Novel construction of Specht modules for Monomial groups

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Novel Construction of Specht Modules for Monomial Groups

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A thesis presented in fulfilment of the requirements for the degree of Doctor of Philosophy

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Abstract

In this thesis, I introduce a new method for constructing Specht modules using Specht matrices for families of symmetric groups, monomial groups, and hyperoctahedral groups. Specht modules are central to the representation theory of symmetric groups, connected to the concepts of Young symmetrizers and the subsequent development of polytabloids. These polytabloids form a basis for the Specht module associated with a particular partition of a natural number.

In this work, I introduce a new combinatorial approach to replace polytabloids with columns of Specht matrices. The columns corresponding to the standard tableaux of shape λ, where λ is a partition of $n \in \mathbb{N}$, form the basis of the Specht module S^{λ}. This new construction is straightforward and easy to grasp, as it relies on basic linear algebra techniques, such as solving systems of linear equations.

A longstanding open question in the representation theory of finite groups is whether a base change matrix can be found for a finite group G that transforms permutation matrices corresponding to irreducible representations into block forms. My thesis addresses this question for specific families of finite groups, including symmetric groups, monomial groups, and hyperoctahedral groups. It shows that a submatrix of the Specht matrix serves as a base change matrix that accomplishes this transformation, enabling us to obtain all irreducible representations of these groups over $\mathbb{C}$.

The methodologies employed in this thesis are computational, involving the development of several algorithms and computer programs to support the theoretical work. These have been implemented in the computer algebra system GAP (version 4).

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Declaration of Authorship

I, Koushik Paul, hereby certify that this thesis is all my own work and that I have not obtained a degree in University of Galway, or elsewhere, on the basis of any of the work described in this thesis.

Signed: ______________________________

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Chapter 1

Introduction

The representation theory of finite groups is a well established subject that was first studied in the late eighteen hundreds. G. Frobenius, I. Schur, and W. Burnside were in the pioneers in this field who were at the forefront of the subject. Contemporary methods often heavily rely on module theory and the Wedderburn theory of semisimple algebras. Currently, despite its potential accessibility through discrete Fourier analysis, we refrain from delving into that aspect of the subject as it does not align closely with the theme of the thesis.

In this thesis we look into the various aspects of representation theory. We seek to elucidate the core components of the representation theory for finite groups over complex numbers, requiring only familiarity with linear algebra and group theory. The primary intent is to provide a robust tool for acquiring insights into finite groups using linear algebraic methods. Most of the notions and notations found in this section are cited from [16].

In the representation theory of symmetric groups, each partition of a natural number corresponds to an irreducible module (Specht module). These modules are of importance because they produce a complete set of irreducible representations of the symmetric group. In the referenced book by Sagan [15] or Fulton [6], the authors discuss the bases of these modules. Traditionally, a complex and lengthy proof has been employed to demonstrate that these basis vectors are associated with standard Young tableaux of a shape of a partition. These results, achieved in 1991, marked a significant breakthrough in the field.

Recently, in the work by Wiltshire-Gordon et al. [18], the authors introduced the concept of Specht matrices. Their approach involves constructing words corresponding to a given partition and its conjugate partition. These matrices can then be derived from these words. However, the sizes of these matrices can grow to very large orders, making manual calculation with pen and paper exceedingly difficult. Fortunately, various computational algebraic tools, such as GAP [9], are available to compute such examples using algorithmic inputs.

I have thoroughly examined both of these concepts and connected them. By establishing a solid link between the Specht matrices and the bases found in the literature, I have introduced a fresh perspective that simplifies the process of obtaining a basis for the irreducible Specht modules through basic linear algebra concepts. This approach unveils numerous intriguing ideas and properties of these matrices, suggesting a shift from the conventional method of understanding finite group representations to a more computational and combinatorial approach. Ultimately, this makes the concepts more accessible and easier to grasp.

I begin by introducing the fundamental concepts of representation theory of finite groups in Chapter 2. In Chapter 3, I explore various facets of symmetric group representations, covering both conventional and modern theories, and bridging any gaps between them. These concepts are developed using straightforward combinatorial ideas. As an example, consider two words SEVENTEEN and ASSASSINS. Any rearrangements of these two words can be paired up together, for instance

$$\begin{pmatrix} E & E & E & E & N & N & S & V & T \\ S & A & I & N & S & A & S & S & S \end{pmatrix}.$$

Can we take any two rearrangements of the words and pair them up so that it fits in the following diagram where the integers in the boxes correspond to the positions of the columns in the word pair?

