top(\mathbb{N})$ . To this end we have  $\bigcup S \in \mathcal{C}$ , a contradiction. □

**Corollary 2.4.6.** For  $\beta \in \omega_1$ , the following are equivalent:

- (a)  $Top(X) \in G_\delta^\beta$ .
- (b)  $Top(X)$  is a  $G_\delta$  set.
- (c)  $Top(X)$  is an  $F_\sigma$  set.
- (d)  $Top(X)$  is Čech complete.
- (e)  $X$  is finite.

## 2.5 Compact infinite subsets of $Top(X)$

In this section, we provide examples of compact infinite subsets of  $Top(X)$ . Note in particular that any countable chain of topologies must converge to its union which may not itself be a topology. For example, consider the nested sequence of finite topologies  $\{\tau_i \mid \tau_i = \mathcal{P}(\{x_0, x_1, \dots, x_i\}) \cup \{X, \emptyset\}\}$ , where  $\{x_0, x_1, \dots\}$  is a countable infinite subset of  $X$ . Then  $\tau_n \rightarrow \bigcup \tau_i$  but notice that  $\bigcup \tau_i$  fails to be a topology as  $\{x_0, x_1, \dots\}$  does not belong to any  $\tau_i$ . Of course,  $LatB(X)$ , as a compactification of  $Top(X)$ , will contain all such limits.

**Example 2.5.1.** For simplicity, take a countable infinite subset  $C$  of  $X$ . Enumerate  $C = \{a_0, a_1, a_2, \dots\}$  and create a sequence in  $\mathcal{P}(C)$  as follows:  $C_0 = \{a_0\}$ ,  $C_1 = \{a_0, a_1\}$ , ...,  $C_\omega = C$ . Now, for any  $n \in \omega$  let  $\tau_n = \{C_m \mid m \leq n\} \cup \{X, \emptyset\}$  and  $\tau_\omega = \{C_m \mid m \in \omega\} \cup \{C, X, \emptyset\}$ ; then  $(\tau_n)_{n \in \omega}$  converges to (non-Hausdorff)  $\tau_\omega$  in  $Top(X)$ . Indeed, let  $B = \bigcap A_i^+ \cap \bigcap B_j^-$  be a basic open set containing  $\tau_\omega$ . Then no  $B_j = C_m$  for any  $m \in \omega$  and any  $A_i$  must be either  $\emptyset$ ,  $X$ ,  $C$  or a  $C_n$ , for some  $n \in \omega$ . Since there are only finitely many  $A_i$  then there exists an  $m \in \omega$  for which all  $A_i \in \tau_m$ .

In view of the above example, we can construct a convergent sequence of compact non-Hausdorff topologies, whose limit is both compact and Hausdorff.

**Example 2.5.2.** Let  $[a, b] \subset \mathbb{R}$ , and define any strictly increasing sequence  $\{x_n \mid a < x_n < b\}_{n \in \omega}$  whose limit is  $b$ . Next, let  $\mathcal{N}_b = \{(c, b] \mid a \leq c \leq b\}$ ,  $\mathcal{N}_a(x) = \{[a, c) \mid c \leq x\}$  and

$$\begin{aligned}\tau_0 &= \langle \{[a, b)\} \cup \mathcal{N}_b \cup \mathcal{N}_a(x_0) \cup \{\emptyset\} \rangle_T \\ \tau_1 &= \langle \{[a, b)\} \cup \mathcal{N}_b \cup \mathcal{N}_a(x_1) \cup \{\emptyset\} \rangle_T \\ &\vdots \\ \tau_\omega &= \bigcup_{i \in \omega} \tau_i\end{aligned}$$

By design,  $\tau_n \rightarrow \tau_\omega$ . Observe also that  $\tau_\omega$  is the usual Euclidean topology on  $[a, b]$  and so is compact and Hausdorff. Indeed, given any  $c \in [a, b]$  we must only check that  $[a, b) \in \tau_\omega$ . Since  $x_n \rightarrow b$  then there exists a  $k \in \omega$  so that  $x_k > c$ , hence  $[a, b) \in \tau_k$ . To show that each  $\tau_n$  ( $n \in \omega$ ) is compact, take an open cover  $\mathcal{C}$  of  $[a, b]$  from  $\tau_k$  ( $k \in \omega$ ). Notice that since  $a$  must be covered then  $[a, c_1) \in \mathcal{C}$  for some  $c_1 \leq x_k$ . Since  $x_k$  must also be covered by some element in  $\mathcal{C}$  then, for some  $c_2 \leq x_k$ , either  $(c_2, b)$  or  $(c_2, b]$  belong to  $\mathcal{C}$ . If  $c_1 > c_2$  then we're done. Otherwise, notice that  $\tau_k \upharpoonright [c_1, c_2]$  is the usual topology on  $\mathbb{R}$  restricted to  $[c_1, c_2]$  (which yields a compact space). Finally, no  $\tau_n$  is Hausdorff (since  $b$  can't be separated from all points in  $[a, b]$ ).

The above example confirms that the collection of compact non- $T_2$  topologies on a set  $X$  fails to be closed given the existence of a countable chain of compact non- $T_2$  topologies whose union (and topological limit) is a  $T_2$  and compact topology.

Even though any compact topology is contained in a maximal compact topology [Kov05] it is possible to construct strictly increasing sequences of compact topologies whose limits are not compact. Consider the half-open half-closed interval  $[a, b]$  equipped with the (convergent) sequence of topologies  $\tau_i$  as in the previous Example, with  $\mathcal{N}_b = \{(c, b) \mid a \leq c\}$  modified accordingly. It is clear that every topology, with the exception of  $\tau_\omega$ , is compact.

Nested sequences are not the only type of compact infinite subsets of  $Top(X)$ . Recall that an *atom* in  $Top(X)$  is a topology of the form  $\{\emptyset, A, X\}$  where  $A$  is a

nonempty and proper subset of  $X$ . Consider then the following theorem where  $\mathcal{I}$  denotes the trivial topology on  $X$ .

**Theorem 2.5.3.** *Let  $\mathcal{T}$  be an infinite collection of atoms in  $Top(X)$ . Then  $\overline{\mathcal{T}} = \mathcal{T} \cup \{\mathcal{I}\}$ .*

*Proof.* Let  $\mathcal{A} \subset \mathcal{P}(X)$  so that  $\emptyset \notin \mathcal{A}$ ,  $\tau_A = \{X, \emptyset, A\}$  and  $T_{\mathcal{A}} = \{\tau_A \mid A \in \mathcal{A}\}$ . Consider the following closed set

$$\mathcal{C} = X^+ \cap \emptyset^+ \cap \left( \bigcap_{D \in \mathcal{P}(X) \setminus \mathcal{A}} D^- \right) \cap \left( \bigcap_{B, C \in \mathcal{A}} (B^- \cup C^-) \right)$$

where, of course,  $B \neq C$ . Then  $T_{\mathcal{A}} \subseteq \mathcal{C}$  and  $\mathcal{I} \in \mathcal{C}$ . Any family that contains any element from  $\mathcal{P}(X) \setminus \mathcal{A}$  can't belong to  $\mathcal{C}$  and any family (and topology) that contains elements from  $\mathcal{A}$  can contain at most one. Thus  $T_{\mathcal{A}} \cup \{\mathcal{I}\} = \mathcal{C}$ . Finally, any neighbourhood of  $\mathcal{I}$  must intersect  $T_{\mathcal{A}}$  and the result follows. □

**Corollary 2.5.4.** *Let  $\mathcal{T}$  be any infinite collection of atoms in  $Top(X)$ . Then  $\mathcal{T} \cup \{\mathcal{I}\}$  is the one-point compactification of  $\mathcal{T}$  in  $2^{\mathcal{P}(X)}$ .*

Given two topologies in  $Top(X)$  we say that they are disjoint provided their intersection is the trivial topology  $\mathcal{I}$  on  $X$ .

**Theorem 2.5.5.** *Let  $\mathcal{T}$  be an infinite collection of pairwise disjoint topologies on  $X$ . Then  $\mathcal{T} \cup \{\mathcal{I}\} = \overline{\mathcal{T}}$  is the one-point compactification of  $\mathcal{T}$  in  $2^{\mathcal{P}(X)}$ .*

*Proof.* In the following expression,  $\sigma$  and  $\rho$  denote topologies in  $\mathcal{T}$  while  $A$  and  $B$  denote certain nonempty and proper subsets of  $X$ . We claim that

$$\mathcal{C} = X^+ \cap \emptyset^+ \cap \left( \bigcap_{D \notin \mathcal{T}} D^- \right) \cap \left( \bigcap_{\substack{(A, B) \in (\sigma, \rho) \\ \sigma \neq \rho}} (A^- \cup B^-) \right) \cap \left( \bigcap_{\substack{A \neq B \\ A, B \in \sigma}} (A^+ \cup B^-) \right) = \mathcal{T} \cup \{\mathcal{I}\}.$$

Clearly  $\mathcal{C}$  is closed and  $\mathcal{T} \cup \{\mathcal{I}\} \subseteq \mathcal{C}$ . Let  $x \in 2^{\mathcal{P}(X)} \setminus \mathcal{T}$  such that  $\mathcal{I} \subset x$ . If  $x \not\subseteq \cup \mathcal{T}$  then there must exist  $O \in x$  so that  $O \notin \cup \mathcal{T}$ ; since  $x \not\subseteq O^-$  then  $x \notin \mathcal{C}$ . If  $x \subseteq \cup \mathcal{T}$  then it is either contained in a topology from  $\mathcal{T}$  or not. In the former case, take  $\rho \in \mathcal{T}$  where  $x \subset \rho$ ,  $U \in \rho \setminus x$  and  $V \in \rho \cap x$  where  $V$  is neither empty nor  $X$ . Then  $x$  fails to belong to  $U^+ \cup V^-$ . Otherwise we are guaranteed a pair of distinct topologies,  $\rho$  and  $\sigma$ , in  $\mathcal{T}$  for which there exists  $V \in x \cap \rho$  and  $U \in x \cap \sigma$  and neither open set is equal to  $X$  or  $\emptyset$ . Hence,  $x \notin V^- \cup U^-$  and we have that  $\mathcal{C} = \mathcal{T} \cup \{\mathcal{I}\}$ . Finally, any neighbourhood of  $\mathcal{I}$  must have a cofinite intersection with  $\mathcal{T}$  and the result follows. □

**Corollary 2.5.6.** *For any (infinite) discrete space  $Y$ , where  $|Y| \leq 2^{|X|}$ ,  $Top(X)$  contains a copy of its one-point compactification.*

*Proof.* There are  $2^{|X|}$  atoms in  $Top(X)$  and all are disjoint from each other. □

## 2.6 $\beta\mathbb{N}$ in $Top(\mathbb{N})$

An *ultratopology*  $\mathcal{T}$  on a set  $X$  is of the form  $\mathcal{T} = \mathcal{P}(X \setminus \{x\}) \cup \mathcal{U}$ , where  $\mathcal{U}$  is an ultrafilter on  $X$  and  $\{x\} \notin \mathcal{U}$ . We shall use the notation  $\mathcal{T}_{\mathcal{U}}$  for such an ultratopology when we wish to identify the associated ultrafilter  $\mathcal{U}$ . We denote by  $Ult(X)$  the set of all ultratopologies on  $X$ . Whenever  $\mathcal{U}$  is a principal (non-principal) ultrafilter,  $\mathcal{T}$  is called a principal (non-principal) ultratopology. Denote by  $TYPE(x)$  the set  $\{\mathcal{F} \mid \mathcal{F} \text{ is an ultrafilter and } \{x\} \notin \mathcal{F}\}$  and by  $TYPE[x]$  the set  $\{\mathcal{T} \in Top(X) \mid \mathcal{T} \text{ is an ultratopology and } \{x\} \notin \mathcal{T}\}$ . Note that  $\{TYPE[x] : x \in X\}$  is a partition of  $Ult(X)$ .

Given  $X$ , define  $\mathcal{U}_X$  to be the set of all ultrafilters on  $X$  and for any  $\mathcal{F} \in \mathcal{U}_X$  and  $A \subset X$  let

$$\mathcal{F}_A = \mathcal{F} \upharpoonright (X \setminus A) = \{N \cap (X \setminus A) \mid N \in \mathcal{F}\}.$$

In other words,  $\mathcal{F}_A$  is the *trace* of  $\mathcal{F}$  on  $X \setminus A$ . Whenever  $A = \{a\}$  we let  $\mathcal{F}_{\{a\}} = \mathcal{F}_a$ .

**Lemma 2.6.1.** *Let  $n \in \mathbb{N}$  and denote  $\mathbb{N}' = \mathbb{N} \setminus \{n\}$  then*

(i)  $\forall \mathcal{F} \in \text{TYPE}(n), \mathcal{F} = \mathcal{F}_n \cup \{M \cup \{n\} \mid M \in \mathcal{F}_n\}$ ,

(ii)  $\forall \mathcal{F} \in \text{TYPE}(n), \mathcal{F}_n \in \mathcal{U}_{\mathbb{N}'}$  and

(iii)  $\mathcal{F}, \mathcal{G} \in \text{TYPE}(n)$  so that  $\mathcal{G} \neq \mathcal{F}$  implies that  $\mathcal{F}_n \neq \mathcal{G}_n$ .

*Proof.* (i) We must only show that  $\mathcal{F}_n \subset \mathcal{F}$ . Indeed, if  $\mathcal{F}_n \subset \mathcal{F}$  then given any  $A \in \mathcal{F}_n$  we have that  $A \cup \{n\} \in \mathcal{F}$  since  $\mathcal{F}$  is a filter. So let  $A \in \mathcal{F}_n$  and notice that either  $A \in \mathcal{F}$  or  $A \cup \{n\} \in \mathcal{F}$ . Since the former case is trivial assume that  $A \cup \{n\} \in \mathcal{F}$ . Notice that since  $\mathcal{F} \in \text{TYPE}(n)$  then  $(\mathbb{N}' \setminus A) \cup \{n\} \notin \mathcal{F}$  and thus  $A = \mathbb{N} \setminus ((\mathbb{N}' \setminus A) \cup \{n\}) \in \mathcal{F}$ .

(ii) Take  $A \in \mathcal{F}_n$ . If  $A \subset B \subseteq \mathbb{N}'$ , then  $B \in \mathcal{F}$  and consequently  $B \in \mathcal{F}_n$ . Next, let  $A, B \in \mathcal{F}_n$  and notice that  $A, B \in \mathcal{F}$  and  $A \cap B \in \mathcal{F}$ . Thus,  $A \cap B \in \mathcal{F}_n$ . Lastly, let  $A \subset \mathbb{N}'$  and notice that either  $A$  or its complement in  $\mathbb{N}$  belong to  $\mathcal{F}$ . In the former case we are done so assume  $\mathbb{N} \setminus A \in \mathcal{F}$ . To this end, we must only notice that  $\mathbb{N}' \setminus A = (\mathbb{N} \setminus A) \cap \mathbb{N}' \in \mathcal{F}_n$ .

(iii) This follows directly from (i). □

**Theorem 2.6.2.** *For any  $n \in \mathbb{N}$  the mapping  $\mathfrak{F} : \text{TYPE}(n) \rightarrow \mathcal{U}_{\mathbb{N}'}$  for which  $\mathcal{F} \mapsto \mathcal{F}_n$  is a bijection.*

*Proof.* Lemma 2.6.1 part (ii) tells us that the range is well-defined. Also, for any filter  $\mathcal{H} \in \mathcal{U}_{\mathbb{N}'}$  it is easy to see that  $\mathcal{F}_{\mathcal{H}} = \mathcal{H} \cup \{M \cup \{n\} \mid M \in \mathcal{H}\} \in \text{TYPE}(n)$  and that  $\mathcal{F}_{\mathcal{H}} \upharpoonright \mathbb{N}' = \mathcal{H}$ . Lastly, from part (iii) of Lemma 2.6.1 we get injectivity. □

**Theorem 2.6.3.** *For any  $n \in \mathbb{N}$ ,  $\text{TYPE}[n]$  is homeomorphic to  $\beta\mathbb{N}$ .*

*Proof.* Since  $\text{TYPE}(n)$  can be bijected with  $\mathcal{U}_{\mathbb{N}'}$  (by  $\mathcal{F} \mapsto \mathcal{F}_n$ ) then the same is true of  $\text{TYPE}[n]$ . That is, for any  $\mathcal{T}_{\mathcal{F}} \in \text{TYPE}[n]$  we canonically map  $\mathcal{T}_{\mathcal{F}} \mapsto \mathcal{F}_n$  so that  $\mathcal{T}_{\mathcal{F}} = \mathcal{P}(\mathbb{N}') \cup \mathcal{F}$ . Recall that a subbase for  $\beta\mathbb{N}'$  is comprised of sets of the form  $A' = \{\mathcal{F} \in \mathcal{U}_{\mathbb{N}'} \mid A \in \mathcal{F}\}$  for all  $A \in \mathcal{P}(\mathbb{N}')$ . We claim that for any  $A \subseteq \mathbb{N}'$ ,

$A' \mapsto (A \cup \{n\})^+ \cap \text{TYPE}[n]$ . Indeed, if  $A \in \mathcal{F}_n$  for some  $\mathcal{F} \in \text{TYPE}(n)$  then by Lemma 2.6.1  $A \cup \{n\} \in \mathcal{F}$  and  $\mathcal{T}_{\mathcal{F}} \in (A \cup \{n\})^+ \cap \text{TYPE}[n]$ .

Similarly, for any  $A \in \mathcal{P}(\mathbb{N}')$ ,  $A^+ \cap \text{TYPE}[n] = \text{TYPE}[n]$  which bijects to  $\mathcal{U}_{\mathbb{N}'}$ . If  $A \subseteq \mathbb{N}$  with  $n \in A$  then for any ultratopology  $\mathcal{T}_{\mathcal{F}}$  in  $A^+ \cap \text{TYPE}[n]$  it must be the case that  $A \setminus \{n\} \in \mathcal{F}_n$ . Consequently,  $A^+ \cap \text{TYPE}[n] \mapsto (A \setminus \{n\})'$ .

□

In a nutshell, we have the following diagram of the above claim:

$$\begin{array}{ccccc}
 \beta\mathbb{N}' & & & & \\
 \uparrow & \swarrow \mathcal{F}_n \mapsto \mathcal{T}_{\mathcal{F}} & & & \\
 \mathcal{U}_{\mathbb{N}'} & \xleftarrow{\mathcal{F}_n \mapsto \mathcal{F}} & \text{TYPE}(n) & \xleftarrow{\mathcal{F} \mapsto \mathcal{T}_{\mathcal{F}}} & \text{TYPE}[n]
 \end{array}$$

**Corollary 2.6.4.** *The  $F_\sigma$  set  $Ult(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} \text{TYPE}[n]$  is not compact.*

*Proof.* Consider the following open cover of  $Ult(\mathbb{N})$ :

$$\{\{n\}^- \mid n \in \mathbb{N}\}.$$

Note that  $\forall n \in \mathbb{N}$ ,  $\{n\}^- \cap Ult(\mathbb{N}) = \text{TYPE}[n]$  and so no finite subcollection of the above cover can cover  $Ult(\mathbb{N})$ .

□

In particular, the discrete topology on  $\mathbb{N}$  is a limit point of  $Ult(\mathbb{N})$ . Lemma 2.6.1 and Theorem 2.6.3 can be extended to any infinite set  $X$ . That said, for  $|X| > \aleph_0$ ,  $Ult(X)$  is not an  $F_\sigma$  set.

**Theorem 2.6.5.** *For  $Y$  a discrete space with  $|Y| \leq |X|$ ,  $Top(X)$  contains a copy of  $\beta Y$ .*

*Proof.* The proof is trivial for  $|Y| = |X|$ . Otherwise, take a copy of  $\beta Y$  within  $Top(Y)$  and an injection  $i : Y \rightarrow X$ . Then  $\forall \rho \in Top(Y)$ ,  $\rho_X = \{A \subset X \mid i^{-1}(A) \in \rho\} \cup \{X\}$

is a topology on  $X$ . Moreover,  $\{\rho_X \mid \rho \in \beta Y\}$  is a homeomorphic copy of  $\beta Y$  in  $Top(X)$ .