	S	A	I	N
E	1	2	3	4
N	5	6		
S	7			
V	8			
T	9			

In Chapter 4, I present a general framework of the construction for symmetric groups, introducing a new method for constructing Specht modules as irreducible representations of monomial groups using Specht matrices. In Chapter 5, I develop a distinct and specialized approach to constructing irreducible Specht modules for hyperoctahedral groups. The thesis concludes in Chapter 6, where I discuss potential directions for future research. All GAP system (version 4) codes developed for this research project are included in the Appendix at the end of the thesis.

Chapter 2

Group Representation

Definition 2.0.1 ([16, Definition 3.1.1: **Representation**]). *Let* G *be a group and* $\mathbb{C}$ *be the set of complex numbers. Let* $GL(V)$ *be the the group of all invertible matrices for some finite dimensional vector space* V*. We say a homomorphism* $\phi : G \to GL(V)$ *is a group representation of* G *over the field* $\mathbb{C}$*. The dimension of* V *is called the degree of* ϕ*. For all* $g \in G$ *under* ϕ*, we denote* $\phi(g)$ *by* ϕ_g*.*

Let $\mathbb{C}^*$ be the set of all nonzero complex numbers. If we define a homomorphism $\phi : G \to \mathbb{C}^*$ given by $\phi_g = 1$ for all $g \in G$ then ϕ is the ***trivial representation*** of the group G.

Now we start to slowly introduce the concepts that will be used throughout the thesis.

Definition 2.0.2 ([16, Definition 3.1.7: **Equivalence**]). *Let* $\phi : G \to GL(V)$ *and* $\psi : G \to GL(W)$ *be two group representations. We say* ϕ *is equivalent to* ψ *if there exists an isomorphism* $\tau : V \to W$ *such that* $\psi_g \tau = \tau \phi_g$ *for all* $g \in G$*. We denote it by* $\phi \sim \psi$*. Therefore the following diagram commutes.*

$$\begin{array}{ccc} V & \xrightarrow{\phi_g} & V \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ W & \xrightarrow{\psi_g} & W \end{array}$$

Definition 2.0.3 ([16, Definition 3.1.11: **Direct Sum of Representations**]). *Let* $\phi : G \to GL(V)$ *and* $\psi : G \to GL(W)$ *be two representations. We define their direct sum* $\phi \oplus \psi : G \to GL(V \oplus W)$ *by*

$$(\phi \oplus \psi)_g(v, w) = (\phi_g(v), \psi_g(w)).$$

We can also interpret the direct sum in terms of matrices: Let us assume the representations $\phi : G \to GL_m(\mathbb{C})$ and $\psi : G \to GL_n(\mathbb{C})$. Then we find that $\phi \oplus \psi : G \to GL_{m+n}(\mathbb{C})$ has the following block matrix form

$$(\phi \oplus \psi)_g = \begin{pmatrix} \phi_g & 0 \\ 0 & \psi_g \end{pmatrix}.$$

Definition 2.0.4 ([16, Definition 3.1.10: **G-invariant Subspace**]). *Let $\phi : G \to GL(V)$ be a group representation and $W \leqslant V$ be a subspace. Then we say W is G-invariant if $\phi_g w \in W$ for all $g \in G$ and $w \in W$.*

Definition 2.0.5 ([16, Definition 3.1.15: **Irreducible Representation**]). *Let G be a group and $\phi : G \to GL(V)$ be a nonzero representation for some finite dimensional vector space V. We say ϕ is irreducible if the only G-invariant subspaces of V are $\{0\}$ and V itself.*

Let $\phi : G \to GL(V)$ be a group representation. Consider a G-invariant subspace W of V. Then we can have the representation $\phi|_W : G \to GL(W)$ by restricting ϕ to W such that $(\phi|_W)_g(w) = \phi_g(w)$ for $w \in W$. As W is G-invariant, we have that $\phi_g(w) \in W$. Sometimes $\phi|_W$ is also referred to as a subrepresentation of ϕ.

Definition 2.0.6 ([16, Definition 3.1.21: **Completely Reducible**]). *Let G be any group and $\phi : G \to GL(V)$ be a group representation. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ where the V_i are G-invariant subspaces. We say ϕ completely reducible if $\phi|_{V_i}$ is irreducible for all $i = 1, \dots, n$. Equivalently, if we have $\phi \sim \phi_1 \oplus \cdots \oplus \phi_n$ we have that ϕ is completely reducible where each ϕ_i is an irreducible representations for $i = 1, \dots, n$.*

Definition 2.0.7 ([16, Definition 3.1.22: **Decomposable Representation**]). *Let G be a group and ϕ be a nonzero representation of G. If V_1 and V_2 be nonzero G-invariant subspaces of V and $V = V_1 \oplus V_2$ then we say that ϕ is decomposable. Otherwise, V is said to be indecomposable.*

Then we have the following lemma. As we only need the statement of Lemma 2.0.8 to be used later, we omit the proof of it.