□

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# Chapter 3

## Metric axioms: a structural study

Jorge L. Bruno and Ittay Weiss

### Abstract

For a fixed set  $X$ , an arbitrary *weight structure*  $d \in [0, \infty]^{X \times X}$  can be interpreted as a distance assignment between pairs of points on  $X$ . Restrictions (i.e., *metric axioms*) on the behaviour of any such  $d$  naturally arise, such as separation, triangle inequality and symmetry. We present an order-theoretic investigation of various collections of weight structures, as naturally occurring subsets of  $[0, \infty]^{X \times X}$  satisfying certain metric axioms. Furthermore, we exploit the categorical notion of adjunctions when investigating connections between the above collections of weight structures. As a corollary, we present several lattice-embeddability theorems on a well-known collection of weight structures on  $X$ .

### 3.1 Introduction

For a fixed set  $X$ , a **standard metric** on it is any  $d \in [0, \infty]^{X \times X}$  for which:

(i)  $\forall x \in X, d(x, x) = 0$

(ii)  $\forall x, y \in X, d(x, y) = 0 = d(y, x) \Rightarrow x = y$  (*separation*)

(iii)  $\forall x, y \in X, d(x, y) = d(y, x)$  (*symmetry*)

(iv)  $\forall x, y, z \in X, d(x, y) + d(y, z) \geq d(x, z)$  (*triangle inequality*).

The collection of all standard metrics on  $X$  can then be identified with a particular subset of  $[0, \infty]^{X \times X}$ . For convenience, we shall refer to axioms (i), (ii), (iii) and (iv) as  $0, s, \Sigma$  and  $\Delta$  respectively. By letting  $P$  denote any collection of these axioms and  $W_P(X) = \{d \in [0, \infty]^{X \times X} \mid d \text{ satisfies all axioms in } P\}$ , it is also possible to view any such collection of axioms as a subset of the ambient set. For convenience, we suppress  $X$  from this notation where there is no danger of ambiguity. In particular,  $W_\emptyset = [0, \infty]^{X \times X}$ . Clearly, for  $P, Q$  (two collections of axioms) one has that  $W_{P \cap Q} \supseteq W_Q \cup W_P$  and  $W_{P \cup Q} = W_Q \cap W_P$ . For simplicity, we suppress the use of brackets where  $P$  is used as a subscript (e.g., if  $P = \{0\}$  then  $W_P = W_0$ ). We refer to any  $d \in [0, \infty]^{X \times X}$  as a *weight structure*. For reasons that will become apparent in the sequel, this paper is concerned with weight structures that satisfy axiom 0.

The ambient set  $[0, \infty]^{X \times X}$  when ordered pointwise (i.e.,  $d \leq m \in [0, \infty]^{X \times X} \Leftrightarrow \forall x, y \in X, d(x, y) \leq m(x, y)$ ) forms a complete lattice; every collection  $W_P$  becomes a poset and the obvious inclusions  $W_P \hookrightarrow W_Q$  (for  $Q \subseteq P$ ) acquire more structure (that of order-preserving functions). The first part of Section 3.3 is concerned with the lattice-theoretic structure of the above defined collections. In particular, we explore a connection between certain first-order predicates and the order-theoretic collection of weight structures they define. The rest of Section 3.3 presents a categorical investigation of the inclusions  $W_P \hookrightarrow W_Q$ . For instance, given any weight structure that does not satisfy symmetry, it is then natural to ask:

*Is there a way to naturally symmetrize such an element? Is such a process unique? If not, is there an optimal one?*

Expressed in the language of categories, we provide answers to the above by means of adjunctions in much the same spirit as how the process of turning a base into a topology can be seen as an adjunction. A comprehensive diagrams of such adjunctions can be found in Figure 3.1 where for  $M \in \{0, s, \Sigma, \Delta\}$  and  $M \notin P \subset \{0, s, \Sigma, \Delta\}$  the symbols  $M_*$  (resp.  $M_l$ ) denote the right (resp. left) adjoint to the inclusion

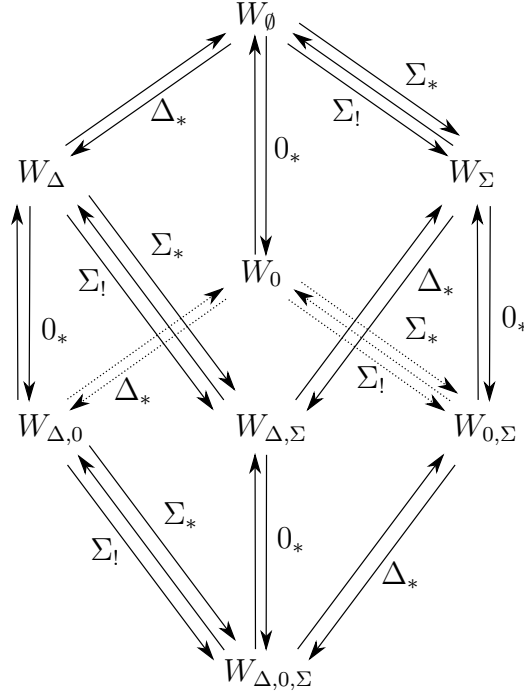


Figure 3.1: Adjunctions

$$W_{P \cup M} \leftrightarrow W_P.$$

For  $A \subseteq X$  and  $d \in W_P$ ,  $A$  is *left  $P$ -open* (resp. *right  $P$ -open*) provided that for any  $x \in A$  one can find  $\epsilon > 0$  so that  $B_L(x)_\epsilon = \{y \in X \mid d(x, y) < \epsilon\} \subseteq A$  (resp.  $B_R(x)_\epsilon = \{y \in X \mid d(y, x) < \epsilon\} \subseteq A$ ). A standard way to generate a topology from any  $d \in W_P$  is given by:  $O \in \tau_L$  (resp.  $O \in \tau_R$ ) iff  $O$  is left  $P$ -open (resp.  $O$  is right  $P$ -open). In other words,

$$d \mapsto (\tau_L, \tau_R).$$

A different way to generate topologies from any  $d \in W_P$  is by mapping  $d$  to the covers its right and left  $\epsilon$ -balls ( $B_R(x)_\epsilon$  and  $B_L(x)_\epsilon$ , respectively) generate. From there one considers such collections as subbases. More precisely,

$$d \mapsto (\langle \{B_L(x)_\epsilon \mid \epsilon > 0\} \rangle, \langle \{B_R(x)_\epsilon \mid \epsilon > 0\} \rangle).$$

Notice that both methods coincided if, and only if,  $\Delta \in P$ . Since  $Top(X)$  (the collection of all topologies on a fixed set  $X$ ) is a complete lattice (see [?] and [?]), the above assignments can be interpreted as order-theoretic functions  $W_P \rightarrow Top(X) \times Top(X)$ . Section 3.4 is devoted to a detailed investigation of the above functions with particular emphasis on meet/join preservation, structure of fibers and order-preservation.

Lastly, Section 3.5 is concerned with the lattice-theoretic structure of the collection of all extended metrics  $Met(X) = W_{0,\Delta,\Sigma}$  on a fixed set  $X$ . We explore Menger convexity (and its dual) within  $Met(X)$  along with several lattice-embeddability properties of  $Met(X)$ .

## 3.2 Preliminaries

For a partial order  $\mathbb{P}$ , we adopt the name *meet semilattice* (resp. *join semilattice*) whenever  $\mathbb{P}$  is closed under all finite meets (resp. finite joins). In particular, if  $\mathbb{P}$  is a meet (resp. join) semilattice, then the empty meet (resp. join) must exist within  $\mathbb{P}$ ; vacuously,  $\bigvee \emptyset = \perp$  and  $\bigwedge \emptyset = \top$  and, consequently, meet and join semilattices have top and bottom elements, respectively. We will refer to  $\mathbb{P}$  as a *lattice* provided it is a meet and join semilattice. Lastly,  $\mathbb{P}$  is a *complete lattice* whenever it contains all meets and joins. Following the above definitions, it is not necessary to distinguish join and meet completeness since both notions yield complete lattices. That is, if  $\mathbb{P}$  is a complete join semilattice (i.e., it has all joins), then it also has all meets, and thus is complete. For lattices  $\mathbb{P}$  and  $\mathbb{Q}$ , an order-preserving map  $f : \mathbb{P} \rightarrow \mathbb{Q}$  is said to *preserve meets* or be *closed under meets* (resp. *preserve joins* or *closed under joins*) iff for any pair  $x, y \in \mathbb{P}$  we have  $f(x \wedge y) = f(x) \wedge f(y)$  (resp.  $f(x \vee y) = f(x) \vee f(y)$ ). Whenever  $f$  is injective and preserves all joins and all meets,  $f$  is said to be an *embedding* of  $\mathbb{P}$  into  $\mathbb{Q}$  and that  $\mathbb{P}$  can be *embedded* within  $\mathbb{Q}$ . Given a function  $g : X \rightarrow Y$  ( $X$  and  $Y$  sets) we adopt the standard meaning of fibers of functions; for  $y \in Y$  the *fiber of  $g$  over  $y$*  is just  $\{x \in X \mid g(x) = y\} = g^{-1}(y)$ . For  $\mathbb{P}$  a complete lattice and  $A \subset \mathbb{P}$ ,  $A$  is a *sublattice* (resp. *complete sublattice*, *complete join sublattice*, *complete meet sublattice*) if the inclusion function is closed under all finite meets and joins (resp. closed under all meets and joins, closed under all joins, closed under all meets). We

note that the latter four are different notions.

All of our basic terminology regarding categories is standard and can be found in [ML98]. For convenience, we will recall that given functors  $F : \mathbf{C} \rightarrow \mathbf{D}$  and  $G : \mathbf{D} \rightarrow \mathbf{C}$ ,  $F$  is said to be *left adjoint* to  $G$  (and  $G$  is *right adjoint* to  $F$ ) provided that there exists a natural bijection (in the variables  $X$  and  $Y$ ) between morphisms  $f : X \rightarrow G(Y)$  in  $\mathbf{C}$  and  $\bar{f} : F(X) \rightarrow Y$  in  $\mathbf{D}$ . The pair  $F, G$  is commonly referred to as an *adjunction* and we denote the adjunction by  $G : \mathbf{D} \rightleftarrows \mathbf{C} : F$ . In the sequel we frequently describe posets as categories where arrows point in the direction of the order. In other words, for  $a, b \in \mathbb{P}$  (a poset) we have  $a \rightarrow b$  iff  $a \leq b$  and order-preserving maps become functors.

For a first-order language  $\mathcal{L}$ , formula  $\phi(x, \dots, x_n)$  from  $\mathcal{L}$  and structure  $M$ , the  $n$ -tuple  $\langle a_1, \dots, a_n \rangle \in M^n$  is said to *satisfy*  $\phi(x, \dots, x_n)$  provided

$$\models_M \phi(a_1, \dots, a_n).$$

The set  $\phi_M = \{\langle a_1, \dots, a_n \rangle \mid \models_M \phi(a_1, \dots, a_n)\}$  is said to be *defined* by  $\phi(x_1, \dots, x_n)$ . If  $\phi_M = N^n$  for some  $N \subseteq M$ , then  $N$  is also said to be *defined* by  $\phi(x_1, \dots, x_n)$ .

### 3.3 Substructures of $[0, \infty]^{X \times X}$

We begin this section by investigating the hierarchy of weight structures  $W_P$  in the lattice  $[0, \infty]^{X \times X}$ , exploring their lattice-theoretic and topological properties and connections between the two.

Elements of  $Met(X)$  are called *metric structures* on  $X$  while  $Top_M(X)$  will denote the collection of topologies generated by elements from  $Met(X)$ .

#### 3.3.1 Compactness and Completeness within $[0, \infty]^{X \times X}$

As a topological space, we consider  $[0, \infty]$  as the one-point compactification of  $[0, \infty)$ . It follows that the product space  $[0, \infty]^{X \times X}$  is compact and Hausdorff, and compact sets and closed sets coincide within  $[0, \infty]^{X \times X}$ . Given an open subset  $O$  of  $[0, \infty]$ ,

and  $a \in X$ , we denote by  $O_a$  the subbasic open set consisting of all functions whose evaluation of  $a$  belongs to  $O$ . Recall that the interval topology on a lattice  $L$  is the one generated by the rays  $x^\uparrow = \{y \in L \mid x \leq y\}$ ,  $x^\downarrow = \{y \in L \mid x \geq y\}$  (for all  $x \in L$ ) as subbasic closed sets.

**Theorem 3.3.1.** *The product topology on  $[0, \infty]^{X \times X}$  is exactly the same as the interval topology on it. Moreover, any sublattice  $L$  of  $[0, \infty]^{X \times X}$  is compact iff  $L$  is closed iff  $L$  is complete.*

*Proof.* Notice that rays are closed in the product space  $[0, \infty]^{X \times X}$  for if we let  $f \in [0, \infty]^{X \times X}$  and take any  $h \in [0, \infty]^{X \times X} \setminus f^\uparrow$ , then for some  $x \in X$  we have that  $h(x) < f(x)$  and we can create an  $\epsilon$ -interval around  $h(x)$  so that  $h(x) - \epsilon < h(x) + \epsilon < f(x)$ . It follows that  $(h(x) - \epsilon, h(x) + \epsilon)_x \cap f^\uparrow = \emptyset$ , where  $(h(x) - \epsilon, h(x) + \epsilon)_x = \{g \in [0, \infty]^{X \times X} \mid g(x) \in (h(x) - \epsilon, h(x) + \epsilon)\}$  is a basic open set, and so  $f^\uparrow$  is closed in the product topology. A similar argument shows that  $f^\downarrow$  is closed in the product topology. The other inclusion works as follows: take a subbasic open set  $O_a$  in the product topology (where we can assume  $O = [0, d)$  or  $(c, \infty]$ ). Since the collection of rays (as explained above) generates all closed sets in the interval topology, the collection of complements of rays generates all open sets. With that in mind, consider  $f_c, f_d \in [0, \infty]^{X \times X}$  so that:

$$f_c(x) = \begin{cases} c & x = a \\ 0 & \text{otherwise.} \end{cases}$$

$$f_d(x) = \begin{cases} d & x = a \\ \infty & \text{otherwise.} \end{cases}$$

The claim is that  $[0, c)_a = [0, \infty]^{X \times X} \setminus f_c^\uparrow$  and  $(d, \infty]_a = [0, \infty]^{X \times X} \setminus f_d^\downarrow$ . Clearly,  $[0, c)_a \subseteq (f_c^\uparrow)^C$  and  $(d, \infty]_a \subseteq (f_d^\downarrow)^C$ . If  $h \in (f_c^\uparrow)^C$  and since  $h \not\geq f_c$ , then it must be that  $h(a) < f_c(a) = c$  (since  $h(x) \geq f_c(x)$  for all other  $x \neq a$ ) and thus equality follows. The same happens with  $f_d$ . The second claim is due in part to Frink ([Fri42], where he shows that any lattice is compact in its interval topology if, and only if, it is complete) and, partly, to the subspace topology on a complete sublattice of

$[0, \infty]^{X \times X}$  being the same as the interval topology. Indeed, take any  $f \in [0, \infty]^{X \times X}$  and a complete sublattice  $L$ ; since  $L$  is complete,  $\bigwedge \{h \in L \mid h \geq f\} = g$  exists and thus  $g^\uparrow \cap L = f^\uparrow \cap L$ .

□

Since  $W_\emptyset = [0, \infty]^{X \times X}$  we will use  $W_\emptyset$  to denote our ambient structure  $[0, \infty]^{X \times X}$ . Clearly,  $W_\emptyset$  and  $W_\Sigma$  are sublattices of  $W_\emptyset$ ,  $W_s$  is closed under non-empty meets and joins from  $W_\emptyset$ , and  $W_\Delta$  is closed under arbitrary joins from  $W_\emptyset$ . If we consider two weight structures  $d, d' \in \text{Met}(X) \cap W_s$  so that for a triple  $x, y, z \in X$ ,  $d(x, z) = d'(x, z)$  and  $d(x, y) = d'(y, z) < \frac{d(x, z)}{2}$ , then  $d \wedge d' \notin W_\Delta$ . Consequently, neither  $\text{Met}(X)$  nor  $W_\Delta$  is closed under finite meets. Notice, also, that for any  $d \in W_\emptyset$  there is a unique  $d' \in W_\emptyset$  for which  $d(x, y) = d'(y, x)$  and we refer to  $d'$  as the *dual* of  $d$ . Of particular importance is to notice that for any  $W_P$ , the function  $f : W_P \rightarrow W_P$  for which  $d \mapsto d'$  is an order isomorphism with  $f^2 = \text{id}$ .

Let  $\mathcal{L}$  be a first-order language on  $X \times [0, \infty]$  (i.e., Boolean algebras, lattices, etc) and  $\phi(x_1, \dots, x_n)$  a formula from the language. A set  $\mathcal{F} \subseteq [0, \infty]^X$  is said to be *expressed* in  $\mathcal{L}$  via  $\phi(x_1, \dots, x_n)$  provided

$$\mathcal{F} = \{F \subseteq X \times [0, \infty] \mid \forall \langle a_1, \dots, a_n \rangle \in F^n, \models_F \phi(a_1, \dots, a_n)\}.$$

In other words,  $\mathcal{F}$  is expressed by the sentence  $\forall x_1 \dots x_n \phi(x_1, \dots, x_n)$ . A formula  $\phi(x_1, \dots, x_n)$  from the language  $\mathcal{L}$  is said to be *relaxed* if for any collection  $A = \{p_i = ((x_i, y_i), a_i)\}_{0 \leq i \leq n}$  (where  $x_i \in X$  and  $a_i \in [0, \infty]$ ) for which  $\not\models_A \phi(p_1, \dots, p_n)$  there exists  $\epsilon > 0$  so that for any other collection  $B = \{q_i = ((w_i, z_i), b_i) \mid a_i - \epsilon < b_i < a_i + \epsilon\}_{0 \leq i \leq n}$  we have that  $\not\models_B \phi(q_1, \dots, q_n)$ .

**Lemma 3.3.2.** *Let  $\mathcal{L}$  be as above. If a subset of  $W_\emptyset$  is expressible by a relaxed quantifier-free formula from  $\mathcal{L}$ , then it is compact.*

*Proof.* Let  $\phi(x_1, \dots, x_n)$  be a relaxed well-formed atomic formula.

Define  $\phi^* = \forall x_1, \dots, x_n \phi(x_1, \dots, x_n)$  and let  $F = \{f \subseteq X \times [0, \infty] \mid \models_f \phi^*\}$ . If  $g \notin F$  we can find  $p_1 = ((x_1, y_1), a_1), \dots, p_n = ((x_n, y_n), a_n) \in g$  (since  $\phi$  is quantifier free) so that  $\not\models_g \phi(p_1, \dots, p_n)$ . Since  $\phi$  is relaxed, there exists an  $\epsilon > 0$  for which any

$h \subseteq X \times [0, \infty]$  so that  $h \supset \{q_i = ((w_i, z_i), b_i) \mid a_i - \epsilon < b_i < a_i + \epsilon\}_{0 \leq i \leq n}$  is such that  $\not\equiv_h \phi(q_1, \dots, q_n)$ . The collection of all such  $h$  is a standard basic open set containing  $g$  and disjoint from  $F$ . Indeed, it can be written as  $\bigcap_{i \leq n} (a_i - \epsilon, a_i + \epsilon)_{x_i}$ . Hence,  $F$  is closed and thus compact.  $\square$

**Corollary 3.3.3.** *Let  $\mathcal{L}$  be as described above. If a subset of  $W_\emptyset$  is expressible by a collection of relaxed quantifier-free formulas from  $\mathcal{L}$ , then it is compact.*

*Proof.* This is a simple consequence of compact and closed sets being identical in  $W_\emptyset$ .  $\square$

Let  $\mathbf{P}$  be a unary predicate symbol for  $\mathcal{L}$  for which  $\mathbf{P}[(x, y), a]$  is true whenever  $x = y$ . Further, let  $\mathbf{Q}$  be also unary so that  $\mathbf{Q}[(x, y), a]$  is true precisely when  $a = 0$ . By letting  $\phi_0(v) := \mathbf{P}(v) \Rightarrow \mathbf{Q}(v)$  we get that  $W_\emptyset$  is compact and thus a complete sublattice of  $W_\emptyset$ . The same is true for  $W_\Sigma$  since we can define the relaxed formula  $\phi_\Sigma(w, v) := \text{“}w = ((x, y), a) \text{ and } v = ((y, x), b) \Rightarrow a = b\text{”}$  and let  $\overline{\phi_\Sigma} := \forall w, v(\phi_\Sigma(w, v))$  be the sentence expressing  $W_\Sigma$ ; notice that  $\text{“}w = ((x, y), a) \text{ and } v = ((y, x), b)\text{”}$  and  $\text{“}a = b\text{”}$  can both be defined by binary predicate symbols. As for  $W_\Delta$  let  $\phi_\Delta(u, v, w) := \text{“}u = ((x, y), a), v = ((y, z), b) \text{ and } z = ((z, x), c) \Rightarrow a + b \geq c\text{”}$  and

$$\overline{\phi_\Delta} := \forall x, y, z \phi_\Delta(u, v, w)$$

then  $\overline{\phi_\Delta}$  expresses  $W_\Delta$ . Hence, all combinations thereof are closed and compact within  $W_\emptyset$ . The same is not true of  $W_s$ . We can define  $\phi_s(v)$  to be  $\text{“}v = ((x, y), 0) \Rightarrow x = y\text{”}$  but this is not a relaxed formula. As a matter of fact  $W_s$  is not closed in  $W_\emptyset$ . Indeed, if we take any collection of weight structures  $\{d_i\}_{i \in \mathbb{R}^+} \subset W_s$  for which there exists a pair  $x, y \in X$  so that  $d_i(x, y) = i$  then  $\bigwedge d_i \notin W_s$ . That said,  $W_s$  is indeed closed under non-empty joins and finite meets from the ambient lattice.

### 3.3.2 Adjunctions

Consider  $2^{\mathcal{P}(X)} = \mathcal{P}(\mathcal{P}(X))$  for a fixed set  $X$ . Here, meets are intersections and joins are unions. Next, say that a collection  $C \in 2^{\mathcal{P}(X)}$  is a *cover* of  $X$  if

$$\bigvee_{c \in C} c = X.$$

Define  $Cov(X)$  to be the subset of  $2^{\mathcal{P}(X)}$  consisting of the covers of  $X$ . It is closed under all non-empty joins from  $2^{\mathcal{P}(X)}$ . The subset of  $2^{\mathcal{P}(X)}$  consisting of the bases for a topology on  $X$  will be denoted by  $Base(X)$  and  $Top(X)$  will denote the set of all topologies on  $X$ . The latter is closed under all non-empty meets from  $Base(X)$ . Since every topology is a base, every base is a cover we get the inclusion sequence  $Top(X) \hookrightarrow Base(X) \hookrightarrow Cov(X) \hookrightarrow 2^{\mathcal{P}(X)}$ . It is possible to move back along the previous sequence by, for instance, taking an element from  $2^{\mathcal{P}(X)}$  and adding  $X$  to it to get a cover. This cover is turned into a base by closing it under finite intersection (including the empty intersection) and turning this base into a topology in the usual way. Hence, omitting  $(X)$ , we get the following diagram

$$Top \longleftarrow Base \longleftarrow Cov \longleftarrow 2^{\mathcal{P}(X)}$$

A number of the above constructions can be viewed as adjunctions: for instance, if we consider  $Top(X)$  and  $Base(X)$  as categories (where arrows point in the direction of the order;  $a \rightarrow b$  iff  $a \leq b$ ) then generating a topology from a base amounts to taking a base and mapping it to the meet of all topologies containing it. That is, take  $b \in Base(X)$  with the inclusion map  $Top(X) \hookrightarrow Base(X)$  and notice that  $F : Base(X) \rightarrow Top(X)$  so that

$$b \mapsto \bigwedge_{b \leq a \in Top(X)} a$$

is the functor that generates a topology from a given base (i.e., the left adjoint to the inclusion mapping). This is a special case of the following theorem (cf. [Joh86] pg. 26).

**Theorem 3.3.4.** *Let  $g : \mathbb{P} \rightarrow \mathbb{Q}$  be an order-preserving map between posets. Then*

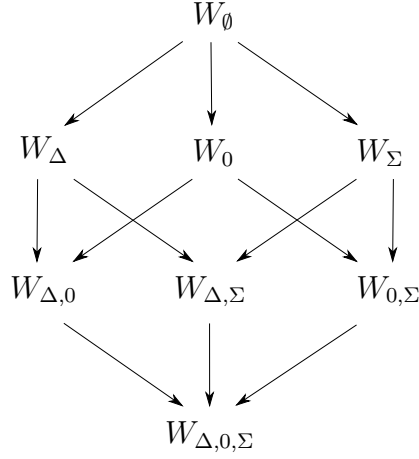
1. *if  $g$  has a left adjoint (resp. right adjoint)  $f : \mathbb{Q} \rightarrow \mathbb{P}$ , then  $g$  preserves all meets (resp. joins) that exist in  $\mathbb{P}$ .*
  
2. *if  $\mathbb{P}$  has all meets (resp. joins) and  $g$  preserves them, then  $g$  has a left (resp. right) adjoint.*

*In particular, if  $\mathbb{P}$  and  $\mathbb{Q}$  are complete lattices then  $g$  has a left (resp. right) adjoint iff  $g$  preserves all meets (resp. joins).*

Hence, the mappings  $Top(X) \leftrightarrow Base(X)$ ,  $Top(X) \leftrightarrow Cov(X)$  and  $Top(X) \leftrightarrow 2^{\mathcal{P}(X)}$  (resp.  $Cov(X) \leftrightarrow 2^{\mathcal{P}(X)}$ ) have left adjoints (resp. right adjoint) corresponding to the generation of topologies based on a subbase, base, cover and an arbitrary collection of sets. The inclusion  $Base(X) \leftrightarrow Cov(X)$  is not covered by Theorem 3.3.4 since  $Base(X)$  has neither all meets nor all joins. That said, this inclusion has a left adjoint: for  $c \in Cov(X)$

$$c \mapsto \bigwedge_{c \subseteq b} b.$$

In much the same spirit we can apply the above to the collection of weight structures. For instance, take any  $d \in W_0$  and send it to  $\bigvee_{d \geq d' \in W_{0,\Delta}} d'$ ; since  $d_0 \in W_{0,\Delta}$  (where  $\forall x, y \in X, d_0(x, y) = 0$ ) this map is well-defined. This mapping is right adjoint to the inclusion map  $W_{0,\Delta} \hookrightarrow W_0$ . The results from Section 3.1 in conjunction with Theorem 3.3.4 yield



Right adjoints

Notice that the inclusion  $W_s \hookrightarrow W_\emptyset$  is not left adjoint and, thus, not included in the above diagram. Its right adjoint, call it  $s_*$ , would have nowhere to map a weight structure  $d \in W_\emptyset$  for which  $d(x, y) = 0$  and  $x \neq y$ . Indeed, for any other  $d' \in W_s$  we need  $d' \leq s_*(d) \Leftrightarrow d' \leq d$  which is impossible. The previous argument can be easily extended to any other inclusion  $W_{P \cup s} \hookrightarrow W_P$  with  $s \notin P \subseteq \{\Delta, \Sigma, 0\}$ . As for left adjoints, the only type of inclusions that are right adjoints are those involving  $\Sigma$ . More to the point,  $W_{P \cup \Sigma} \hookrightarrow W_P$  is right adjoint and only this type of inclusions are right adjoint. Notice that  $W_{P \cup \Delta}$  is never closed under meets from  $W_P$  and that this fact settles the cases where  $W_{P \cup \Delta} \hookrightarrow W_P$ . For  $W_s \hookrightarrow W_\emptyset$ , notice that for a left adjoint,  $s_!$ , to exist it must be that if  $d \in W_\emptyset$  so that  $d(x, y) = 0$  with  $x \neq y$ , then for any other  $d' \in W_s$  we must have  $d' \geq s_!(d) \Leftrightarrow d' \geq d$ . The meet of all  $d' \in W_s$  for which  $d' \geq d$  does not exist in  $W_s$ . Lastly, a left adjoint to  $W_{P \cup 0} \hookrightarrow W_P$  does not exist since the inclusion map does not map the top element in  $W_{P \cup 0}$  to that of  $W_P$ .

Back to  $W_{0,\Delta} \hookrightarrow W_0$ , for any  $d \in W_0$  that doesn't satisfy the triangle inequality the right adjoints outlined above describe the process for turning  $d$  into one that does, call it  $m$ , and where  $d > m$ . For a pair  $x, y \in X$ ,  $m(x, y)$  must then be the shortest distance, travelling over all finite paths  $\gamma : (x = y_0, \dots, y_n = y)$  on  $X$ , from  $x$  to  $y$ . For every such path  $\gamma$ , the distance of each segment  $(y_k, y_{k+1})$  will be taken from  $d$ . In other words,  $m(x, y)$  is the infimum of all *lengths* over all finite paths  $\gamma$  from  $x$  to  $y$ , where the length of each segment of  $\gamma$  is given by  $d$ . Thus (a)  $m < d$ , (b)  $m$

satisfies the triangle inequality and (c)  $m$  is the largest weight structure to satisfy (a) and (b).

**Lemma 3.3.5.** *For  $d \in W_0$  and  $G : W_0 \rightarrow W_{0,\Delta}$  given by*

$$d \mapsto \bigvee_{d \geq d' \in W_{0,\Delta}} d',$$

*then for all  $x, y \in X$ ,  $G(d)(x, y) = \bigwedge_{\gamma} \sum_0^n d(y_i, y_{i+1})$  where the meet operation ranges over all paths  $\gamma : (x = y_0, y_1, \dots, y_n = y)$ .*

*Proof.* Let  $m(x, y) = \bigwedge_{\gamma} \sum_0^n d(y_i, y_{i+1})$  as defined above and note that  $m \in W_{0,\Delta}$ . Clearly,  $m \leq d$  and so  $G(d) \geq m$ . If  $G(d) > m$  then  $\exists x, y \in X$  so that  $G(d)(x, y) > m(x, y) = \bigwedge_{\gamma} \sum_0^n d(y_i, y_{i+1})$  and consequently, there exists a  $\gamma : (x = y_0, \dots, y_n = y)$  for which  $\sum_0^n G(d)(y_i, y_{i+1}) \geq G(d)(x, y) > \sum_i d(y_i, y_{i+1})$ . To this end, we have that for some  $i \leq n$ ,  $G(d)(y_i, y_{i+1}) > d(y_i, y_{i+1})$  which is a contradiction.  $\square$

In Section 3.3.3 we present a complete and comprehensive survey of all adjunctions between collections of weight structures (i.e., adjoints of adjoints, etc). In particular, we express the above outlined adjunction's explicitly just as it was done in the previous lemma. In [Kel63] Kelly defines the notion of a pair of *conjugate* pseudo-quasi-metrics on a set  $X$ ; that is, a pair of weight structures  $p, q \in W_{0,\Delta}$  for which  $p$  and  $q$  are duals of each other (i.e.,  $q(x, y) = p(y, x)$  for all  $x, y \in X$ ). The set  $X$  endowed with the conjugate pair  $(X, p, q)$  defines a bitopology on  $X$ ; a set with two associated topologies, one generated by  $p$  and the other by  $q$  (i.e., generated by their left  $\{0, \Delta\}$ -open sets). We can take the join of  $p$  and  $q$  and obtain a symmetric weight structure on  $X$  (which Kelly denotes as  $d$ ). The left adjoint to the inclusion  $W_{0,\Delta,\Sigma} \hookrightarrow W_{0,\Delta}$  (for which  $m \mapsto \bigwedge \{r \in W_{0,\Delta,\Sigma} \mid r \geq m\}$ ) is the one that takes  $p, q \mapsto d$ . Clearly the join of the topologies generated by  $p$  and  $q$  coincides with the one generated by  $d$ . Hence, the study of bitopological spaces generated by pseudo-quasi-metrics is closely related to that of the adjunction  $W_{0,\Delta} \rightleftarrows W_{0,\Delta,\Sigma}$ .

### 3.3.3 Adjoints: the full picture

For  $P \subseteq \{s, 0, \Delta, \Sigma\}$  and  $M$  a property from  $\{s, 0, \Delta, \Sigma\} \setminus P$ , let  $M^* : W_{P \cup M} \hookrightarrow W_P$ ; the collection  $P$  will be understood from context and thus its absence from the notation  $M^*$ . Its left adjoint, if it exists, will be denoted by  $M_!$ , while its right adjoint will be denoted by  $M_*$ .

For the property  $0$  and any  $P$  not containing  $0$ ,  $0^* : W_{P \cup 0} \rightarrow W_P$  has a right adjoint. The right adjoint is given, for  $d \in W_\emptyset$ , by  $0_*(d)(x, y) = d(x, y)$  if  $x \neq y$ , and  $0$  otherwise. Clearly, for any  $d' \in W_0$  we have that  $d'(x, y) \leq 0_*(d)(x, y) \iff 0^*(d')(x, y) \leq d(x, y)$ , establishing the adjunction. Further,  $0_*$  itself has a right adjoint if and only if  $\Delta \notin P$ . Namely,  $\infty_! : W_0 \rightarrow W_\emptyset$  given by  $\infty_!(d')(x, y) = d'(x, y)$  if  $x \neq y$  and  $\infty$  otherwise. Then,  $0_*(d)(x, y) \leq d'(x, y) \iff d(x, y) \leq \infty_!(d')(x, y)$  is the proof of the adjunction. Since  $\infty_!$  does not map the bottom element of  $W_0$  to the bottom element of  $W_\emptyset$  we see that  $\infty_!$  does not have a right adjoint, and this is the end of the line for the adjunctions.

For the property  $\Sigma$ , the functor  $\Sigma^* : W_\Sigma \rightarrow W_\emptyset$  has both a left adjoint and a right adjoint. The left adjoint  $\Sigma_!$  is given by  $\Sigma_!(d)(x, y) = d(x, y) \vee d(y, x)$  while the right adjoint  $\Sigma_*$  is given by  $\Sigma_*(d)(x, y) = d(x, y) \wedge d(y, x)$ .

**Lemma 3.3.6.** *For any  $P \subseteq \{0, \Delta, s\}$ ,  $\Sigma_*$  (resp.  $\Sigma_!$ ) has no right adjoint (resp. no left adjoint).*

*Proof.* This proof is constructed so as to satisfy any  $P \subseteq \{0, \Delta, s\}$ . We prove it for  $|X| = 2$  and extend it to any cardinality at the end of this proof.

( $\Sigma_!$ ) Assume as given and recall that if we can show that there exists a join in  $W_P$  that  $\Sigma_!$  doesn't preserve then it doesn't have a right adjoint. Let  $m, d \in W_P$  so that  $2 > m(x, y) > d(y, x) > m(y, x) > d(x, y) > 1$ . Since we also want  $m, d \in W_P$  for any  $P$ , we can make  $m$  and  $d$  so as to satisfy  $\Delta, 0$  and  $s$ : clearly  $\Delta$  and  $s$  are satisfied and we let  $m(x, x) = m(y, y) = d(x, x) = d(y, y) = 0$ . That is, the condition presented above on their distances between  $x$  and  $y$  does not interfere with any axiom from  $\{0, \Delta, s\}$ , only symmetry. Next, we show that

$$\Sigma_!(m \wedge d)(x, y) < (\Sigma_!(m) \wedge \Sigma_!(d))(x, y)$$

since that proves that  $\Sigma_!(m \wedge d) < \Sigma_!(m) \wedge \Sigma_!(d)$ . Notice that  $(m \wedge d)(x, y) = d(x, y)$  and  $(m \wedge d)(y, x) = m(y, x)$  (where  $d \wedge m$  is given point-wise) and so  $\Sigma_!(m \wedge d)(x, y) = \Sigma_!(m \wedge d)(y, x) = m(y, x)$ . Similarly,  $\Sigma_!(m)(x, y) = \Sigma_!(m)(y, x) = m(x, y)$  and  $\Sigma_!(d)(x, y) = \Sigma_!(d)(y, x) = d(y, x)$  and so  $(\Sigma_!(m) \wedge \Sigma_!(d))(x, y) = d(y, x) > m(y, x) = \Sigma_!(m \wedge d)(x, y)$ .

This proof also works for any other  $X$ . The reason we restricted the distances above between 1 and 2 is so that if there are other points in  $X$ , then we can just make their distance to  $x$  and  $y$  and between themselves = 1.

( $\Sigma_*$ ) The above example also works for  $\Sigma_*$ ; one must only show that it doesn't preserve that above metrics' join and can also be extended to any other  $X$  just as it was done with the left adjoint.  $\square$

Next we explore  $\Delta^* : W_{P \cup \Delta} \hookrightarrow W_P$ ; it has only a right adjoint,  $\Delta_*$ , and we show this right adjoint of  $\Delta^*$  to have no right adjoint (in very much the same spirit as with the previous property).

**Lemma 3.3.7.** *If  $\Delta \notin P$  and  $\Delta^* : W_{P \cup \Delta} \rightarrow W_P$ , then  $\Delta_*$  has no right adjoint.*

*Proof.* To avoid degenerate cases we demand  $|X| = 3$  and show how to extend it to any other cardinality. We create two weight structures that will satisfy all axioms from  $\{s, 0, \Sigma\}$  so as to prove the statement for any  $P$ . Let  $X = \{x, y, z\}$  and  $m, d \in W_{\{s, 0, \Sigma\}}$  so that all distances are bounded below by 1 and above by 2 and

- $m(x, y) + m(x, z) < m(y, z)$ ,
- $d(x, y) + d(x, z) < d(y, z)$ , and
- $m(y, z) < m(x, z) + d(x, y)$ ,  $m(x, y) < d(x, y)$ ,  $m(x, z) > d(x, z)$  and  $m(y, z) > d(y, z)$ .

First notice that  $(m \vee d)$  is taken point-wise since  $W_{\{s, 0, \Sigma\}}$  is closed under non-empty joins from  $W_\emptyset$ . It follows that, by design,  $m \vee d \in W_{P \cup \Delta}$  and thus  $\Delta_*(m \vee d) = m \vee d$ . Next, observe  $\Delta_*(m)(x, y) = m(x, y)$ ,  $\Delta_*(m)(x, z) = m(x, z)$  and  $\Delta_*(m)(z, y) =$

$m(x, y) + m(x, z)$ ;  $\Delta_*(d)(x, y) = d(x, y)$ ,  $\Delta_*(d)(x, z) = d(x, z)$  and  $\Delta_*(d)(z, y) = d(x, y) + d(x, z)$ . To this end we have

$$\Delta_*(m)(y, z) \vee \Delta_*(d)(y, z) < m(y, z) = \Delta_*(m \vee d)(x, y).$$

Again, to extend this example to any cardinality, we just evaluate all other distance to and from  $x, y, z$  to other points in  $X$  and between all other points in  $X$  to be  $= 1$ .  $\square$

### 3.4 Properties of $\psi_P : W_P \rightarrow Top(X)$

For any  $P \subseteq \{\Delta, \Sigma, 0\}$  we let  $\psi_P^+ : W_P \rightarrow Top(X)$  (resp.  $\psi_P^- : W_P \rightarrow Top(X)$ ) so that  $d$  is mapped to the topology its corresponding cover  $\{B_L(x)\epsilon \mid x \in X \text{ and } \epsilon > 0\}$  (resp.  $\{B_R(x)\epsilon \mid x \in X \text{ and } \epsilon > 0\}$ ) generates. It is clear that if  $\Sigma \in P$  then  $\psi^- = \psi^+$  and we assume that  $0 \in P$  throughout this section.

Several order-theoretic questions about each function  $\psi_P^+, \psi_P^-$  arise naturally. For instance, when are  $\psi_P^+, \psi_P^-$  order-preserving? Are meets and/or joins preserved? What can be said about the fibers of  $\psi_P^+, \psi_P^-$ ?