Lemma 2.0.8 ([16, Lemma 3.1.23]). *Let G be a group and $\phi : G \to GL(V)$ be a representation equivalent to a decomposable representation. Then ϕ is decomposable.*

Let G be any finite group. For a vector space V of dimension n and for $v \in V$, let us define the representation $\phi : G \to GL(V)$ such that for any $a \in G$ we have $\phi_a \in GL(V)$. Then $\phi_a : V \to V$ is a linear operator. Now let $W \leqslant V$ be a subspace of dimension m. If we have $\phi_a(w) \in W$ for all $a \in G$ and $w \in W$ then the group G acts on W.

Let us choose a basis $\mathbb{B} = (b_1, \ldots, b_n)$ of V. Then for $\phi_a \in GL(V)$ we get a matrix $[\phi_a]_{\mathbb{B}}$ with respect to the basis $\mathbb{B}$ so that $[\phi_a]_{\mathbb{B}} \in GL_n(\mathbb{C})$. Let us denote the k-th column of $[\phi_a]_{\mathbb{B}}$ by $[\phi_a(b_k)]_{\mathbb{B}}$.

Now if we choose $\mathbb{B}$ in such a way so that the first m vectors in $\mathbb{B}$ span W, then $\mathbb{B}_W = (b_1, \ldots, b_m)$ is a basis of W. Therefore the first m columns of the matrix $[\phi_a]_{\mathbb{B}}$ are generated by the basis of W and the rest of the columns are generated by $\mathbb{B}' = (b_{m+1}, \ldots, b_n)$. Then we can write the matrix $[\phi_a]_{\mathbb{B}}$ in a block-matrix form

$$[\phi_a]_{\mathbb{B}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is an $m \times m$ square matrix and D is an $(n-m) \times (n-m)$ square matrix. Then the matrix $A = [\phi_a|_W]_{\mathbb{B}_W}$ and the matrix C is a zero-matrix.

Now $0 < W < V$ and W is G-invariant. Therefore, G acts on the quotient space $V/W = \{v + W : v \in V\}$ as well. Then

$$\phi_a(v + W) = \phi_a(v) + \phi_a(W).$$

Now $\phi_a(W) = W$ and therefore

$$\phi_a(v) + W \in V/W.$$

Let $\mathbb{B}'' = (b_{m+1} + W, \ldots, b_n + W)$ be a basis of V/W, and $v + W = \{v + w : w \in W\}$. Then $\phi_a(b_k + W) = \phi_a(b_k) + W$.

We know that $[\phi_a(b_k)]_{\mathbb{B}}$ is k-th column of $[\phi_a]_{\mathbb{B}}$. Then for $m_{ik} \in \mathbb{C}$ we have

$$\phi_a(b_k) = \sum_{i=1}^{n} m_{ik} b_i$$

and therefore

$$\phi_a(b_k) + W_1 = \sum_{i=m+1}^{n} (m_{ik} b_i + W_1)$$

as $\mathbb{C}$-spaces $V/W \cong W'$ where $W' = \langle b_{m+1}, \dots, b_n \rangle$ over the $\mathbb{C}$-space. Therefore $V = W \oplus W'$ but W' is not G-invariant. Now we can say that the matrix D in the block-matrix $[\phi_a]_{\mathbb{B}}$ is the matrix $[\phi_a|_{V/W}]_{\mathbb{B}''}$.

Two of the important lemmas are included in the following which are relevant to the study but we omit the proofs.

Lemma 2.0.9 ([16, Lemma 3.1.24]). *Let* G *be a group and* $\phi : G \to GL(V)$ *be a group representation equivalent to an irreducible representation. Then* ϕ *is irreducible.*

Lemma 2.0.10 ([16, Lemma 3.1.25]). *Let* G *be a group and* $\phi : G \to GL(V)$ *be a group representation equivalent to a completely reducible representation. Then* ϕ *is completely reducible.*

Now we are moving on to one of the most important concepts of the chapter followed by Maschke's Theorem.

Corollary 2.0.11 ([16, Corollary 3.2.5]). *Let* G *be a finite group and* $\phi : G \to GL(V)$ *be a nonzero representation of* G *over* $\mathbb{C}$. *Then* ϕ *is either irreducible or decomposable.*

Therefore we can conclude that any irreducible representation is indecomposable but the converse is not always true. Now we are going to get introduced to one of the most important theorems, Maschke's theorem. We omit the proof in here but one can refer to any book on representation theory including [16] for a detailed proof.

Theorem 2.0.12 ([16, Theorem 3.2.8: **Maschke**]). *Over the field of complex numbers, every representation of a finite group is completely reducible.*

Then as a conclusion we can say, if $\phi : G \to GL_n(\mathbb{C})$ is any representation of a finite group G, then

$$\phi \sim \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \phi_m \end{pmatrix}$$

where the ϕ_i are irreducible for all i.