Notice that the above questions need only be solved by either one of any pair  $\psi_P^+, \psi_P^-$ . Indeed, recall from Section 3.3 that the function  $f : W_P \rightarrow W_P$  sending a generalized weight structure to its corresponding dual is an order isomorphism. In particular, we have that  $\psi_P^+ = f \circ \psi_P^-$ . In view of this, we will only concern ourselves with  $\psi_P^+$  and, for convenience, we denote it plainly by  $\psi_P$ .

We begin by noting that if the function  $\psi_P$  is closed under any property from  $\{\text{binary meets, binary joins, arb. meets, arb. joins}\}$ , then so is each fiber  $\psi_P^{-1}(\tau)$  for any  $\tau \in Top(X)$ . For instance, if  $\psi_P$  is closed under binary joins, then take any pair  $d, m \in \psi_P^{-1}(\tau)$  for some  $\tau \in Top(X)$ . Since  $\psi_P$  is closed under binary joins, then  $\psi_P(d \vee m) = \psi_P(d) \vee \psi_P(m) = \tau \vee \tau$  which then implies that  $d \vee m \in \psi_P^{-1}(\tau)$ . Similar arguments prove the above for the remaining properties.

**Lemma 3.4.1.** *For  $P \subseteq \{\Delta, 0, \Sigma\}$  and  $M \in \{\text{binary meets, binary joins, arb. meets, arb. joins}\}$  if  $\psi_P$  is closed under  $M$ , then each fiber of  $\psi_P$  is a sub  $M$ -semilattice of*

$W_P$ .

Incidentally, for  $\Delta \in P$   $\epsilon$ -balls form a base and the function  $\psi_P$  is order-preserving and preserves finite non-empty joins (thus  $\psi_P$  fibers are closed under binary joins). To see this, take  $d$  and  $m$  and denote  $\psi_P(d) \vee \psi_P(m) = \tau$  and  $p = m \vee d$ . Since  $p \geq m, d$  then  $\psi_P(p) \geq \tau$ . Let  $\epsilon > 0$  and take any  $y \in B_\epsilon^d(x) \cap B_\epsilon^m(x)$ . Since  $p(x, y) = \max\{m(x, y), d(x, y)\}$  then  $p(x, y) < \epsilon$ . It follows that  $y \in B_\epsilon^p(x)$ ,  $\psi_P(p) = \tau$  and thus  $\psi$  preserves joins. Next, take  $d \in W_P$  so that  $d(x, y) = 0$  or  $\infty$  for all  $x, y \in X$ ; the topology generated by  $d$  partitions  $X$  into basic open sets (i.e. open sets where  $d(x, y) = 0$  for all  $x, y$  in the open set). Notice that if  $d' > d$  then  $\psi_P(d') > \psi_P(d)$  and, thus,  $d = \bigvee(\psi_P^{-1} \circ \psi_P(d)) = \bigvee\{d' \in W_P \mid \psi_P(d) = \psi_P(d')\}$ . Consequently,  $\psi_P^{-1}(d)$  is closed under non-empty joins. Call any such weight structure (i.e., one whose evaluations are all either 0 or  $\infty$ ) a *partition weight structure*.

**Theorem 3.4.2.** For  $\Delta \in P \subseteq \{\Delta, 0, \Sigma\}$

- (a) (i)  $\psi_P$  preserves binary joins, and fibers along  $\psi_P$  are join sublattices of  $W_P$ .
- (ii) With the exception of topologies generated by partition weight structures,  $\psi_P$  fibers are not closed under non-empty joins. Consequently,  $\psi_P$  does not preserve arbitrary joins.
- (b) In general,  $\psi_P$  fibers are not lattices and  $\psi_P$  does not preserve binary meets.

*Proof.* (a)(i) has been proved already. For (a)(ii) it is clear that fibers that contain a partition weight structure are closed under non-empty joins. Otherwise, if for some  $\tau$  whose fiber does not contain a partition weight structure we have  $d = \bigvee \psi_P^{-1}(\tau) \in \psi_P^{-1}(\tau)$ , then  $d < 2d \in \psi_P^{-1}(\tau)$  (a contradiction). By Lemma 3.4.1,  $\psi$  does not preserve arbitrary joins. For (b) we make use of Figure 3.2, where all points are taken to depict points from the plane and the dashed lines represent concentric arcs centered at  $x$  (the radii of these arcs converges to 0). Let  $X_1$  (resp.  $X_2$ ) denote the collection  $\{x\} \cup \{y_i \mid 1 \leq i\}$  in addition to all red sequences (resp. blue sequences). We also let  $d : W_P(X_1)$  and  $m : W_P(X_1)$  be the usual planar metrics on their respective sets of points. Notice that for any fixed  $j \geq 1$  neither  $X_1$  nor  $X_2$  contains the limit of any  $\{j_n\}_{n \in \mathbb{N}}$  sequence and the same is true of the  $\{y_n\}_{n \in \mathbb{N}}$  sequence for both sets. The obvious function

$F : X_1 \rightarrow X_2$  where  $F(y_i) = y_i$ ,  $F(x) = x$  and all red points are mapped to their corresponding blue points generates a homeomorphism between  $(X_1, d)$  and  $(X_2, m)$ . The claim is true on all points other than  $x$  itself (since both metrics are discrete on all points other than  $x$ ). For  $x$  and any  $\delta > 0$  if  $\delta$  is the same as the radius of any one of the dashed arcs, then  $B_\delta^d(x) = B_\delta^m(f(x))$ . Let  $c_1 < \delta < c_2$  where  $c_1$  and  $c_2$  represent the radii of a pair of adjacent arcs. Obviously,  $B_\delta^d(x) \subseteq B_\delta^m(x)$  and since  $\{y\}$  ( $y \neq x$ ) is open in  $d$  then  $B_\delta^m(x) = B_\delta^d(x) \cup \{y \mid f(y) \in B_\delta^d(x)\}$ . Hence, both metrics generate homeomorphic topologies, in which case we will consider  $X_1$  and  $X_2$  as being the same set and refer to it as  $X$ .

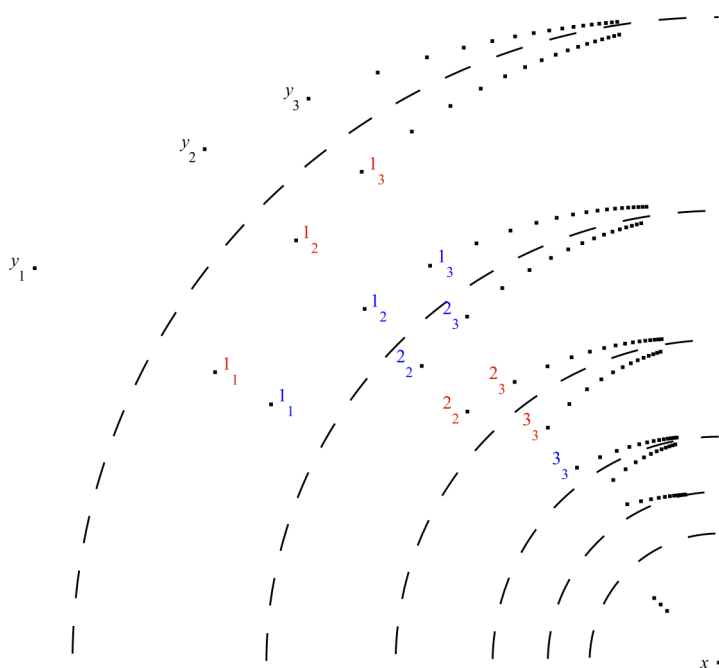


Figure 3.2: The metrics  $d$  and  $m$

Let  $p = d \wedge m$ : we will show that  $\{y_n\}_{n \in \mathbb{N}}$  converges to  $x$  in  $(X, p)$ . For  $p$ , the shortest way to travel from  $x$  to  $y_1$  is to go from  $x$  to  $1_1$  via  $m$  and from  $1_1$  to  $y_1$  via  $d$ . The same happens between  $x$  and  $y_2$  (i.e.  $p(x, y_2) = d(x, 2_2) + m(2_2, 1_2) + d(1_2, y_2)$ ). The sequences are taken to represent sets of points that become arbitrarily close to each other. For instance, for the blue  $\{1_i\}$  sequence and the blue  $\{2_i\}$  sequence, we

demand that  $\lim_{j \rightarrow \infty} m(1_j, 2_j) = 0$ . Thus  $\lim_{n \rightarrow \infty} p(x, y_n) = 0$  and the sequence  $\{y_n\} \xrightarrow{p} x$ . This tells us that there is a convergent sequence in  $(X, p)$  that converges in neither  $(X, d)$  nor  $(X, m)$  and thus the topology generated by  $p$  is strictly coarser than the one generated by  $d$  and  $m$ .  $\square$

The scenarios where  $\Delta \notin P$  are not as clear since  $\epsilon$ -balls do not form a base of open sets;  $\psi_P$  is actually a function from  $W_P$  into  $Cov(X)$ . In particular, as the following example shows,  $\psi_P$  in general is not order-preserving.

**Example 3.4.3.** Let  $X = \{x_n\} \cup \{y_n\} \cup \{x\}$  and  $d, m \in W_{0, \Sigma}$  so that for all  $a, b \in X$

$$d(a, b) = \begin{cases} 0 & a, b \in \{x_n\} \cup \{y_n\} \\ 2^{-n} & a = x, b = x_n \text{ or } a = x, b = y_0 \\ \frac{d(x, x_n) + d(x, x_{n+1})}{2} & a = x, b = y_n, n > 0. \end{cases}$$

and

$$m(a, b) = \begin{cases} 0 & a, b \in \{x_n\} \cup \{y_n\} \\ 2^{-n} & a = x, b = x_n \text{ or } a = x, b = y_0 \\ \frac{d(x, x_{n-1}) + d(x, x_n)}{2} & a = x, b = y_n, n > 0. \end{cases}$$

Clearly,  $m > d$  and neither of the topologies generated by them contains the other.

Example 3.4.3 also shows that neither  $\psi_0$  nor  $\psi_{0, \Sigma}$  are closed under either meets or joins. Indeed,  $\psi_P(m \vee d) = \psi_P(m) < \psi_P(d) \vee \psi_P(m)$  and  $\psi_P(m \wedge d) = \psi_P(d) > \psi_P(d) \wedge \psi_P(m)$ . Showing that fibers are not sublattices is more involved.

**Theorem 3.4.4.** For  $\Delta \notin P$

(a)  $\psi_P$  is not order-preserving.

(b)  $\psi_P$  preserves neither meets nor joins.

(c) Fibers along  $\psi_P$  are, in general, not closed under meets in  $W_P$ .

*Proof.* Example 3.4.3 proved part (a), and (a)  $\Rightarrow$  (b). We prove (c) for the case where  $P = \{0, \Sigma\}$  and note that the scenario where  $P = \{0\}$  follows immediately. Take  $X = \{x_n\} \cup \{y, z\}$  (all distinct points) and define  $d, m \in W_{0, \Sigma}$ , with  $0 < \alpha < 1$ , as follows

$$d(a, b) = \begin{cases} \alpha & a = y, b = z \\ 2^{-n} & a = y, b = x_n \\ 1 & a = z, b = x_n \\ |d(y, x_i) - d(y, x_j)| & a = x_i, b = x_j. \end{cases}$$

and

$$m(a, b) = \begin{cases} d(a, b) & a = y, b = z \\ d(y, x_n) & a = z, b = x_n \\ d(z, x_n) & a = y, b = x_n \\ |d(y, x_i) - d(y, x_j)| & a = x_i, b = x_j. \end{cases}$$

Notice that both weight structures generate the same topology on  $X$ . In particular, the set  $\{y, z\}$  can be separated from the sequence  $\{x_n\}$ . Next, let  $p = d \wedge m$  and notice that in the topology generated by  $p$  we have  $\{x_n\} \rightarrow \{y, z\}$ . Indeed, in  $p$  any  $\epsilon$ -ball about either  $z$  or  $y$  contains an infinite tail of  $\{x_n\}$  and, thus, the cover generated by  $p$  cannot separate  $\{y, z\}$  from the sequence.  $\square$

Weight structures that do not satisfy the triangle inequality are usually referred to as *premetrics* and *semimetrics* (depending on whether or not they satisfy symmetry). A standard way of generating a topology based on a premetric (in contrast to generating a cover that then generates a topology) is as follows

**Definition 3.4.5.** The function  $\phi_P : W_P \rightarrow Top(X)$  is the one for which given  $d \in W_P$  then  $O \in \phi_P(d)$  iff for all  $x \in O$  we can find  $\epsilon > 0$  so that  $B_\epsilon(X) \subseteq O$ .

The following result can be easily verified.

**Lemma 3.4.6.** *For any  $P \subseteq \{0, \Sigma, \Delta\}$  the function  $\phi_P$  is order-preserving. Moreover, given any  $d \in W_P$ ,  $\psi_P(d) \geq \phi_P(d)$ .*

Notice that if  $\Delta \in P$ ,  $\psi_P = \phi_P$ . Otherwise, even with this stronger definition of an open set,  $\epsilon$ -balls are not guaranteed to be open sets.

**Theorem 3.4.7.** *For  $0 \in P$*

(a) (i)  $\phi_P$  preserves binary meets, and fibers along  $\phi_P$  are sub meet semilattices of  $W_P$ .

(ii) *Fibers along  $\phi_P$  are not closed under binary joins, and  $\phi_P$  does not preserve joins.*

(b) *In general,  $\phi_P$  fibers are not closed under arbitrary meets, and  $\phi_P$  does not preserve arbitrary meets.*

*Proof.* Theorem 3.4.2 deals with the case where  $\Delta \in P$  and we assume that  $\Delta \notin P$  throughout the proof. In particular, this means that given a pair of weight structures from  $W_P$  then their meet is taken pointwise.

For (a)(i) first notice that since  $\phi_P$  is order-preserving it follows that for any pair  $d, m \in W_P$  we have  $\phi_P(d \wedge m) \leq \phi_P(d) \wedge \phi_P(m)$ . Next, let  $O \in \phi_P(d) \wedge \phi_P(m)$  and denote  $p = d \wedge m$ . In turn, for any  $x \in O$  we can find  $\epsilon_1, \epsilon_2 > 0$  so that  $B_{\epsilon_1}^d(x), B_{\epsilon_2}^m(x) \subseteq O$ . Letting  $\epsilon = \min\{\epsilon_1, \epsilon_2\}$  it is simple to verify that  $B_\epsilon^p = B_\epsilon^d \cup B_\epsilon^m \subseteq B_{\epsilon_1}^d(x) \cup B_{\epsilon_2}^m(x) \subseteq O$ . Hence,  $\phi_P(p) = \phi_P(d) \wedge \phi_P(m)$ .

For (a)(ii) we make use of part of the proof of Theorem 3.4.4 where  $X = \{x_n\} \cup \{z, y\}$  as follows

$$d(a, b) = \begin{cases} 0 & a = y, b = z \\ 2^{-n} & a = y, b = x_n \\ 1 & a = z, b = x_n \\ |d(y, x_i) - d(y, x_j)| & a = x_i, b = x_j. \end{cases}$$

and

$$m(a, b) = \begin{cases} d(a, b) & a = y, b = z \\ d(y, x_n) & a = z, b = x_n \\ d(z, x_n) & a = y, b = x_n \\ |d(y, x_i) - d(y, x_j)| & a = x_i, b = x_j. \end{cases}$$

Clearly,  $\phi_P(d \vee m)$  contains  $\{y, z\}$  as an open set and both weight structures generate the same topology on  $X$ . Since any open set containing  $y$  in  $\phi_P(d)$  contains an infinite tail of the sequence  $\{x_n\}$  then  $\phi_P(d)$  cannot separate  $y$  from the sequence (the same is true for  $z$  w.r.t.  $m$ ). Consequently, the topology generated by  $d \vee m$  is strictly larger than the one generated by  $\phi_P(d) \vee \phi_P(m)$ . It is important to notice that (just as with Theorem 3.4.4) the non-symmetric case follows immediately.

That last claim is simpler: take a two-point set  $X = \{x, y\}$  and  $d_n \in W_P$  so that  $d_n(x, y) = \frac{1}{n}$ . Each weight structure generates the discrete topology on  $X$  but their meet generates the trivial topology.  $\square$

## 3.5 $Met(X)$

### 3.5.1 Lattice Properties

Recall that for a set  $X$ ,  $Met(X) = W_{0, \Delta, \Sigma} = W_0 \cap W_\Delta \cap W_\Sigma$  and since all of the latter objects are sub join complete semilattices of the ambient lattice, so is  $Met(X)$ . In the previous section we constructed an instance where two elements from  $W_\Delta$  have a meet outside  $W_\Delta$ . Furthermore, the construction allows for the weight structures to be in  $Met(X)$  as long as  $|X| > 2$ . Hence,  $Met(X)$  is not a sublattice of  $W_0$ . Of course, since  $Met(X)$  is a join complete lattice, it is a complete lattice in its own right. The meet operation is different to that of the ambient lattice; for a collection  $\{d_i\}_{i \in I} \subset Met(X)$  the meet  $d$  (within  $Met(X)$ ) is given by

$$d(x, y) = \bigwedge_{\gamma: (x=y_0, \dots, y_n=y)} \sum_j \left( \bigwedge_i d_i(y_j, y_{j+1}) \right),$$

for any pair  $x, y \in X$ .

It is not hard to see that  $Met(X)$  does not contain atoms (resp. anti-atoms); simply choose any element  $d \in Met(X)$  and let  $d' \in Met(X)$  be the one that for all  $x, y \in X$  yields  $d'(x, y) = \frac{d(x, y)}{2}$  (resp.  $d'(x, y) = 2d(x, y)$ ). However, there exists a very special collection of metric structures that closely resembles anti-atoms in  $Met(X)$ .

**Definition 3.5.1.** For  $x \in X$  and  $\alpha \in (0, \infty]$  let  $d^{\alpha, x, y} \in Met(X)$  with  $d_{\alpha, x, y}(x, y) = \alpha$  and,  $d_{\alpha, x, y}(z, w) = \infty$  otherwise. We will refer to these objects as *pseudo-anti-atoms*.

Pseudo-anti-atoms are, in a sense, the collection of the *largest* elements from  $Met(X)$ . That is, for any metric structure  $d \in Met(X)$  we can find a pseudo-anti-atom, call it  $d^{\alpha, x, y}$ , so that  $d < d^{\alpha, x, y} < d_\infty$ . In particular, we can describe any metric structure as a meet of pseudo-anti-atoms; take  $d \in Met(X)$  and notice that

$$d = \bigwedge_{x, y \in X} d^{d(x, y), x, y}.$$

In the literature on  $Top(X)$  one also looks at basic intervals [VL72] (i.e., chains where each link is obtained by a minimal change).

**Definition 3.5.2.** For  $d, m \in Met(X)$  write  $d \prec_n m$  if  $d < m$  and if there exists a finite collection  $\{x_j \mid 1 \leq j \leq n\}$  of distinct elements from  $X$  such that  $d(u, v) = m(u, v)$  for all  $u, v \in X$  so that  $\{u, v\} \not\subseteq \{x_j\}$ . For  $d, m \in Met_1(X)$  we say that the interval  $[d, m]$  is an *n-elementary interval* if  $\forall d_3, d_4 \in [d, m]$  holds that if  $d_3 < d_4$  then  $d_3 \prec_n d_4$ . Say that  $d$  is *n-elementarily maximal* if no  $d'$  exists with  $d \prec_n d'$  and *n-elementarily minimal* if no  $d'$  exists so that  $d' \prec_n d$ .

If  $d \prec_n m$  then  $d \prec_m m$  for all  $m \geq n$ . Consequently, an *n-elementary interval* (resp. a metric structure is *n-elementary maximal*, *n-elementary minimal*) is *m-elementary* (resp. *m-elementary maximal*, *m-elementary minimal*) for all  $m \geq n$

(resp.  $m \leq n$ ). In Section 3.5.2 we shall prove that any finite lattice embeds in an  $n$ -elementary interval for some  $n \in \mathbb{N}$ .

**Lemma 3.5.3.** *For any  $n \in \mathbb{N}$  if  $d \prec_n m$  with  $0 < d(x, y) \forall x, y \in X$  and  $d(x, y) < m(x, y)$ , then*

(a) *The metric structures  $d$  and  $m$  generate the same topology on  $X$ .*

(b) *All metric structures in  $[d, m]$  generate the same topology on  $X$ .*

*Proof.* Clearly (a) iff (b) and we prove (a). We begin by letting  $\{x_j\}_{j \leq n}$  be the set for which  $d(x_i, x_j) < m(x_i, x_j)$  ( $i \neq j$ ). All open balls about any  $z \in X \setminus \{x_j\}_{j \leq n}$  are the same for both metric structures. Also, any  $m$   $\epsilon$ -ball about any  $x_i$  is finer than a  $d$   $\epsilon$ -ball about  $x_i$ . Lastly, for any  $\epsilon > 0$  let  $0 < \delta < \min\{m(x_i, x_j) \mid i \neq j\}$  and notice that  $B_\delta^d(x) \subseteq B_\epsilon^m(x)$ . Since both metric structures generate the same topology the proof is complete.  $\square$

Recall that a metric space  $(X, d)$  is *Menger convex* if for all  $x \neq y \in X$  and  $0 < r < d(x, y) = L$  there exists a point  $p \in X$  satisfying  $d(x, p) = r$  and  $d(p, y) = L - r$ . Similarly, we define the dual of Menger convexity: a metric space  $d$  is *Menger\* convex* provided for any pair  $x \neq y \in X$  there exists a  $z \in X$  so that  $d(y, z) = d(x, y) + d(x, z)$ . We adopt both definitions for the collection  $Met(X)$ .

**Theorem 3.5.4.** *A metric structure  $d$  is 1-elementary maximal (resp. 1-elementary minimal) if, and only if,  $\forall x, y \in X$  and all  $\epsilon > 0$  there exists a  $z \in X$  so that  $d(x, y) > d(x, z) + d(y, z) - \epsilon$  (resp.  $d(x, z) > d(x, y) + d(y, z) - \epsilon$ ).*

*Proof.* We prove the claim for elementary maximality since its dual follows immediately. For sufficiency of any such  $d$ , it is clear that if for any pair  $x \neq y \in X$  and an arbitrary  $\epsilon > 0$  we can find  $z \in X$  so that  $d(x, y) > d(x, z) + d(y, z) - \epsilon$ , then it is impossible to find any  $m$  so that  $d \prec_1 m$ . Indeed, any such  $m$  would violate the triangle inequality by only extending the distance of a distinct pair  $x, y$  and nothing else. Necessity is also straight forward; the only axiom that inhibits a 1-elementary extension on any such  $d$  is the triangle inequality.  $\square$

The following is simple to verify.

**Corollary 3.5.5.** *Any Menger (resp. Menger\*) metric structure is 1-elementarily maximal (resp. 1-elementarily minimal).*

### 3.5.2 Lattice Embeddability

In [Whi46] Whitman proved that any lattice can be embedded in the lattice of equivalence relations,  $Eq(X)$ , for some set  $X$ . We exploit this remarkable result in the following where  $\psi : Met(X) \rightarrow Top(X)$  and  $\top$  is the discrete topology on  $X$ .

**Theorem 3.5.6.** *Any lattice can be embedded in  $\psi^{-1}(\top)$  for some set  $X$ .*

*Proof.* Take  $1 < \alpha < 2$  and let  $\phi : Eq(X) \rightarrow Met(X)$  so that  $\sim \in Eq(X)$   $\phi(\sim) = d_\sim \in Met(X)$  where

$$d_\sim(a, b) = \begin{cases} \alpha & \text{if } a \sim b \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Then for  $\{\sim_i\}_{i \in I} \subset Eq(X)$  and  $\sim = \bigwedge \sim_i$  we have  $a \sim b$  if and only if  $(a, b) \in \sim_i$  for all  $i$ . The same happens with  $\phi(\sim) = \phi(\bigwedge \sim_i)$ . That is,  $\phi(\sim)(a, b) = \alpha$  if and only if  $\phi(\sim_i)(a, b) = \alpha$  for all  $i$ . Hence,  $\phi(\sim) = \bigwedge \phi(\sim_i)$ . The same occurs with joins; for  $\sim = \bigvee \sim_i$  then  $\phi(\sim) = \bigvee \phi(\sim_i)$ . Notice that the above joins and meets in  $\psi^{-1}(\top)$  are in agreement with the meets and joins from the ambient lattice. Thus we have embedded  $Eq(X)$  within  $\psi^{-1}(\top)$ .  $\square$

The authors of [KGM97] show that all finite distributive lattices occur as intervals between Hausdorff topologies; we show something similar occurs in  $Met(X)$ . Until 1980, one of the outstanding questions in lattice theory was a conjecture by Whitman: *every finite lattice can be embedded in some  $Eq(X)$ , for a finite set  $X$* . The conjecture was turned into a theorem by P. Pudlák and J. Tůma in [PT80].

**Theorem 3.5.7.** *Any finite lattice can be embedded in an  $n$ -elementary interval within  $Met(X)$  for some finite set  $X$ .*

*Proof.* We will embed  $Eq(X)$  in an  $n$ -elementary interval, where  $n = \frac{|X|(|X|-1)}{2}$ . Let  $d, d_\alpha \in Met(X)$  so that for all  $x, y \in X$ ,  $d(x, y) = 1 < d_\alpha(x, y) = \alpha < 2$ . In particular,  $d \prec_n d_\alpha$  and  $[d, d_\alpha]$  is  $n$ -elementary. Just as with Theorem 3.5.6 we let  $\phi : Eq(X) \rightarrow Met(X)$  so that  $\sim \in Eq(X)$   $\phi(\sim) = d_\sim \in Met(X)$  where

$$d_\sim(a, b) = \begin{cases} \alpha & \text{if } a \sim b \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Since any lattice embeds in the lattice  $Eq(X)$  for some finite set  $X$  and we've embedded  $Eq(X)$  in an  $n$ -elementary interval, the proof is complete. □

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# Chapter 4

## A Metric Approach to Topology

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### Abstract

We construct two categories  $\mathbf{M}$  and  $\mathbf{P}$ , with functors  $\mathcal{M} : \mathbf{M} \rightarrow \mathbf{Top}$  and  $\mathcal{P} : \mathbf{P} \rightarrow \mathbf{Top}$  so that  $\mathcal{M}$  witnesses an equivalence of categories and  $\mathcal{P}$  is a faithful, continuous (and lifts limits) and surjective on objects extension of  $\mathcal{M}$ . In particular, these categories are designed so as to capture the essence of topology in terms of distance assignments between points by naturally extending the categories of all pseudoquasimetrics and premetrics (with  $\epsilon$ - $\delta$  continuous functions as morphisms), respectively. In view of the equivalence  $\mathbf{M} \rightarrow \mathbf{Top}$  we also study several reformulations of well-known topological properties in terms of naturally occurring full subcategories of  $\mathbf{M}$ .

### 4.1 Introduction

Given a set  $X$ , a *weight structure*  $d \in [0, \infty]^{X \times X}$  is understood to denote a distance assignment between pairs of points from  $X$ . Several restrictions on the behaviour of any such  $d$  naturally arise:

- (a)  $\forall x \in X, d(x, x) = 0$ .
- (b)  $\forall x, y \in X, d(x, y) < \infty$  (*finite-valued*).
- (c)  $\forall x, y \in X, d(x, y) = 0 \Rightarrow x = y$  (*separation*).
- (d)  $\forall x, y \in X, d(x, y) = 0 = d(y, x) \Rightarrow x = y$  (*disjointness*).
- (e)  $\forall x, y \in X, d(x, y) = d(y, x)$  (*symmetry*).
- (f)  $\forall x, y, z \in X, d(x, y) + d(y, z) \geq d(x, z)$  (*triangle inequality*).

If one assumes (e) then (c) and (d) are equivalent. Traditionally, a weight structure is referred to as a metric provided it satisfies all six axioms. Other collections of weight structures can be identified by adjoining a prefix (e.g. para, pre, quasi, pseudo, semi, etc) to the word metric where each prefix denotes a particular set of axioms to be assumed true (or not). Some prefixes are universally understood while others are left to the author's discretion. For any weight structure,  $d$ , that satisfies (a) there exists an obvious topology,  $\tau$ , generated in terms of  $\epsilon$ -balls. That is,  $A \in \tau$  iff for all  $x \in A$  there exists  $\epsilon > 0$  so that  $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\} \subseteq A$  (as a matter of fact, if  $d$  doesn't satisfy symmetry then there are two obvious topologies associated with  $d$ : the right and left  $\epsilon$ -ball topologies). These  $\epsilon$ -balls are only guaranteed to be open (and thus form a base for  $\tau$ ) provided  $d$  satisfies (f). From now on we assume that any weight structure satisfies (a) and only concern ourselves with axioms (c)-(f). In the literature one can find an ample selection of candidates for morphisms between weight structures: continuous maps, contractions, uniformly continuous maps, Lipschitz maps, etc. We adopt continuous functions as morphisms between weight structures<sup>1</sup>. More precisely, we adopt the  $\epsilon$ - $\delta$  definition of continuity between weight structures. It is important to notice that topological continuity and  $\epsilon$ - $\delta$  continuity are equivalent iff the weight structure satisfies the triangle inequality. Otherwise we can only assume that  $\epsilon$ - $\delta$  continuity implies topological continuity.

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<sup>1</sup>Continuous functions are not considered to be the most *natural* choice of morphisms for metric spaces (in particular in view of F. W. Lawvere's description of metric spaces as  $[0, 1]$ -enriched categories [Law02]). Our reasons for choosing continuous maps will become apparent in the sequel.

It is natural then to distinguish between two collections (categories) of weight structures: those that satisfy the triangle inequality and those that do not. Let  $\mathbf{W}^\Delta$  (resp.  $\mathbf{W}$ ) denote the category of weight structures that satisfy (a) and (f) (resp. (a)) with  $\epsilon$ - $\delta$  continuous functions. Let  $\mathcal{M}$  and  $\mathcal{P}$  denote the functors that take a weight structure from  $\mathbf{W}^\Delta$  and  $\mathbf{W}$ , respectively, and send it to the topology it generates in  $\mathbf{Top}$ . For reasons outlined above,  $\mathcal{P}$  is a faithful extension of  $\mathcal{M}$  and  $\mathcal{M}$  is fully faithful. Moreover, since all objects from  $\mathbf{W}$  generate sequential spaces then  $\mathcal{P}[\mathbf{W}]$  is a proper subcategory of  $\mathbf{Top}$ . The following question then arises naturally:

*Is there a way to naturally extend either or both categories so as to capture the abstract concept of topology in terms of the intuitive notion of ‘distance assignment’ between points? More precisely*

- (a) Locate an extension  $\mathbf{W}^\Delta \rightarrow \mathbf{M}$  of  $\mathbf{W}^\Delta$  such that the functor  $\mathbf{W}^\Delta \rightarrow \mathbf{Top}$  extends to an equivalence  $\mathcal{M} : \mathbf{M} \rightarrow \mathbf{Top}$ .
- (b) Locate an extension  $\mathbf{W} \rightarrow \mathbf{P}$  of  $\mathbf{W}$  such that the functor  $\mathbf{W} \rightarrow \mathbf{Top}$  extends to a faithful surjective functor  $\mathcal{P} : \mathbf{P} \rightarrow \mathbf{Top}$ .

One of the strongest attempts at solving (a) was achieved by R. Kopperman ([Kop88]) in terms of his continuity spaces (sets valued on *value semigroups*). This theory captures many of the properties from  $([0, \infty], \leq, +)$  that make metric techniques more powerful than topological ones. Unfortunately, in Kopperman’s theory, the concept of positive elements (which is naturally found in  $([0, \infty], \leq, +)$ ) is not an intrinsic one. A refinement of Kopperman’s category is the one developed in this paper whose objects are based on R. C. Flagg’s (also labeled as) *continuity spaces* where the concept of positivity is an intrinsic one ([Fla97]). It is this latter construction which will give rise to the categories  $\mathbf{P}$  and  $\mathbf{M}$  mentioned above. In particular, since Flagg’s construction essentially gives rise to  $\text{Ob}(\mathbf{M})$  (we only define the morphisms in  $\mathbf{M}$  and note the equivalence of categories) our aim within  $\mathbf{M}$  is to provide a topological characterisation of an obvious collection of full subcategories of  $\mathbf{M}$  (that of symmetric, disjoint and separated continuity spaces). On the other hand,  $\mathbf{P}$  will

be constructed based on  $\mathbf{M}$  so as to naturally extend  $\mathbf{W}$ . We will investigate  $\mathbf{P}$  in greater detail than  $\mathbf{M}$ : we provide constructions for all topological limits and some colimits within  $\mathbf{P}$ . Even though  $\mathcal{P}$  will turn out not to be full,  $\mathbf{Top}$  and  $\mathbf{P}$  will be shown to be surprisingly similar; hereafter we show that  $\mathcal{P}$  is continuous and lifts all small limits and that  $\mathcal{P}$  preserves and lifts all small coproducts and direct limits.

## 4.2 Constructing $\mathbf{M}$ and $\mathbf{P}$

### 4.2.1 The Category $\mathbf{M}$

We begin by illustrating the construction of  $\mathbf{M}$  (based on Flagg’s continuity spaces) and later note the equivalence with  $\mathbf{Top}$ . For a lattice  $L$  and any pair  $x, y \in L$ ,  $y \succ x$  is the *well-above* relation defined by  $y \succ x$  if whenever  $x \geq \bigwedge S$ , with  $S \subseteq L$ , there exists some  $s \in S$  such that  $y \geq s$ . A well-known characterization of completely distributive lattices is as follows ([Ran53])

**Theorem 4.2.1.** *A lattice  $L$  is completely distributive iff for all  $y \in L$*

$$y = \bigwedge \{a \in L \mid a \succ y\}.$$

**Definition 4.2.2.** A *value distributive lattice* is a completely distributive lattice  $L$  for which  $L_{\prec} = \{a \in L \mid a \succ 0\}$  forms a filter.

**Definition 4.2.3.** A *value quantale* is a completely distributive lattice  $L$  together with an associative and commutative binary operation  $+$  :  $L \times L \rightarrow L$  such that

- $x + 0 = x$  for all  $x \in L$  (here 0 is the empty join).
- $\bigwedge(x + S) = x + \bigwedge S$  for all  $x \in L$  and  $S \subseteq L$ .

Value quantales will often be denoted with the letters  $V$  and  $W$  in order to syntactically separate them from standard completely distributive lattices.

**Definition 4.2.4.** Let  $V$  be a value quantale. A  $V$ -space is a pair  $(X, d)$  with  $X$  a set and  $d : X \times X \rightarrow V$  such that

- $d(x, x) = 0$  for all  $x \in X$
- $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$

hold.

Given any value quantale,  $V$ , and a  $V$ -space  $(X, d)$  we follow Flagg's terminology of the triple  $(V, X, d)$  as a *continuity space*. The category  $\mathbf{M}$  will be that of all continuity spaces; its objects are the triples  $(V, X, d)$  where  $V$  is a value quantale and  $(X, d)$  is a  $V$ -space. A morphism  $(V, X, d) \rightarrow (W, Y, m)$  is a function  $f : X \rightarrow Y$  such that for every  $x \in X$  and for every  $\epsilon \in W_{\prec}$  there exists  $\delta \in V_{\prec}$  such that for all  $x' \in X$ : if  $d(x', x) \prec \delta$  then  $m(f(x'), f(x)) \prec \epsilon$ . Morphisms in  $\mathbf{M}$  are then in perfect agreement with those in  $\mathbf{W}^{\Delta}$  and thus  $\mathbf{W}^{\Delta}$  is a full subcategory of  $\mathbf{M}$ : every ordinary metric space  $(X, d)$  is a  $V$ -space for  $V = [0, \infty]$  with  $+$  being ordinary addition.

**Definition 4.2.5.** Let  $(X, d)$  be a  $V$ -space and  $\epsilon \in V$  with  $\epsilon \succ 0$ .  $B_{\epsilon}(x) = \{y \in X \mid d(y, x) \prec \epsilon\}$  is the *open ball* with radius  $\epsilon$  about the point  $x \in X$ .

**Theorem 4.2.6.** Let  $(X, d)$  be a  $V$ -space. Declaring a set  $U \subseteq X$  to be *open* if for every  $x \in U$  there exists  $\epsilon \succ 0$  such that  $B_{\epsilon}(x) \subseteq U$  defines a topology on  $X$ .

*Proof.* The proof of the above is straightforward and can be found in Flagg's paper. □

This construction clearly gives rise to a functor  $\mathcal{M} : \mathbf{M} \rightarrow \mathbf{Top}$  which obviously extends the usual open balls topology  $\mathcal{M} : \mathbf{W}^{\Delta} \rightarrow \mathbf{Top}$ .

**Theorem 4.2.7.**  $\mathcal{M} : \mathbf{M} \rightarrow \mathbf{Top}$  is an equivalence of categories.

*Proof.* Verifying that  $\mathcal{M}$  is fully faithful (i.e. surjective on morphisms) is done just as in the standard case of ordinary metric spaces. To finish the proof it suffices to show that  $\mathcal{M}$  is essentially surjective. Something stronger is true:  $\mathcal{M}$  is actually surjective on objects. The proof is given by Flagg but we repeat the construction here.

Given a set  $A$  consider  $\mathcal{P}_f(A)$ , the set of all finite subsets of  $A$ . A collection  $a \subseteq \mathcal{P}_f(A)$  is downwards closed if for all  $B, C \in \mathcal{P}_f(A)$ ,  $B \subseteq C$  and  $C \in a$  implies  $B \in a$ . Let  $\Omega(A)$  be the set of all downwards collections  $a \subseteq \mathcal{P}_f(A)$ , ordered by reverse inclusion and with  $+$  taken to be intersection. It is easy to verify that  $\Omega(A)$  is a value quantale.

Now, given a topological space  $(X, \tau)$  construct an  $\Omega(\tau)$ -space  $(X, d)$  for which

$$d(x, y) = \{F \subseteq \mathcal{P}_f(\tau) \mid \text{for all } U \in F \text{ if } x \in U \text{ then } y \in U\}.$$

It distills into a tautology to show that the open balls topology induced by  $(X, d)$  is  $(X, \tau)$ . This shows that  $\mathcal{M} : \mathbf{M} \rightarrow \mathbf{Top}$  is surjective on objects.

□

*Remark 4.2.8.* There is nothing new in what we've said so far, except for constructing  $\mathbf{M}$  and noticing that  $\mathcal{M} : \mathbf{M} \rightarrow \mathbf{Top}$  is actually an equivalence of categories.

## 4.2.2 The Category $\mathbf{P}$

There is the obvious functor  $\mathbf{W} \rightarrow \mathbf{M}$  that follows the topologies generated by objects from both categories. This is not nearly as satisfying as the inclusion  $\mathbf{W}^\Delta \rightarrow \mathbf{M}$ . Therefore, we construct the category  $\mathbf{P}$  whose objects are triples  $(V, X, d)$  where  $V$  is a value distributive lattice and  $d : X \times X \rightarrow V$  so that  $d(x, x) = 0$ . An associative and commutative binary operation is no longer required. For simplicity we also refer to objects in  $\mathbf{P}$  as continuity spaces; in the sequel, it will be clear from context which type of continuity space is under consideration.

Notice that the above is enough to capture all of the necessary properties required for a general notion of a premetric (i.e. an object from  $\mathbf{W}$ ). The obvious way to generate a topology follows that of premetrics:  $(X, \tau)$  is generated by a triple  $(V, X, d)$  provided that  $A \in \tau$  iff for all  $x \in A$ , there is  $\epsilon \succ 0$  so that  $B_\epsilon(x) = \{y \in X \mid d(x, y) \prec \epsilon\} \subseteq A$ . Obviously  $\epsilon$ -balls are not guaranteed to be open: take the three-point set  $X = \{a, b, c\}$  and premetric  $d : X \times X \rightarrow \mathbb{R}$  so that  $d(a, b) = d(b, c) = 0$  and  $d(a, c) = d(b, a) = 1$  and notice that  $B_{\frac{1}{2}}(a)$  is not open. Morphisms will again be  $\epsilon$ - $\delta$

continuous functions. This is fundamentally different from the previous construction: all  $\epsilon$ - $\delta$  continuous functions are topologically continuous but the converse can be easily shown to fail. A faithful functor, surjective on objects, is the best we can hope for.