We now get to the idea of group algebra. It can be looked at from the perspective of all formal linear combinations of the groups elements.

Definition 2.0.13 ([16, Section 4.4: **Group Algebra**]). *Let* G *be a finite group. Define a vector space with basis* G *such that*

$$\mathbb{C}[G] = \{\sum_{g \in G} c_g g \mid c_g \in \mathbb{C}\}.$$

Then $\mathbb{C}[G]$ *consists of all formal linear combination of elements of* G*. Any two elements* $\sum_{g\in G} a_g g$ *and* $\sum_{g\in G} b_g g$ *in* $\mathbb{C}[G]$ *are said to be equal if and only if* $a_g = b_g$ *for all* $g \in G$*. The addition is defined by* $\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)g$.

If we take two elements from the group algebra and multiply them then we obtain another element in the group algebra. This same idea is used in the literature [6] to multiply basis elements and add them to get all other elements in the entire space.

Definition 2.0.14 ([16, Definition 4.4.1: **Regular Representation**]). *Let* G *be a finite group. For* $h \in G$*, we define the regular representation* $\phi : G \to GL(\mathbb{C}[G])$ *of* G *by*

$$(\sum_{g\in G} c_g g)\phi_h = \sum_{g\in G} c_g gh.$$

Definition 2.0.15 (**Right** G**-Module**). *Let* G *be a group and* V *be a vector space. Then we say that* V *is a* G*-module if there exists a linear map* $V \times G \to V$ *defined by* $(v, g) \mapsto v \cdot g$ *such that the following criterion are satisfied*

1. $(v_1 + v_2) \cdot g = v_1 \cdot g + v_2 \cdot g$ *for all* $v_1, v_2 \in V$ *and* $g \in G$,
2. $\alpha(v \cdot g) = (\alpha v) \cdot g$ *for all* $v \in V$, $g \in G$ *and* $\alpha \in \mathbb{C}$,
3. $(v \cdot g_1) \cdot g_2 = v \cdot (g_1 \cdot g_2)$ *for all* $v \in V$ *and* $g_1, g_2 \in G$.

In the above definition one can also define a left G-module by taking the left-action of G on V. Let G be a group and $\phi : G \to GL(V)$ be a group representation for some finite-dimensional vector space V. For all $g \in G$ we have $\phi_g \in GL(V)$. Then for any $v \in V$, we have $v \cdot \phi_g = v \cdot g \in V$. This gives us that a natural matrix multiplication is nothing but a right-action of a group G on a vector space V. Therefore we have that the representation of a group G for a vector space V is equivalent to a G-module.

Definition 2.0.16. *Let* V *be a right* G*-module and* W *be a left* G*-module for some group* G*. The tensor product* $V \otimes_G W$ *of* V *and* W *is a* G*-module with basis elements* $v \otimes w$ *for* $v \in V$ *and* $w \in W$ *such that the following hold*

1. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$ *and* $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$ *for* $v_1, v_2 \in V, w_1, w_2 \in W$,
2. $vg \otimes w = v \otimes gw$ *for* $g \in G$.

Definition 2.0.17 (Induced Representation). *Let* G *be a group and* $H \leqslant G$ *be a subgroup. Let* W *be a finite dimensional vector space. Let* $\phi : H \to GL(W)$ *be a representation equipped with a* $\mathbb{C}[H]$ *module. By setting* $V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$*, we define an induced representation* $\mathrm{Ind}_H^G(\phi_g)(\pi \otimes w) = g\pi \otimes w$ *where* π *is a class representative of* G/H.

Definition 2.0.18 ([16, Definition 4.3.1: **Character**])**.** *Let* G *be a group and* $\phi : G \to GL(V)$ *be a representation. Then the character* $\chi_\phi : G \to \mathbb{C}$ *of* ϕ *is defined by* $\chi_\phi(g) = \mathrm{tr}(\phi_g)$ *where* $\mathrm{tr}(\phi_g)$ *is the trace of* ϕ_g*. We say that the character* χ_ϕ *is irreducible if* ϕ *is irreducible.*

Therefore, if $\phi : G \to GL_n(\mathbb{C})$ is a group representation given by $\phi_g = (\phi_{ij}(g))$, then we have $\chi_\phi(g) = \sum_{i=1}^{n} \phi_{ii}(g)$.

Some properties of the character are given below. However we omit the proofs of them in here.

Proposition 2.0.19 ([16, Proposition 4.3.4])**.** *If two representations* ϕ *and* ρ *are equivalent, then* $\chi_\phi = \chi_\rho$.