**Lemma 4.2.9.** *The functor  $\mathcal{P} : \mathbf{P} \rightarrow \mathbf{Top}$  is a faithful functor that is surjective on objects.*

*Proof.* Faithfulness is clear. Since  $\mathbf{M}$  is a full subcategory of  $\mathbf{P}$  then object surjectivity is proved.  $\square$

Any premetric is a continuity space and  $\mathbf{W}$  is a full subcategory of  $\mathbf{P}$  with the obvious inclusion functor  $\mathbf{W} \rightarrow \mathbf{P}$ . The full picture is in the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathbf{W} & \longleftarrow & \mathbf{W}^\Delta \\
 & \swarrow \mathcal{P} & \downarrow & & \downarrow & \searrow \mathcal{M} \\
 \mathbf{Top} & \xleftarrow{\mathcal{P}} & \mathbf{P} & \longleftarrow & \mathbf{M} & \xrightarrow{\mathcal{M}} & \mathbf{Top}
 \end{array}$$

where all unlabeled arrows denote inclusions.

### 4.3 Full subcategories of $\mathbf{M}$ : symmetry, disjointness and separation

Since symmetry is not assumed within  $\mathbf{M}$  it is possible to consider the following cases separately.

**Definition 4.3.1.** Given a continuity space  $(V, X, d)$

- (a) if for all  $x, y \in X$ ,  $d(x, y) = 0$  and  $d(y, x) = 0$  implies  $x = y$  then  $(V, X, d)$  is *disjoint*.

(b) if for all  $x, y \in X$ ,  $d(x, y) = 0$  implies  $x, y$  then  $(V, X, d)$  is *separated*.

(c) if for all  $x, y \in X$ ,  $d(x, y) = d(y, x)$  then  $(V, X, d)$  is *symmetric*.

The full subcategory of  $\mathbf{M}$  of all continuity spaces satisfying (a) (resp. (b), (c)) will be denoted by  $\mathbf{M}_D$  (resp.  $\mathbf{M}_S, \mathbf{M}_\Sigma$ ). Any combination thereof will be denoted in the usual way. For instance,  $\mathbf{M}_{\Sigma, S}$  is comprised of all continuity spaces satisfying (c) and (b). Clearly,  $\mathbf{M}_S$  is a full subcategory of  $\mathbf{M}_D$  and under symmetry  $\mathbf{M}_{\Sigma, S} = \mathbf{M}_{\Sigma, D}$ . Giving purely topological characterisations for  $\mathbf{M}_D$  and  $\mathbf{M}_S$  is a relatively simple task.

**Lemma 4.3.2.** *For a continuity space  $(V, X, d)$*

(a)  $\mathcal{M}[(V, X, d)]$  is  $T_0$  iff  $(V, X, d)$  is disjoint and  $\mathcal{M}[\mathbf{M}_D]$  is the full subcategory of **Top** of all  $T_0$  topological spaces.

(b)  $\mathcal{M}[(V, X, d)]$  is  $T_1$  iff  $(V, X, d)$  is separated and  $\mathcal{M}[\mathbf{M}_S]$  is the full subcategory of **Top** of all  $T_1$  topological spaces.

*Proof.* Flagg shows (see [Fla97]) surjectivity of objects. Morphisms follow immediately from Lemma 4.2.9.  $\square$

### 4.3.1 Symmetry

Characterizing symmetry purely in topological terms is considerably more involved than separation and disjointness. As the following lemmas show, symmetry is equivalent to complete regularity. This is far from unexpected result: symmetry in Kopperman's value semigroups is also equivalent to complete regularity (see [Kop88]). Moreover, uniformities yield only completely regular spaces and any completely regular topology has a corresponding uniform space. Dyadics are frequently employed when showing complete regularity.

**Definition 4.3.3.**  $DY = \{\frac{i}{2^j} \mid i, j \in \mathbb{N}\}$  and  $DY_1 = DY \cap [0, 1]$ .

The proof of the following can be found in [Fla97] pg. 264.

**Lemma 4.3.4.** *For a value quantale  $(V, \leq, +)$  if  $\epsilon \succ 0$  then there exists  $\delta \succ 0$  so that  $\epsilon \succ 2\delta$ .*

This is of key importance to show complete regularity. Loosely speaking, given any  $\epsilon \succ 0_V$  in a value quantale  $V$ , the previous lemma is enough to guarantee an order-preserving function  $f : DY_1 \rightarrow \{\delta \in V \mid \delta \leq \epsilon\}$ . In turn, as we shall see, it is possible to generate a nested collection of open sets (indexed by  $DY_1$ ) about any point  $x \in X$ . Take any  $\epsilon \succ 0_V$ . We are guaranteed at least one  $\delta_1 \succ 0_V$  so that  $2\delta_1 \prec \epsilon$ . Similarly, since  $\delta_1 \succ 0_V$  we can find  $\delta_2 \succ 0_V$  so that  $2\delta_2 \prec \delta_1$  and so on. Thus, we have  $\{\delta_i \mid i \in \mathbb{N}\}$  so that  $2\delta_{i+1} \prec \delta_i$ . Next we define,  $\frac{\epsilon}{2^i} := \delta_i$  (where we let  $\delta_0 := \epsilon$ ). Clearly,  $2\frac{\epsilon}{2} \leq \epsilon$  and equality is not at all guaranteed: take the lattice  $\{\perp, \top\}$  as an example. At this stage all powers of  $\frac{1}{2}$  have been suitably defined: what about all remaining dyadics in  $[0, 1]$ ? Such numbers will be defined in terms of powers of  $\frac{1}{2}$ : for instance, we let  $\frac{3}{4}\epsilon := \frac{\epsilon}{2} + \frac{\epsilon}{4}$ . Recall that any dyadic number can be expressed uniquely as a sum of products of  $\frac{1}{2}$ ; given any  $n \in DY_1$  we can express  $n = \sum_{i=0}^{\infty} f(i)_n \frac{1}{2^i}$  where  $f(i)_n = 0$  or  $1$  (in a sense this function defines  $n$ ). In general we let, for  $n \neq \frac{1}{2^i}$  (since we've already defined those)  $n\epsilon := \sum_{i=0}^{\infty} f_n(i)\epsilon$  and it is simple to show that for  $n \leq m \in DY_1$  then  $n\epsilon \leq m\epsilon$ . Next, take  $\delta, \epsilon$  so that  $2\delta \prec \epsilon$  and any  $x \in X$ . If  $y \in \overline{B_\delta(x)}$  then for any  $p \in V$  we can find a  $z \in B_\delta(x) \cap B_{p \wedge \delta}(y) \subseteq B_\delta(x) \cap B_p(y)$ . If  $d : X \times X \rightarrow V$  is symmetric then  $d(x, y) \leq d(x, z) + d(z, y) \leq \delta + \delta \prec \epsilon$  and  $y \in B_\epsilon(x)$ . In other words,  $\overline{B_\delta(x)} \subseteq B_\epsilon(x)$ . The following proof is partly modeled on a result by Kopperman ([Kop88] pg. 97).

**Lemma 4.3.5.** *Any symmetric continuity space yields a completely regular topology.*

*Proof.* Take any symmetric continuity space  $(V, X, d)$ , with  $V = (V, \leq_V, +_V)$  and let  $x \in O \in \tau_V$  (i.e. the topology generated by  $(V, X, d)$ ) and notice that there exists an  $\epsilon \succ 0_V$  so that  $\overline{B_\epsilon(x)} \subseteq O$  (that is, there is a closed set of radius  $\epsilon$  about  $x$  entirely contained within  $O$ ). Indeed, since  $x \in O$  then there exists a  $p \succ 0_V$  for which  $B_p(x) \subseteq O$ . Since  $p \succ 0$  we can find  $\epsilon \succ 0$  for which  $2\epsilon \prec p$ . In turn,  $d(x, y) \leq \epsilon \Rightarrow d(x, y) \prec p$  and  $y \in B_p(x)$ . For this  $\epsilon$  we can, as shown above, generate at least one order-preserved copy of  $DY_1$  within  $\epsilon_\downarrow$ . Let us fix any one such copy and continue with the proof.

Let  $M_\epsilon : V \rightarrow [0, 1]$  by

$$M_\epsilon(a) = \begin{cases} 1 & \text{if } \{n \in DY_1 \mid a \leq n\epsilon\} = \emptyset. \\ \inf\{n \in DY_1 \mid a \leq n\epsilon\} & \text{otherwise.} \end{cases}$$

What we seek, and shortly prove, is for  $M_\epsilon$  to preserve  $+_V$  and  $\leq_V$  in  $DY_1$ . Thus, if  $c \leq_V a +_V b$  with  $a \leq_V n\epsilon$  and  $b \leq_V m\epsilon$  (for  $n, m \in DY_1$ ) then  $c \leq_V a +_V b \leq_V n\epsilon +_V m\epsilon \leq_V (n+m)\epsilon$ . Hence,  $M_\epsilon(c) \leq M_\epsilon(a) + M_\epsilon(b)$ . Next define  $g : X \rightarrow [0, 1]$  by  $g(y) = \min\{M_\epsilon(d(x, y)), 1\}$  and we show  $g$  is continuous. Recall that continuous functions between continuity spaces are defined in terms of  $\epsilon$ - $\delta$  arguments. In this case we must show that for  $x \in X$  and any  $p > 0$  we can find  $\delta \succ 0_V$  so that  $d(x, y) \prec \delta \Rightarrow |g(x) - g(y)| < p$ .

Since  $d(x, z) \leq d(x, y) + d(y, z)$  we have  $g(z) \leq g(y) + M_\epsilon(d(y, z))$  and consequently  $g(z) - g(y) \leq M_\epsilon(d(y, z))$ . By symmetry of  $d$  we have  $|g(y) - g(z)| \leq M_\epsilon(d(y, z))$ . Next, choose any  $p > 0$  and take any  $n \in DY_1$  so that  $n < p$ . Notice that if  $d(y, z) \prec n\epsilon$  then  $|g(y) - g(z)| \leq M_\epsilon(d(y, z)) \leq n < p$  and thus  $g$  is continuous. Lastly, define  $f : X \rightarrow [0, 1]$  as  $f(x) = \max\{0, 1 - g(x)\}$ . Hence,  $x \mapsto 1$  and any  $y \notin \overline{B_\epsilon(x)}$  gets mapped to 0.  $\square$

Next we show the converse is also true. Recall that a completely regular space coincides with initial topology from its collection of all continuous real-valued functions. Equivalently, the smallest one making all of its point/closed set separating functions continuous. The idea behind the following theorem is to construct a symmetric continuity space based on such a collection. The construction relies on making all such real-valued functions continuous and generating the original completely regular topology.

**Lemma 4.3.6.** *Any completely regular topology can be generated by a symmetric continuity space.*

*Proof.* Let  $(X, \tau)$  be a completely regular topology and collect all continuous  $[0, 1]$ -valued functions on  $X$  that separate points from closed sets in a set denoted by  $F$ . Let  $V = [0, 1]^F$  and  $K = \{f \in (0, 1]^F \mid f(x) = 1 \text{ for all but finitely many } x \in F\}$ . As it stands, neither  $V$  nor  $K$  are value quantales (the well-above 0 elements do not form a filter). What we do (combining Kopperman's and Flagg's paper) is to embed

$V$  into a suitable value quantale where the well-above 0 elements will be exactly the images of  $K$ . For a pair  $f, g \in V$ ,  $f \ll g$  ( $g$  is *way-above*  $f$ ) if  $g(x) > f(x)$  for all  $x$  for which  $f(x) < 1$ .

A *round filter* (cf [Fla97] pg. 275)  $p \subseteq K$  is one for which:

- $\top \in p$ ,
- for all  $f, g \in K$  if  $f \in p$  and  $f \ll g$  then  $g \in p$ , and
- for any  $f \in p, \exists g \in p$  so that  $f \gg g$ .

Following Flagg's notation, we let  $\Gamma(K)$  denote the collection of all round filters on  $K$ ; Flagg shows that  $\Gamma(K)$  is indeed a value quantale when ordered by reverse inclusion and addition is taken to be intersection. Next, we embed  $V$  within  $\Gamma(K)$  and generate the desired topology using  $\Gamma(K)$ . The embedding (call it  $\Psi$ ) is the obvious one: for any  $f \in V$  let  $\hat{f} = \{g \in K \mid g \gg f\}$ . The embedding  $\Psi : V \rightarrow \Gamma(K)$  will work by  $f \mapsto \hat{f}$  (notice that  $\bigwedge \hat{f} = f$  so that  $\Psi$  is an embedding). It is not obvious but true that within  $\Gamma(K)$  a round filter  $p$  is well-above  $0_{\Gamma(K)}$  iff for some  $f \in K, p \subseteq \hat{f}$  (cf [Fla97]). In particular, for that  $f$  we have  $\bigwedge p \geq f$ . In order to simplify the proof we define two functions: one will go from  $X \times X$  into  $V$  and it will generate the other (the actual metric on  $X$ ) from  $X \times X$  into  $\Gamma(K)$ . The first function  $m : X \times X \rightarrow V$  is done coordinate-wise. That is, for any pair  $x, y \in X$  we have  $m(x, y)(f) = |f(x) - f(y)|$  (since  $m$  literally evaluates functions from  $V$  into  $[0, 1]$ ). The second one works as follows:  $d : X \times X \rightarrow \Gamma(K)$  so that  $(x, y) \mapsto \Psi(m(x, y)) = \widehat{m(x, y)}$ .

Next we show that  $(\Gamma(K), X, d)$  generates  $(X, \tau)$ . First we show that for any  $p \in \Gamma(K)_{\prec}, B_p(x) \in \tau$  for any  $x \in X$ . Notice that  $y \in B_p(x)$  iff  $d(x, y) \prec p$  iff  $\widehat{\bigwedge p} \subseteq d(x, y)$  iff  $\bigwedge p \gg m(x, y)$ . By design,  $\bigwedge p \gg m(x, y)$  iff for all  $h_i \in F$  ( $1 \leq i \leq n$ ) so that  $m(x, y)(h_i) < 1, \bigwedge p(h_i) = p_i > m(x, y)(h_i) = |h_i(x) - h_i(y)|$ . In turn we get  $\bigwedge p \gg m(x, y)$  iff  $h_i(x) - p_i < h_i(y) < h_i(x) + p_i$  iff  $y \in \bigcap_1^n h_i^{-1}[(h_i(x) - p_i, h_i(x) + p_i)]$ . Thus,  $B_{\hat{f}}(x)$  is indeed open in  $\tau$ .

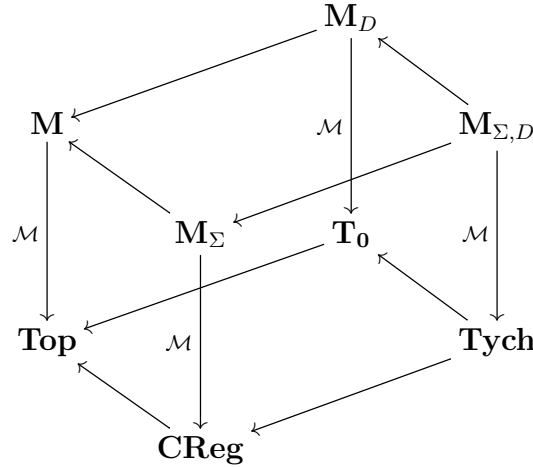
Lastly, take  $x \in O \in \tau$  and any  $f \in F$  so that  $f(x) = 1$  and  $f(y) = 0$  for all  $y \notin O$ . Let  $h \in K$  so that  $1 > h(f) > 0$  and  $h(g) = 1$  for all  $g \neq f$ . If  $y \in B_{\hat{h}}(x)$  then  $d(x, y) \prec \hat{h} \Rightarrow \hat{h} \subseteq d(x, y)$ . Consequently, for  $m(x, y)(g) < 1$  we get

$h(g) > m(x, y)(g) \Rightarrow h(f) > m(x, y)(f) = |f(x) - f(y)|$ . By design,  $h(f) < 1$  and thus  $f(y) > 0$ . Hence,  $y \in O$  and  $B_{\hat{h}}(x) \subseteq O$ .

□

In the following theorem: **T<sub>0</sub>**, **CReg** and **Tych** denote the full subcategories of **Top** composed of all **T<sub>0</sub>**, completely regular and Tychonoff spaces, respectively.

**Theorem 4.3.7.**  $\mathcal{M}[\mathbf{M}_\Sigma] = \mathbf{CReg}$ ,  $\mathcal{M}[\mathbf{M}_{\Sigma,S}] = \mathbf{Tych}$  and



commutes. Moreover, **M<sub>Σ</sub>**, **M<sub>Σ,D</sub>** and **M<sub>D</sub>** are reflective subcategories of **M**.

*Proof.* The first part of the theorem is clear. From Lemma 4.3.2 and the adjunction **Top**  $\Leftrightarrow$  **T<sub>0</sub>** we get that **M<sub>D</sub>** has a left adjoint. Next, we show **M<sub>Σ</sub>**  $\rightarrow$  **M** has a left adjoint. To do so we make use of the equivalence between **M<sub>Σ</sub>** and **CReg** and the adjunction **CReg**  $\Leftrightarrow$  **Top**. Take any non-symmetric  $(V, X, d) \in \mathbf{M}$ . If  $\mathcal{M}[(V, X, d)] = \tau$  is completely regular we can generate a symmetric continuity space that also generates  $\tau$ , in which case we map  $(V, X, d)$  to the latter symmetric space. If  $\mathcal{M}[(V, X, d)] = \tau$  is not completely regular we can associate to  $\tau$  a universal completely regular  $\tau'$ ; the initial topology on  $X$  generated by  $C_{\tau'}(X)$ . Since  $\mathcal{M}$  is fully faithful the adjunction **CReg**  $\Leftrightarrow$  **Top** lifts up to **M<sub>Σ</sub>**  $\rightarrow$  **M**. Lastly, we need only notice that the adjunctions **M<sub>Σ</sub>**  $\Leftrightarrow$  **M** and **M<sub>D</sub>**  $\Leftrightarrow$  **M** yield a left adjoint for the inclusion **M<sub>Σ,D</sub>**  $\rightarrow$  **M**. □

There exist at least three ways to *symmetrise* a continuity space (and thus ‘com-

pletely regularise' a topology): given a topology  $\tau$ , which is not completely regular, we can generate its corresponding continuity space, say  $(V, X, d)$ . Since  $d$  is not symmetric (for then  $\tau$  would be completely regular) we can construct its dual  $d^* : X \times X \rightarrow V$  so that  $d(x, y) = d^*(y, x)$  for all  $x, y \in X$ . Let  $\tau^*$  be the topology generated by  $(V, X, d^*)$ . We then have three candidates for a *symmetrisation* of  $(V, X, d)$  (where the first two will yield the same topology):

(a)  $(V, X, m_\vee)$  so that  $m_\vee(x, y) = d(x, y) \vee d^*(x, y)$ ,

(b)  $(V, X, m_+)$  so that  $m_+(x, y) = d(x, y) + d^*(x, y)$  and

(c)  $(V, X, m_\wedge)$  so that

$$m_\wedge(x, y) = \bigwedge_{\gamma} \left( \sum_i^n \min\{d(a_i, a_{i+1}), d^*(a_i, a_{i+1})\} \right)$$

where the sums are indexed over all finite paths  $\gamma = (x = a_1, \dots, y = a_n)$ .

Flagg proves that (b) generates  $\tau \vee \tau^*$ . The same is true for (a): clearly,  $\mathcal{M}[(V, X, m_\vee)] \geq \tau \vee \tau^*$  and since  $\forall p \succ 0, B_p^m(x) = B_p^d(x) \cap B_p^{d^*}(x)$  then  $\mathcal{M}[(V, X, m_\vee)] \leq \tau \vee \tau^*$ . Notice that in (c),  $\mathcal{M}[(V, X, m_\wedge)] \leq \tau \vee \tau^*$  (and equality is not at all guaranteed).