Proposition 2.0.20 ([16, Proposition 4.3.5])**.** *Let* G *be a group and* ϕ *be a representation of* G*. Then for all* $g, h \in G$*, we have that* $\chi_\phi(g) = \chi_\phi(hgh^{-1})$ *holds.*

Lemma 2.0.21 ([16, Lemma 4.3.13])**.** *If* $\phi = \rho \oplus \psi$ *then* $\chi_\phi = \chi_\rho + \chi_\psi$.

The lemma mentioned above suggests that every character can be expressed as a linear combination of irreducible characters with integral coefficients.

Corollary 2.0.22 ([16, Corollary 4.4.8])**.** *The number of equivalence classes of irreducible representations of* G *is same as the number of conjugacy classes of* G.

The information pertaining to the representation theory of a finite group G can be encapsulated within a matrix referred to as its character table. Consequently, the subsequent concept emerges.

Definition 2.0.23 ([16, Definition 4.4.11: **Character Table**]). *Let* G *be a finite group. Let* $\chi_1, \ldots, \chi_s$ *and* $C_1, \ldots, C_s$ *be the irreducible characters and conjugacy classes of* G *respectively. Then the character table of* G *is the matrix* X *of dimensions* $s \times s$ *with* $X_{ij} = \chi_i(C_j)$.

Let X be the character table of a finite group G. Then we have that the rows of X are indexed by the characters of G and the columns by the conjugacy classes of G. We also have that the ij-entry in X is the value of the i-th character on the j-th conjugacy class.

Finally, the following is a well-known and very useful theorem in representation theory of finite groups. One can find its proof in any literature including [16].

Theorem 2.0.24 ([16, Corollary 4.4.5]). *Let* G *be a finite group and the* $\rho_1, \rho_2, \ldots, \rho_s$ *are the irreducible representations of* G*. If* d_i *be the dimensions of the irreducible representation* ρ_i*, then we have that*

$$|G| = \sum_{i=1}^{s} d_i^2 = d_1^2 + d_2^2 + \cdots + d_s^2.$$

If we replace the vector space V in the Lemma 2.0.8 by $\mathbb{C}[G]$ for a finite group G then we get from Theorem 2.0.12 that the regular representation decomposes into irreducible representations. The representing matrices from the regular representation are permutation matrices of very large dimensions. Base change transforms these matrices into block forms. An open problem that naturally follows from this is to find such base change matrices that transform these permutation representations into block matrices. In this thesis, I take this job of finding the base change matrices as a motivation, and give solution to this problem for the symmetric groups, monomial groups and hyperoctahedral groups.

Chapter 3

Symmetric Groups

In this chapter we construct the Specht modules for the symmetric groups in a similar fashion to that of shown in [18]. In the family of Coxeter groups, we have that symmetric group is same as the type A group. More specifically, symmetric group on n points is same as the type A group on $n-1$ generators. Here in the following we start in a systematic way by introducing the relevant idea slowly by describing all the important concepts needed to approach the construction of irreducible Specht modules for symmetric groups.

3.1 Symmetric Groups, Generators and Class Representatives

Definition 3.1.1 ([15, **Symmetric Group**]). *The symmetric group S_n of degree n is the group of all bijections from the set $\{1, 2, \ldots, n\}$ to itself where composition is multiplication. So we have, S_n as a permutation group of order $n!$.*

The symmetric group can be generated in a few ways. For $n \geqslant 3$, we have that the permutations $\{(1\ 2), (1\ 2\ 3\ \cdots\ n)\}$ generate S_n. We can also consider the generating set for S_n to be the set of all *transpositions* of the form $(i, i+1)$ for $1 \leqslant i < n$, i.e., we have the generating set to be $\{(1\ 2), (2\ 3), \ldots, (n-1\ n)\}$.

For each element in the symmetric group we have an associated *cycle-type*. Any permutation can be written as a product of disjoint cycles. If we write the product of disjoint cycles sequentially in a non-increasing order of each of its cycle-lengths, then we find the cycle type of the permutation. It is important to include the fixed points which are cycles of length 1 in this process as it contributes to the cycle-type.

As an example, let us consider the permutation $\pi = (1\ 2)(3\ 4) \in S_5$. It is clear that π is a product of two disjoint transpositions of length 2 and one cycle of length 1 which is omitted in the product. If we put the cycle lengths in a non-increasing sequence, we find that π is of cycle-type $(2, 2, 1)$.

Definition 3.1.2. *Let $\sigma \in S_n$ be a permutation. We say σ is an even permutation if it is the product of an even number of permutations, and an odd permutation if it is the product of an odd number of transpositions.*

Definition 3.1.3. *The sign of a permutation σ for all $\sigma \in S_n$ is defined as*

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Definition 3.1.4 ([16, Definition 10.1.1: **Partition**])**.** *A partition of n is a tuple $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$ and $\lambda_1 + \cdots + \lambda_l = n$. To indicate that λ is a partition of n, we simply write $\lambda \vdash n$.*

In the previous example it can be seen that the cycle-type $(2, 2, 1)$ is also a partition of 5, and so it is evident how cycle-type maps a permutation into a partition.

Two group elements $a, b \in G$ are conjugates, denoted by $a \sim b$ if there exists some $g \in G$ such that $b = g^{-1}ag$. The conjugacy classes of S_n are entirely decided by the cycle-types of the elements in the group. If two distinct permutations have same cycle-type then they belong to the same conjugacy class. In other words, partitions parameterize the conjugacy classes of permutations. The following is a well-known property.

Lemma 3.1.5 ([16, Theorem 10.1.3])**.** *Two permutations $\sigma, \rho \in S_n$ have the same cycle type if and only if σ is conjugate to ρ.*

Now for any conjugacy class, we need a representative in order to understand and denote the classes. For an element $\sigma \in S_n$ having certain cycle type $(k_1, k_2, \ldots, k_l)$ which is a partition of n, all the other elements with the same cycle type belongs to the same class, this implies that σ with its cycle structure can be considered as the representative of the class. Let us denote the set of conjugacy classes of S_n by $\mathcal{C}_n$.

Let $P(n)$ be the set of all partitions of $n \in \mathbb{N}$. Contrary to the idea of cycle-type as a function $f : S_n \to P(n)$, we define a function $g : P(n) \to S_n$ defined by the property that $g(\lambda)$ is conjugacy class representative of S_n for $\lambda \in P(n)$ and the set of conjugacy classes $\mathcal{C}_n \subset S_n$, we have $|P(n) \cap \mathcal{C}_n| = 1$. This clearly is an injection as for each partition we have a conjugacy class and for each of them we have a class representative. Now let us understand how this function works in making a choice for our favourite representative.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P$, be any partition of n. We know that the partition is a sequence of non-increasing natural numbers and they add up to n. Each of these λ_i are therefore the cycle lengths of the cycles that make up the permutations belonging to the same class. In order to find the class representative c for the conjugacy class λ, we start making cycles of length λ_i by filling them with numbers $\{1, 2, \ldots, n\}$ in this way:

$$\Big(1\ 2 \cdots \lambda_1\Big)\Big(\lambda_1{+}1\ \lambda_1{+}2 \cdots \sum_{i=1}^{2} \lambda_i\Big)\Big(\sum_{i=1}^{2} \lambda_i{+}1\ \sum_{i=1}^{2} \lambda_i{+}2 \cdots \sum_{i=1}^{3} \lambda_i\Big) \cdots \Big(\sum_{i=1}^{n-1} \lambda_i{+}1 \cdots n\Big).$$

This mechanism can be considered as an algorithm as shown below.

Algorithm 1 From partition to class representative

Input: Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of $n \in \mathbb{N}$.
Output: Class representative of conjugacy class corresponding to the cycle-type.
Step 1: Define a function cycle $C : (n, o) \mapsto (1 + o, 2 + o, \ldots, n + o)$ where both arguments $n, o \in \mathbb{N}$.
Step 2: Initialize $o := 0$ and $\pi :=$ identity permutation.
Step 3: Loop over $\lambda_1, \lambda_2, \ldots, \lambda_l$ as explained from Step-4 to Step-6.
Step 4: Make cycle $C(\lambda_i, o)$ for $i = 1, 2, \ldots, l$.
Step 5: $\pi := \pi \cdot C(\lambda_i, o)$.
Step 6: Change o to $o + \lambda_{i-1}$ for $i > 1$.
Step 7: Return the final π.

As we are just picking numbers from $\{1, 2, \ldots, n\}$ in that order and assigning them sequentially as points in the permutations, Algorithm 1 is a self-explanatory and easily understood process which requires no formal proof.

As an example, let S_5 be the symmetric group on five points. All the conjugacy classes of S_5 therefore has the partitions

$$\{(1, 1, 1, 1, 1), (2, 1, 1, 1), (2, 2, 1), (3, 1, 1), (3, 2), (4, 1), (5)\}.$$

Then by using the mechanism explained for finding the representative for each of these conjugacy classes we find they are respectively

$$\{(1)(2)(3)(4)(5), (1\ 2)(3)(4)(5), (1\ 2)(3\ 4)(5), (1\ 2\ 3)(4)(5), (1\ 2\ 3)(4\ 5), (1\ 2\ 3\ 4)(5), (1\ 2\ 3\ 4\ 5)\}.$$

Usually for efficiency we omit the 1-cycles whenever writing a permutation without forgetting that they exist as 1-cycles. Therefore we finally find the class representatives as shown in Table 3.1.