*Remark 4.3.8.* Neither of the above symmetrisations is equivalent to completely regularising a topology. Since (a) and (b) generate topologies finer than their original ones the claim is then clear for such cases. In contrast, (c) is not as straight forward: let  $X = \{x\} \cup \{x_n\}_{n \in \mathbb{N}}$  and  $d : X \times X \rightarrow \mathbb{R}$  so that for all  $y, z \in X$ :  $d(y, z) = \frac{1}{n}$  provided  $y = x_n$  and  $z = x$ , and  $d(y, z) = 1$  otherwise. The functor  $\mathcal{M}$  maps  $(X, d)$  to the discrete topology on  $X$  (a completely regular topology) but the symmetrisation of  $(X, d)$  via (c) is mapped by  $\mathcal{M}$  to a strictly coarser topology than the discrete topology on  $X$ .

## 4.4 Limits in $\mathbf{P}$

Here we show that  $\mathcal{P}$  is continuous (i.e. it preserves all small limits) and in turn, since  $\mathcal{P}$  is surjective on objects, that it also lifts limits. This is certainly surprising given that  $\mathcal{P}$  is not full. We begin by presenting explicit constructions for equalizers and products from  $\mathbf{Top}$  within  $\mathbf{T}$  purely in terms of continuity spaces. These constructions will turn out to be exactly equalizers and products of their corresponding continuity spaces. For instance, consider a collection  $\{(X_j, \tau_j) \mid j \in J\}$  in  $\mathbf{Top}$  with a corresponding collection  $\{(V_j, X_j, d_j) \mid j \in J\}$  in  $\mathbf{P}$  (for which  $\mathcal{P}[(V_j, X_j, d_j)] = (X_j, \tau_j)$  for all  $j \in J$ ). In this section we construct  $(R, Z, d_R)$  in  $\mathbf{P}$  (with the obvious associated projections) so that  $\mathcal{P}[(R, Z, d_R)] = \prod(X_j, \tau_j)$ . Moreover, we show that  $\prod(V_j, X_j, d_j) = (R, Z, d_R)$  also. In a sense, when concerned with topological limits,  $\mathbf{P}$  morphisms carry all of the necessary topological information between topological spaces.

In any category, all limits can be constructed in terms of products and equalizers. In our case, we are guaranteed explicit constructions for all other (topological) limits in  $\mathbf{T}$ . That said, we will present a detailed construction of inverse limits for they are inextricably linked with topological joins and initial topologies. To facilitate notation, we will suppress the subscript dummy indexing in the product notation. For instance,  $\prod_{j \in J} V_j$  will become  $\prod V_j$  (where the indexing set will be understood from context).

### 4.4.1 Completely distributive lattices and the well-above relation

Let  $(L, \leq_L)$  be a completely distributive lattice. Raney ([Ran53]) tells us that this is equivalent to the condition that  $p = \bigwedge \{y \in L \mid p \prec y\}$ , for any  $p \in L$ . Let  $L_{\prec} = \{y \in L \mid y \succ 0\}$  and order  $L_{\prec}$  with the order inherited from  $L$ . Clearly,  $L_{\prec}$  is upwards closed since the well-above relation is upwards-transitive:  $x \succ 0$  and  $y \geq x \Rightarrow y \succ 0$ . It is this latter set that will give rise to the constructions of limits and colimits in  $\mathbf{P}$ .

## 4.4.2 Products

Take an arbitrary collection of continuity spaces  $\{(V_j, X_j), d_j\}$ , where  $j \in J$ , let  $Z = \prod X_j$  and  $U = \prod_f (V_j)_{\prec}$ , so that if  $a \in U$  then for only finitely many  $i \in J$ ,  $a_i \neq 1_i$ . We seek to order-embed  $\prod V_j$  into a suitable value quantale  $R$  so as to capture all possible combination of distances between points in  $Z = \prod X_j$ . As it stands, with the obvious ordering,  $(U, \leq_U)$  is not value distributive (the well-above elements do not form a filter). In order to fix that, consider the following, where  $\Omega(U) = \{A \subset \mathcal{P}_f(U) \mid A \text{ is a lower set}\}$  and  $\mathcal{P}_f(U)$  is the collection of all finite subsets of  $U$ .

**Lemma 4.4.1.** *Let  $R = \Omega(U)$*

- (a) *Ordering  $R$  by reverse inclusion yields  $(R, \leq_R)$  a completely distributive lattice. Moreover,  $p \succ 0_R$  if and only if  $p$  is finite.*
- (b)  *$(R, \leq_R)$  is a value distributive lattice.*
- (c) *The function  $+_R : R \times R \rightarrow R$  for which*

$$x +_R y = x \cap y$$

*is associative and commutative. In particular,  $(R, \leq_R, +_R)$  is a value quantale.*

*Proof.* The first part of (a) is clear from Section 4.2. Next, if  $p$  is finite and for some  $S = \{S_k\}_{k \in I} \subseteq R$  we have that  $\bigwedge S_k \leq 0_R$  (i.e.  $\bigcup S_k = \mathcal{P}_f(U)$ ) then for at least one  $k \in I$ ,  $\cup p \in S_k$ . Indeed, each  $S_k$  is a downward-closed collection of finite subsets of  $U$  whose union must yield  $\mathcal{P}_f(U)$  and, hence, at least one must contain  $\cup p$ . Since we are dealing with lower sets, then  $\cup p \in S_k \Rightarrow p \subseteq S_k$  and thus  $p \geq S_k$ . Conversely, notice that

$$0_R = \bigcup_{z \in \mathcal{P}_f(U)} \mathcal{P}_f(z) = \bigwedge_{z \in \mathcal{P}_f(U)} \mathcal{P}_f(z)$$

and that no infinite set is contained in any of the previous sets.

To show value distributivity, notice that  $1_R \succ 0_R$ . Next, if  $p, q \succ 0_R$  then clearly  $p \wedge q = p \cup q$  is finite and, by part (a), well-above  $0_R$ .

For (c) it's clear that  $\emptyset = 0_R = 0_R \cap a = 0_R +_R a$  for all  $a \in R$ . The rest follows from distributivity of unions over intersections.  $\square$

The injection  $\phi : \prod V_j \rightarrow R$  is defined as follows: for a given  $x \in \prod V_j$  let

$$\phi(x) = x_{\uparrow} = \{A \in \mathcal{P}_f(U) \mid A \subseteq x^{\uparrow}\}$$

and  $x^{\uparrow} = \{a \in U \mid \forall j \in J, a_j \succ x_j\}$ . Notice that since all  $V_j$  are completely distributive lattices then for any  $x \in \prod V_j$ ,  $x^{\uparrow}$  uniquely determines  $x$ . Consequently, we have  $\wedge(\cup x_{\uparrow}) = \wedge(x^{\uparrow}) = x$  in  $\prod V_j$ .

**Theorem 4.4.1.** *The function  $\phi : \prod V_j \rightarrow R$  so that  $x \mapsto x_{\uparrow}$  is an order-embedding.*

*Proof.* Take  $x = (x_j)$  and  $y = (y_j)$  in  $\prod V_j$  so that  $x \neq y$ . Notice that  $\wedge(\cup x_{\uparrow}) = \wedge\{a \in \cup x_{\uparrow} \mid a_i \geq x_i\} = x \neq y = \wedge\{a \in \cup y_{\uparrow} \mid a_i \geq y_i\} = \wedge(\cup y_{\uparrow})$  and, hence, that  $\phi$  is injective. Also, if  $x > y$ , then clearly  $x_{\uparrow} \subset y_{\uparrow}$  and  $x_{\uparrow} > y_{\uparrow}$ .  $\square$

To this end, we have only to define  $d_Z : Z \times Z \rightarrow R$  in a suitable manner. For  $x = (x_j), y = (y_j) \in Z$  let  $d(x, y) \in \prod V_j$  so that  $\pi_j(d(x, y)) = d_j(x_j, y_j)$  and

$$d_Z(x, y) = \phi(d(x, y)).$$

*Remark 4.4.2.* Notice that  $d_Z$  need not satisfy the triangle inequality for continuity spaces; this particular construction does not belong to **M**.

In the following theorem  $\tau_{\pi}$  denotes the product topology on  $Z = \prod X_j$  from a collection of topological spaces  $\{(X_j, \tau_j) \mid j \in J\}$ . Also, for each  $i \in J$ ,  $(X_i, \tau_i) = \mathcal{P}[(V_i, X_i, d_i)]$  and  $(Z, \tau_R) = \mathcal{P}[(R, Z, d_Z)]$ .

**Theorem 4.4.2.**  $\tau_R = \tau_{\pi}$  .

*Proof.* ( $\supseteq$ ) For a fixed  $i \in J$  let  $O \in \tau_i$ . We will show that  $O_{\pi} = \pi^{-1}(O) \in \tau_R$ . Take  $x \in O_{\pi}$  and notice that since  $\pi_i(x) = x_i \in O$  then we can find  $\epsilon \succ 0_i$  for which  $B_{\epsilon}(x_i) \subseteq O$ . Let  $\hat{\epsilon} \in U$  for which  $\hat{\epsilon}_j = \epsilon$  if  $j = i$  and  $1_j$  otherwise. Consider  $\bar{\epsilon} = \{\{\hat{\epsilon}\}, \emptyset\}$  and  $B_{\bar{\epsilon}}(x)$  and notice that  $y \in B_{\bar{\epsilon}}(x)$  iff  $d_Z(x, y) \supseteq \bar{\epsilon}$ . Since  $d_Z(x, y) =$

$d(x, y)_\uparrow = \{A \in \mathcal{P}_f(U) \mid A \subseteq d(x, y)^\uparrow\}$  and  $d(x, y)^\uparrow = \{a \in U \mid \forall j \in J, a_j \succ d(x_j, y_j)_j\}$  then  $d_i(x_i, y_i) \prec \epsilon$  and  $d_j(x_j, y_j)$  can be any choice within  $V_j$  for  $j \neq i$ . Hence,

$$y \in \pi_i^{-1}(y_i) \subseteq \pi_i^{-1}(B_\epsilon(x_i)) \subseteq \pi_i^{-1}(O) = O_\pi$$

and thus  $B_\epsilon(x) \subseteq O_\pi$ .

( $\subseteq$ ) Take any basic open  $O \in \tau_R$  and  $x \in O$ . Since  $O$  is open we can find  $p \succ 0_R$  so that  $B_p(x) \subseteq O$ . It is immediate that since  $\cup p$  is finite then for only a finite  $F \subseteq J$   $\pi_i(O) \neq X_i$  provided  $i \in F$ . In light of the previous remark we show that  $\forall i \in F$ ,  $O_i = \pi_i(O) \in \tau_i$  and consequently, obtain that  $O = \bigcap_{i \in F} \pi_i^{-1}(O_i) \in \tau_\pi$ .

Fix an  $i \in F$  and take  $x_i \in O_i$  for which  $\pi_i(x) = x_i$ . Since  $O$  is open then there exists  $p \succ 0_R$  so that  $B_p(x) \subseteq O$ . Let us inspect the  $i^{\text{th}}$  projection of  $\cup p$ : let  $p_i = \{a \in (V_i)_\prec \mid a = \pi_i(b) \text{ for some } b \in \cup p\}$  and notice that such a set is finite. In particular,  $\hat{p}_i = \wedge p_i \succ 0_i$  since  $V_i$  is value distributive. If  $y \in B_{\hat{p}_i}(x_i)$  then  $d_i(x_i, y) \prec \epsilon$ ,  $\forall \epsilon \in p_i$  and  $y \in \pi_i(B_p(x)) \subseteq O_i$ . This completes the proof for then  $B_{\hat{p}_i}(x) \subseteq O_i$  and  $\tau_R \subseteq \tau_\pi$ .

□

As mentioned before, **P** and **Top** are surprisingly similar. This can be seen in the following result, where we show that topological products and continuity space products coincide in **P**.

**Theorem 4.4.3.** *For a set-indexed collection of continuity spaces  $\{(V_j, X_j, d_j) \mid j \in J\}$  we have  $\prod(V_j, X_j, d_j) = (R, Z, d_Z)$ .*

*Proof.* Take a continuity space  $(W, Y, m)$  with a collection

$$\{f_j \mid f_j \in \text{hom}_{\mathbf{P}}((W, Y, m), (V_j, X_j, d_j)) \text{ and } j \in J\}.$$

First we show that the projections  $\pi_j : Z \rightarrow X_j$  are  $\epsilon$ - $\delta$  continuous. Fix an  $i \in J$ , let  $x \in Z$  and  $\epsilon \succ 0_i$  for some  $i \in J$ . Let  $\bar{\epsilon} = \{\{\hat{\epsilon}\}, \emptyset\}$  where  $\hat{\epsilon} \in \prod V_j$  so that  $\hat{\epsilon}_j = \epsilon$  if  $j = i$  and  $1_j$  otherwise. In this case,

$$\begin{aligned}
d_R(x, y) \prec \bar{\epsilon} &\Rightarrow d_R(x, y) \supseteq \bar{\epsilon} \\
&\Rightarrow \forall j \in J, d_j(x_j, y_j) \prec \hat{\epsilon}_j \\
&\Rightarrow d_i(x_i, y_i) \prec \epsilon.
\end{aligned}$$

Hence,  $\pi_j \in \text{hom}_{\mathbf{P}}((R, Z, d_Z), (V_j, X_j, d_j))$  for all  $j \in J$ . Next, let  $h : Y \rightarrow Z$  so that  $\pi_j(h(x)) = f_j(x)$  for all  $j \in J$ . Let  $x \in Y$  and choose any  $p \succ 0_R$ . By design, for only a finite subset  $F$  of  $J$  we have  $\pi_i(\cup p) \neq \{1_i\}$  with  $i \in F$ . For each  $i \in F$  let  $p_i = \wedge \pi_i(\cup p)$  and notice that  $p_i \succ 0_i$ . Therefore, for only  $i \in F$  is  $p_i \neq 1_i$ . Since each  $f_i$  is  $\epsilon$ - $\delta$  continuous then for each  $p_i$  we can find  $\delta_i \succ 0_W$  so that  $m(x, y) \prec \delta_i \Rightarrow d_i(f_i(x), f_i(y)) \prec p_i$  (and for only the  $i \in F$  is  $\delta_i < 1_i$ ). To this end we let  $\delta = \wedge \delta_i$  (notice that  $\delta \succ 0_W$ ) and observe that

$$\begin{aligned}
m(x, y) \prec \delta &\Rightarrow \forall i \in F, m(x, y) \prec \delta_i \\
&\Rightarrow \forall i \in F, d_i(f_i(x), f_i(y)) \prec p_i \\
&\Rightarrow \forall i \in F, d_i(f_i(x), f_i(y)) \prec r, \forall r \in \pi_i(\cup p) \\
&\Rightarrow d_R(h(x), h(y)) \supseteq p \\
&\Rightarrow d_R(h(x), h(y)) \prec p
\end{aligned}$$

□

**Corollary 4.4.4.** *The functor  $\mathcal{P}$  is preserves and lifts all small products.*

### 4.4.3 Equalizers

Equalizers are the simplest of all constructions. Take a pair of continuity spaces  $(V, X, d_X)$  and  $(W, Y, d_Y)$  with continuous functions  $f, g : (V, X, d_X) \rightarrow (W, Y, d_Y)$ . The topological equalizer of  $V, W, g, f$  is simply the subspace topology on  $Z = \{x \in X \mid f(x) = g(x)\}$ . This is only the restriction of  $d_X : X \times X \rightarrow V$  to  $Z$  as one can easily verify.

**Theorem 4.4.5.** *For a pair of continuity spaces  $(V, X, d_X)$  and  $(W, Y, d_Y)$  with morphisms  $f, g : (V, X, d_X) \rightarrow (W, Y, d_Y)$  then the equalizer  $(U, Z, d_Z)$  is as follows:*

- $Z = \{x \in X \mid f(x) = g(x)\},$
- $U = V,$  and
- $d_Z$  is the restriction of  $d_X$  onto  $Z \times Z.$

*Remark 4.4.6.* Observe that the above construction (unlike topological products in **P**) also works for equalizers in **M**. That is, Theorem 4.4.5 provided equalizers for **P**, **M** and topological equalizers in both categories.

**Corollary 4.4.7.** *The functor  $\mathcal{P}$  is continuous and lifts all small limits.*

#### 4.4.4 Initial Topology, Inverse Limits and Topological Joins

Finding a direct (resp. inverse) limit in the category **Top** is equivalent to placing the final (resp. initial) topology on the set-theoretic limit of a direct (resp. inverse) system of objects and morphisms. Since our objective is to do so in **P** we must develop a suitable construction for initial and final topologies on a given set in terms of the continuity spaces and morphisms involved.

We begin with a collection of continuity spaces  $(W_j, Y_j, d_j)$  and continuous functions  $f_{ji} : Y_i \rightarrow Y_j$  whenever  $j \leq i \in J$  for some directed set  $J$ . Further, let  $((X, \tau), f_j : X \rightarrow Y_j)$  ( $j \in J$ ) be the above system's topological inverse limit. We seek a value quantale  $V$  (in terms of the  $W_j$ ) and metric assignment  $d : X \times X \rightarrow V$  (also in terms of the  $d_j$ ) so that  $\mathcal{P}[(V, X, d)] = (X, \tau)$ . Topologically, let  $(Y_j, \tau_j) = \mathcal{P}[(W_j, Y_j, d_j)]$  and

$$\varprojlim Y_j = \{a \in \prod Y_j \mid a_i = f_{ij}(a_j) \text{ for all pairs } i \leq j\}$$

then  $\varprojlim (Y_j, \tau_j) = (\varprojlim Y_j, \tau_{\leftarrow})$  where  $\tau_{\leftarrow}$  is the subspace topology inherited from the product topology on  $\prod Y_j$ . The above construction for products yields a value

distributive lattice  $V_\Pi$  and distance assignment  $d_\Pi : \prod Y_j \times \prod Y_j \rightarrow V_\Pi$  purely in terms of the pairs  $(W_j, d_j)$  for which  $\mathcal{P}[(V_\Pi, \prod Y_j, d_\Pi)] = \prod (X_j, \tau_j)$ . The same holds for equalizers. Regarding  $(V, X, d)$ , it is then clear that  $X = \varprojlim Y_j$ ,  $V = V_\Pi$ ,  $d$  is the restriction of  $d_\Pi$  to  $X$  and each  $f_j = \pi_j \upharpoonright X$ .

Let us consider a fixed set  $X$  and a collection  $\{\tau_j\}_{j \in I} \subset \text{Top}(X)$  (where  $I$  has the discrete order) with a corresponding collection  $(W_j, X, d_j)$  so that  $\mathcal{P}[(W_j, X, d_j)] = (X, \tau_j)$  for all  $j \in I$ . The task is then to construct a value distributive lattice  $V$  and distance assignment  $d : X \times X \rightarrow V$  so that  $\mathcal{P}[(V, X, d)] = (X, \bigvee \tau_j)$ . Notice that we can complete  $\{(X, \tau_j)\}_{j \in I}$  into a directed system by adding  $Y = \prod X_j$  (where  $X_j = X$  for all  $j$ ) and morphisms being the canonical projections  $\pi_i : Y \rightarrow X_i$ . To complete  $I$  into a directed set we let  $J = I \cup \{\top\}$  so that  $\top > j$  for all  $j \in I$  and  $Y$  is identified with  $X_\top$ . In turn, the inverse limit  $((V, X, d), (g_j) \mid j \in J)$  of the system  $((X_j, \tau_j), (f_{ij}) \mid i \leq j \in J)$  generates the topological join of the  $\tau_j$ . Indeed, letting  $g_\top : X \rightarrow \Delta_Y$  be the obvious diagonal mapping (for all other  $j \in J$ ,  $g_j = \text{id} : X \rightarrow X$ ) then the diagram

$$\begin{array}{ccc} \prod X_j & \xrightarrow{\pi_i} & X_i \\ g_\top \uparrow & \nearrow \text{id} & \\ X & & \end{array}$$

commutes for all  $i \in J$ . This means that  $(V, X, d)$  generates the initial topology on  $X$  for the collection  $\{((X, \tau_j), \text{id}) \mid j \in J\}$  and that  $\mathcal{P}[(V, X, d)] = (X, \bigvee \tau_j)$ . That is, for all  $i \in J$ , let  $\mathcal{P}[(W_i, X, d_i)] = (X, \tau_j)$  and  $(V, \prod X_j, m_i)$  be a continuity space that generates the initial topology on  $\prod X_j$  making  $\pi_i$  continuous. Next let

- $V = \Omega[\prod_f (W_j)_{\prec}]$  and
- $d(x, y) = \bigvee m_j(x, y)$ .