$\lambda \in P(5)$	$(1,1,1,1,1)$	$(2,1,1,1)$	$(2,2,1)$	$(3,1,1)$	$(3,2)$	$(4,1)$	(5)
$f(\lambda)$	id	(1 2)(3)(4)(5)	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3)(4\ 5)$	$(1\ 2\ 3\ 4)$	$(1\ 2\ 3\ 4\ 5)$

Table 3.1: Partitions of 5 and corresponding class representatives of S_5

3.2 Coxeter Groups

Symmetric group S_n is same as Coxeter group of type A_{n-1} generated by transpositions $(i, i+1)$ for $i \in \{1, 2, \ldots, n-1\}$. It gives us a clear indication that this method of construction of irreducible representations can therefore be extended further in the Coxeter system.

Definition 3.2.1 ([10, **Coxeter System**]). *A Coxeter system is a pair* (W, S) *consisting of a group* W *and a set of generators* $S \subset W$*, subject only to relations of the form*

$$(ss')^{m(s,s')} = 1,$$

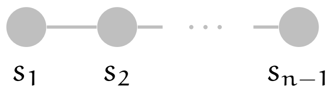
where $m(s,s) = 1$*,* $m(s,s') = m(s',s) \geqslant 2$ *for* $s \neq s'$ *in* S*. In case no relation occurs for a pair* s, s'*, we make the convention that* $m(s,s') = \infty$*. Formally,* W *is the quotient* F/N*, where* F *is a free group on the set* S *and* N *is the normal subgroup generated by all elements*

$$(ss')^{m(s,s')}.$$

The rank of (W, S) *is* $|S|$.

All finite Coxeter groups have corresponding Coxeter graphs which classify these groups. The classification of irreducible finite Coxeter groups is well-known and they can be found listed in [10]. They are of types $A_{n-1} (n > 2), B_n (n > 2), D_n (n > 4), E_6, E_7, E_8, F_4, H_3, H_4$ and $I_2(m)$.

Definition 3.2.2. *Coxeter group of type A_{n-1} is an irreducible finite group with the following Coxeter diagram*



A_{n-1} *($n \geqslant 3$)*

where the nodes are the generators s_i of the group such that $s_i = (i, i+1)$ and $i \in \{1, 2, \dots, n-1\}$.

These A_{n-1} groups are symmetries on n points. The nodes $s_i = (i, i+1)$ for $i \in \{1, 2, \dots, n-1\}$ are the transpositions $(1\,2), (2\,3), \dots, (n{-}1\;n)$ which generate the symmetric group S_n. An edge connecting any two of these generators s_i, s_j is labeled by $m_{i,j}$. There exsits an edge if and only if $m_{i,j} > 2$. For any $i, j \in \{1, 2, \dots n-1\}$ if $|i-j| = 1$ then $m_{i,j} = 3$, i.e., for all i we have $(s_i s_{i+1})^3 = 1$. For $|i-j| \neq 1$ we have $m_{i,j} = 2$, i.e., $(s_i s_j)^2 = 1$. Whenever $m_{i,j} = 2$, the generators s_i, s_j commute, i.e., $s_i s_j = s_j s_i$.

3.3 Partitions and Words

In this section of the thesis, our objective is to establish a connection between the representation theory of the symmetric group and elementary combinatorics. Here, we introduce the innovative notion of Specht matrices initially introduced in [18]. We will delve into several concepts outlined in the article, which will subsequently serve as fundamental components for this thesis.

A question in combinatorics is to determine the number of unique arrangements possible for the letters in a word. For example, let us consider the word ARRAY of length 5 with three distinct letters A, R and Y, and repetitions occurring for A and R twice each. The word be rearranged in 5! ways, but due to the repetitions of letters, we find the total number of rearrangements to be

$$\#\{\text{ rearrangements of ARRAY }\} = \frac{5!}{2! \cdot 2! \cdot 1!}.$$

The notion of reorganizing letters can be formally expressed as an action of the symmetric group. Symmetric group S_n acts from the right naturally on all words of length n, i.e., on a word with n letters by rearranging the letters. For one such group action on a word with n

letters, we find an orbit as a set consisting of all the rearrangements of one word. The action on a single orbit results into a transitive action of S_n.

Definition 3.3.1 ([16, Definition 7.1])**.** *An action of a group* G *on a set* X *is a homomorphism* $\sigma : G \longrightarrow S_X$ *where* S_X *is the symmetric group on the set* X.

Definition 3.3.2. *Let* $w = w_1 w_2 \ldots w_n$ *be a word of length* n *and let* $\sigma \in S_n$. *We define an action of* σ *on* w *by setting*

$$w \cdot \sigma = w_{1\cdot\sigma^{-1}} w_{2\cdot\sigma^{-1}} \cdots w_{n\cdot\sigma^{-1}}.$$

Then we can find the arrangements of the word ARRAY as an orbit of the right-action of the elements in S_5 on the letters of the word as shown

{ARRAY, ARRYA, ARYAR, ARYRA, AYARR, AYRAR, AYRRA, RAARY, RAAYR, RARAY, RARYA, RAYAR, RAYRA, RRAAY, RRAYA, RRYAA, RYAAR, RYARA, RYRAA, YAARR, YARAR, YARRA, YRAAR, YRARA, YRRAA, AARRY, AARYR, AAYRR, ARARY, ARAYR}.