Lastly, we must only identify  $X$  with the diagonal in  $\prod X_j$ .

## 4.5 Colimits in $\mathbf{P}$

We turn our attention to topological coproducts and direct limits as continuity spaces. As with Section 4.4, topological coproducts and direct limits will coincide with coproducts of their corresponding continuity spaces from  $\mathbf{P}$ . With regard to the remaining colimits in  $\mathbf{P}$ , a suitable construction for topological coequalizers remains unknown to the authors. In turn, although highly probable, it remains unknown whether or not  $\mathcal{P}$  is cocontinuous and/or lifts small colimits.

### 4.5.1 Coproducts

For an arbitrary set-indexed family  $\mathcal{A} = \{(V_j, X_j, d_j)\}$  ( $j \in J$ ) of continuity spaces, we seek a construction of an  $R$ -space  $(R, Z, d_Z)$  that generates the topological sum of the topologies generated by the collection  $\mathcal{A}$ . In particular, we want  $R$  and  $d_Z$  to be defined in terms of the  $V_j$ s and the  $d_j$ s respectively. Let  $U = \bigsqcup_{j \in J} (V_j)_{\prec}$  and  $Z = \bigsqcup_{j \in J} X_j$  (where the symbol  $\sqcup$  denotes disjoint union). It is not difficult to verify that  $U$  is not a value distributive lattice (in fact, it's not even meet complete). That said, all information about the  $V_j$ s is contained in  $U$  and  $R$  will be constructed from  $U$ .

**Lemma 4.5.1.** *For  $R = \Omega(U)$*

1. *Ordering  $R$  by reverse inclusion yields  $(R, \leq_R)$  a completely distributive lattice.*
2. *If  $p \in R$  then  $p \succ 0_R$  if and only if  $p$  is finite.*
3.  *$(R, \leq_R)$  is a value distributive lattice.*

*Proof.* There is nothing new to prove here. This theorem is the same as with products with a different choice of  $U$ . □

For completeness we define  $+_R : R \times R \rightarrow R$  so that for all  $a, b \in R$ :

$$a +_R b = a \cap b.$$

The following is easy to prove.

**Lemma 4.5.2.** *For  $R$  as defined above,  $(R, \leq_R, +_R)$  is a value quantale.*

The last step is to construct  $d_Z$ . The next theorem is crucial for such a construction and the underlying reason works as follows: consider a pair of topological spaces  $(X, \tau)$ ,  $(Y, \sigma)$  (generated by a pair  $(V, X, d_X)$ ,  $(W, Y, d_Y)$ ) and their topological sum  $(Z, \rho)$ . In other words,  $\rho$  is the finest topology on  $Z$  for which the injections

$$i_X : (X, \tau) \rightarrow (Z, \rho) \text{ and } i_Y : (Y, \sigma) \rightarrow (Z, \rho)$$

are continuous (i.e.  $O \in \rho$  iff  $i_X^{-1}(O) \in \tau$  or  $i_Y^{-1}(O) \in \sigma$ ). In terms of continuity spaces, we seek a continuity space  $(Q, Z, d_Z)$  with suitable order-embeddings  $\phi_V : V \rightarrow Q$  and  $\phi_W : W \rightarrow Q$  so that open sets about points in  $X$  (resp.  $Y$ ) translate to open sets about points within  $i_X(X)$  (resp.  $i_Y(Y)$ ). Simply put, we require for points that are near each other in  $X$  and  $Y$  to remain close in  $Z$  and distances between points from the disjoint images of  $i_X$  and  $i_Y$  to be as large as possible.

Back to  $\mathcal{A}$  and in light of the above, for any  $i \in I$ , consider the function  $\phi_i : V_i \rightarrow R$  so that for  $a \in V_i$ ,

$$a \mapsto a_{\uparrow} = \mathcal{P}_f(a^{\uparrow} \cup \bigsqcup_{i \neq j} (V_j)_{\prec})$$

where  $a^{\uparrow} = \{\epsilon \succ 0_i \mid \epsilon \succ a\}$  and  $0_i$  is the bottom element of  $V_i$ . Lastly, define  $d_Z : Z \times Z \rightarrow R$  for all  $x, y \in Z$  as follows:

$$d_Z(x, y) = \begin{cases} \phi_j(d_j(i_j^{-1}(x), i_j^{-1}(y))) & x, y \in i_j(X_j), \\ 1_R & \text{otherwise.} \end{cases}$$

where, for all  $j \in J$ ,  $i_j : X_j \rightarrow Z$  denotes the obvious injection. In order to lighten some of the notational burden, given any point  $x \in X_j$  (resp.  $A \subseteq X_j$ ) we let  $x^* = i_j(x)$  and  $A^* = i_j(A)$ , for all  $j \in J$ .

**Lemma 4.5.3.** *Given any  $j \in J$*

1. *The function  $\phi_j$  is an order-embedding.*
2. *For all  $\epsilon \succ 0_j$  and any pair  $x, y \in X_j$  then*

$$d_j(x, y) \prec \epsilon \Leftrightarrow d_Z(x^*, y^*) \prec \bar{\epsilon} = \{\{\epsilon\}, \emptyset\}.$$

3. If  $1_R > \epsilon \succ 0_R$  and  $x^*, y^* \in Z$  then  $d_Z(x^*, y^*) \prec \epsilon$  if and only if for some  $j \in J$

(a)  $x, y \in X_j$  and

(b)  $d_j(x, y) \prec \delta, \forall \delta \in \cup \epsilon \cap V_j$ .

*Proof.* Since  $V_j$  is completely distributive then  $a \neq b \in V_j$  implies that  $\wedge a^\uparrow \neq \wedge b^\uparrow$ . Thus  $a_\uparrow \neq b_\uparrow$  and  $\phi_j$  is injective. Next notice that  $a > b$  iff  $\cup a^\uparrow \subset \cup b^\uparrow$  iff  $a_\uparrow > b_\uparrow$  and (1) is proved. Notice that (1) in conjunction with the definition of  $\phi_j$  tells us that for  $a \in V_j$  and  $p \in R_\prec$  then  $a_\uparrow \prec p$  iff  $\cup p \subseteq \cup(a_\uparrow)$  iff  $\cup p \cap V_j \subseteq \cup(a_\uparrow) \cap V_j$ . This observation is not as banal as it may seem and it's crucial for what comes next.

Part (2) is immediate from the definition of  $d_Z$ . For (3) necessity comes from (2) and the definition of  $d_Z$ . Next, let  $1_R > \epsilon \succ 0_R$  so that  $d_Z(x^*, y^*) \prec \epsilon$ . If  $\{x^*, y^*\} \not\subseteq X_{j^*}$  for some  $j \in J$  then  $d_Z(x^*, y^*) = 1_R$  and we get a contradiction. Otherwise, for all  $\delta \notin \cup \epsilon \cap V_j$  we have  $\delta \in \cup(d_Z(x^*, y^*))$  since  $V_k \subset \cup(d_Z(x^*, y^*))$  for all  $k \neq j$ . If  $\delta \in \cup \epsilon \cap V_j$  then it must be (b). Otherwise,  $\epsilon \not\subseteq d_Z(x^*, y^*)$  and then  $\epsilon \not\prec d_Z(x^*, y^*)$ .

□

Next, we show that the above construction does indeed generate the topological sum. In the following theorem  $(Z, \tau_\sqcup) = \coprod (X_j, \tau_j)$  ( $Z = \sqcup X_j$ ) from a collection of topological spaces  $\{(X_j, \tau_j) \mid j \in J\}$ . Also, for each  $i \in J$ ,  $(X_i, \tau_i) = \mathcal{P}[(V_i, X_i, d_i)]$  and  $(Z, \tau_R) = \mathcal{P}[(R, Z, d_Z)]$ .

**Theorem 4.5.1.**  $\tau_R = \tau_\sqcup$ .

*Proof.* ( $\supseteq$ ) For a fixed  $i \in J$ , let  $x \in O \in \tau_j, \epsilon \succ 0_i$  and take any  $y \in B_\epsilon(x) \subseteq O$ . Since  $d_i(x, y) \prec \epsilon$  then  $\phi_j(d_i(x, y)) \prec \bar{\epsilon} = \{\{\epsilon\}, \emptyset\}$  (by Lemma 4.5.3(2)) and, more importantly,

$$y^* \in B_{\bar{\epsilon}}(x^*) = \{z \in Z \mid d_Z(x^*, z) \prec \bar{\epsilon}\} = \{z \in X_j^* \mid d_Z(x^*, z) \prec \bar{\epsilon}\}.$$

Similarly, take  $y^* \in B_{\bar{\epsilon}}(x^*)$  and notice that  $d_Z(x^*, y^*) \prec \bar{\epsilon} \Rightarrow y \in X_y$  and  $d_j(x, y) \prec \epsilon$  and that  $y \in B_\epsilon(x)$ ; for all  $\epsilon \succ 0_i$ ,  $[B_\epsilon(x)]^* = B_{\bar{\epsilon}}(x^*)$  and  $\tau_\sqcup \subseteq \tau_R$ .

( $\subseteq$ ) Let  $x^* \in O \in \tau_R$  and  $p \in R_{\prec}$  so that  $B_p(x^*) \subseteq O$ . If  $1_R = p$  then there is nothing to prove so we assume  $1_R > p$ . Without loss of generality, assume that  $x \in X_k$  for some  $k \in J$  and let  $p_k = \vee(\cup p \cap V_k)$ . Since  $\cup p$  is finite then  $p_k \succ 0_k$ . If  $d_k(x, y) \prec p_k$  then  $d_k(x, y) \prec \delta, \forall \delta \in (\cup p \cap V_k)$ . Hence,  $d_R(x, y) \prec p$  and the injection  $i_k$  is  $\epsilon$ - $\delta$  continuous.  $\square$

In the same spirit as with products we show next that coproducts in  $\mathbf{P}$  coincide with those from  $\mathbf{Top}$

**Theorem 4.5.4.** *For a set-indexed collection of continuity spaces  $\{(V_j, X_j, d_j) \mid j \in J\}$  we have  $\coprod(V_j, X_j, d_j) = (R, Z, d_Z)$ .*

*Proof.* The proof of Theorem 4.5.1 shows that the canonical injections are  $\epsilon$ - $\delta$  continuous. Next, take a continuity space  $(W, Y, m)$  in conjunction with a collection

$$\{f_j \mid f_j \in \text{hom}_{\mathbf{P}}((V_j, X_j, d_j), (W, Y, m)) \text{ and } j \in J\}$$

and define  $h : Z \rightarrow Y$  so that  $x^* \mapsto f_i(x)$  for  $x \in X_i$ . Take  $x^* \in Z$  and  $\epsilon \succ 0_W$  where, wlog,  $x \in X_k$  for some  $k \in J$ . Since  $f_k$  is  $\epsilon$ - $\delta$  continuous there exists  $\delta \succ 0_k$  so that  $d_k(x, y) \prec \delta \Rightarrow m(f_k(x), f_k(y)) \prec \epsilon$ . Letting  $\bar{\delta} = \{\{\delta\}, \emptyset\}$  we have that  $y^* \in B_{\bar{\delta}}(x^*) \Rightarrow y \in X_k$  and most, importantly,  $y \in B_\delta(x)$ . Thus,  $m(h(x^*), h(y^*)) = m(f_k(x), f_k(y)) \prec \epsilon$ .

$\square$

**Corollary 4.5.5.** *The functor  $\mathcal{P}$  preserves and lifts all small coproducts.*

## 4.5.2 Final Topology, Direct Limits and Topological Meets

In any category direct limits can be derived from coproducts and coequalizers (where topological meets can be seen as a biproduct of direct limits within **Top**). The aforementioned lack of a suitable construction for coequalizers in **P** (and, consequently, direct limits) does not undermine that of meets of topologies as continuity spaces.

Take any set-indexed collection of topological spaces  $\{(X, \tau_j) \mid j \in J\}$  with its corresponding collection of continuity spaces  $\{(W_j, X, m_j) \mid j \in J\}$ . What follows is the construction of a continuity space  $(V, X, m)$  that generates  $\bigwedge(X, \tau_j)$ . Let  $U = \prod_{j \in J} (W_j)_{\prec}$  and  $V = \Omega(U)$ . Next, given  $x \in W_i$  (for some fixed  $i \in J$ ) denote  $x_{\uparrow} = \prod_{j \in J} A_j$  so that  $A_k = x^{\uparrow} = \{a \in W_i \mid a \succ x\}$  when  $k = i$  and  $A_j = (W_j)_{\prec}$ , otherwise. We define for all  $j \in J$  the functions  $\phi_j : W_j \rightarrow V$  so that  $x \mapsto x_{\uparrow}$  (all order-embeddings) and  $m : X \times X \rightarrow V$  so that for any pair  $x, y \in X$ :

$$m(x, y) = \bigwedge \phi_j(m_j(x, y)).$$

Without loss of generality, let  $1_V > \epsilon \succ 0_V$  and recall that since  $\epsilon$  must be finite then so must be  $\cup \epsilon$ . In particular, for each  $i \in J$  we can form the finite collection  $\epsilon_i = \pi_i(\cup \epsilon) = \{\delta_1, \delta_2, \dots, \delta_n\}$  (i.e.  $\epsilon_i$  is the set containing all of the  $i^{\text{th}}$  coordinates of all elements from  $\cup \epsilon$ ). Furthermore, for all  $j \in J$  we let  $\bar{\epsilon}_j = \bigwedge \epsilon_j$  and note that  $\bar{\epsilon}_j \succ 0_J$ . For the sake of simplicity let us denote  $(X, \tau_{\bigwedge}) = \bigwedge(X, \tau_j)$  and the one generated by  $(V, X, m)$  as  $(X, \tau_V)$ .

**Lemma 4.5.6.** *For all  $j \in J$  and any  $\epsilon \succ 0_V$  we have  $B_{\bar{\epsilon}_j}(x) \subseteq B_{\epsilon}(x)$ .*

*Proof.* Take any  $z \in B_{\bar{\epsilon}_j}(x)$  and notice that since  $m_j(x, z) \prec \bar{\delta}_j$  then  $m_j(x, z) \prec \delta$  for all  $\delta \in \epsilon_j$ . Hence,  $\phi_j(m_j(x, z)) \supseteq \epsilon$  and thus  $m(x, z) \supseteq \epsilon$ . More importantly,  $m(x, z) \prec \epsilon$ .  $\square$

**Lemma 4.5.7.**  $\tau_V = \tau_{\bigwedge}$ .

*Proof.* ( $\subseteq$ ) This is immediate from the above lemma; take  $x \in O \in \tau_V$  and choose an  $\epsilon \succ 0_V$  for which  $B_{\epsilon}(x) \subseteq O$ . Since for all  $j \in J$ ,  $B_{\bar{\epsilon}_j}(x) \subseteq B_{\epsilon}(x)$  then  $O$  is open in all  $(X, \tau_j)$ . Consequently it must be that  $O \in \tau_{\bigwedge}$ .

( $\supseteq$ ) Take any  $x \in O \in \tau_\wedge$ . It is immediate that since for any  $j \in J$  we can find  $\epsilon_j \succ 0_j$  so that  $B_{\epsilon_j}(x) \subseteq O$  then  $\bigcup B_{\epsilon_j}(x) \subseteq O$ . To this end, let  $\delta = \{\{(\epsilon_j)_{j \in J}\}, \emptyset\}$  and notice that if  $z \in B_\delta(x)$  then  $m(x, z) \prec \delta$ . Since by design  $m(x, z) = \bigwedge \phi_j(m_j(x, y)) = \bigcup \phi_j(m_j(x, y))$  then for at least one  $i \in J$ ,  $\{(\epsilon_j)_{j \in J}\} \in \phi_i(m_i(x, z))$  and, thus,  $m_i(x, z) \prec \epsilon_i$ . Hence,  $B_\delta(x) \subseteq \bigcup B_{\epsilon_j}(x) \subseteq O$  and the proof is complete.  $\square$

## 4.6 Conclusion

Limits and colimits for  $\mathbf{P}$  are fairly simple to construct. The same is not true for their counterparts in  $\mathbf{M}$ ; suitable constructs for topological limits and colimits within  $\mathbf{M}$  also remain unknown to the authors. Potentially the hardest construction within  $\mathbf{P}$  (resp.  $\mathbf{M}$ ) is that for coequalizers and also remains unknown. This continuity space must be able to (intrinsically) measure distances between subsets of a given topological space in agreement with the quotient topology. Moreover, in general, it must be coarser than the Hausdorff pseudo-metric on  $\mathcal{P}(X)$  and strictly finer than the quotient metric on it. The former observation is simple to verify while the latter can be appreciated by noting that the quotient topology of a pseudo-metrizable space need not be pseudo-metrizable.

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# Conclusion

Chapter 1 presents the results of a preliminary enquiry into the topological and order-theoretic properties of  $Top(X)$  as a naturally occurring subset of  $2^{\mathcal{P}(X)}$ . Chapter 2 develops substantially this line of enquiry. In particular, as outlined in this chapter, and for reasons relating to regular cardinals, the proof that  $Top(X)$  is not an  $F_\sigma$  set cannot be extended any further. Thus, the question of whether  $Top(X)$  can be attained as from  $F_\sigma$  sets (or even as a Borel set) within  $2^{\mathcal{P}(X)}$  remains open.

Chapter 3 explores in great detail metric axioms as collections of weight structures within the ambient space  $[0, \infty]^{X \times X}$ . The choice of  $[0, \infty]^{X \times X}$  as a natural setting for this investigation is justified in the first section where it is shown that this ambient space possesses highly desirable topological (compact Hausdorff) and lattice-theoretic (completeness) properties. Moreover, a strong connection between the two sets of properties is also shown to exist. The categorical notion of adjunctions is exploited when moving between the aforementioned collections of weight structures and in describing *optimal* ways of adding more structure to an arbitrary weight structure. Having all of the above tools at one's disposal, the rest of Chapter 3 focuses on lattice-embeddability theorems and in describing connections between collections of weight structure and their corresponding generated topologies. Chapter 3 is a self-contained paper and in which all of the motivating questions are answered.

The equivalence  $\mathbf{M} \rightarrow \mathbf{Top}$  outlined in Chapter 4 should be interpreted as a complete reformulation of topology purely in terms of distance assignments between pairs of points. This new approach to topology can be exploited in searching for answers to open problems and in particular with regard to compactness; net convergence (and thus compactness) is surprisingly simple to define in  $\mathbf{M}$ . The first steps in doing so

must be taken in the direction of  $\mathcal{M}$  and  $\mathbf{M}$ , investigating both in greater detail. For instance,

- *What other topological properties naturally arise in  $\mathbf{M}$ ?*
- *Are there any naturally occurring properties in  $\mathbf{M}$  that have not been considered in  $\mathbf{Top}$ ?*

Clearly, the above should be answered explicitly in terms of value distributive lattices and distance assignments. More general and ambitious tasks would entail

- *Given any value distributive lattice, characterise all topologies that can be generated from it.*
- *Characterise purely in lattice theoretic terms which value distributive lattices yield the same collection of topologies.*
- *What does the category of value distributive lattice modulo the above restriction look like?*

In particular, the latter two demand the use of morphisms between completely distributive lattices that *naturally* preserve the associative and commutative binary operations between value distributive lattices and their filters of well-above zero elements.

Back to  $\mathbf{M}$ , suitable constructions for limits (with the exception of equalizers) and colimits remain unknown. Of particular interest are those for coequalizers since they also remain unknown for  $\mathbf{P}$ . Lastly, the functor  $\mathcal{P}$  was shown to be surprisingly strong and since the category  $\mathbf{P}$  is better understood than  $\mathbf{M}$ , one must ask: *is it possible to utilise  $\mathbf{P}$  in doing everyday topology? If so, then to what extent?*

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