The isomorphism type of a transitive action of any finite group is given by the stabilizer of one point, up to conjugacy. In case of S_n, the stabilizer of a word is a Young subgroup $S_{l_1} \times S_{l_2} \times \cdots \times S_{l_k} = S_\lambda$ where $(l_1, l_2, \ldots, l_k) = \lambda$ is a partition of n.

In the example S_5 acts on the rearrangements of word ARRAY. For the word ARRAY, we have that $S_2 \times S_2 \times S_1$ is isomorphic to the stabilizer subgroup of S_5 that corresponds to the word. So we obtain an isomorphism of S_5-sets:

$$\{\text{ rearrangements of ARRAY }\} \cong \frac{S_5}{S_2 \times S_2 \times S_1}.$$

One can associate a shape to the word w by counting the number of times the individual letters appear in w. This simply will give a partition of n. Let us call such a shape associated to a word w to be $\lambda(w)$. For example, $\lambda(ARRAY) = (2, 2, 1)$.

Conversely, for a given partition $\lambda \vdash n$, we construct a canonical word w_λ corresponding to it. We choose the letters to be natural numbers $1, 2, 3, \ldots$ rather than ordinary letters, each letter l in w_λ repeating as many times as the l-th number λ_l in the sequence λ. By choosing the letters in this way, w_λ (and any of its rearrangements) has the property that the letter 1 is the most frequent letter, letter 2 being the second most frequent letter and so on.

For example, if $\lambda = (2, 2, 1) \vdash 5$ then we have the canonical word as per our construction to be $w_\lambda = 11223$.

The word w =ARRAY corresponds to $\lambda = (2, 2, 1)$ as A and R are repeated twice and Y once. The canonical word is $w_\lambda = 11223$ where we are assigning the most frequent letter to 1, the second most frequent letter to 2 and so on, i.e., $A = 1, R = 2$ and $Y = 3$. We then obtain ARRAY$= 12213$, of which w_λ is the canonical form, another rearrangement of the letters of w.

Let G be any group and X, Y be two sets. We take S_X and S_Y to be the symmetric groups on the set X and Y. Two group actions $\sigma : G \longrightarrow S_X$ and $\tau : G \longrightarrow S_Y$ are isomorphic if there is a bijection $\psi : X \longrightarrow Y$ such that $\psi\sigma_g = \tau_g\psi$ for all $g \in G$. This gives us a bijection which is compatible with the action of the symmetric group. Therefore it shows how two permutation actions are isomorphic to each other. We are going to use this idea to prove the following proposition.

Proposition 3.3.3. *Let u and v be two words of same length n. Then as permutation action of S_n the rearrangements of u and the rearrangements of v are isomorphic if and only if the words u and v have the same shape, i.e., if and only if $\lambda(u) = \lambda(v)$.*

Proof. We have that u and v are two words of same length n. Then S_n acts on the words by rearranging the letters of each word. Let A be the set of all rearrangements of the word u and B be the set of all rearrangements of the word v.

Then we have that under the right action of S_n on each of the words u and v we have the orbits A and B respectively. Each of the words $u' \in A$ has the same shape as $\lambda(u)$ and likewise each of the words $v' \in B$ has the same shape $\lambda(v)$. Then without loss of generality we can also denote the sets A and B by $\lambda(u)$ and $\lambda(v)$ respectively.

Let $F : A \longrightarrow B$ be a bijection compatible with the action of the symmetric group S_n defined by $u' \mapsto F(u')$ where $u' \in A$ and $F(u') = v' \in B$.

$(\Rightarrow)$ Let F be an isomorphism from A to B. Then the cardinalities of A and B are same. In each of the sets we have a canonical word, say u_λ and v_λ respectively. As A and B are isomorphic, there is a correspondence between their canonical words. By the property of canonical words then we have that in each of the words the most frequent letter comes first followed by the second most frequent letter and so on. Now as the length of the words are same and their sets of rearrangements are isomorphic, then we have a one to one correspondence between the distinct letters of the words u_λ and v_λ.

Now by the permutation action of S_n, we have that $\mathrm{Stab}_{S_n}(u') = \mathrm{Stab}_{S_n}(F(u'))$ which implies that $\lambda(u') = \lambda$ and $\lambda = \lambda(F(u')) = \lambda(v')$ for some $\lambda \vdash n$. Therefore, $\lambda(u) = \lambda(v)$.

($\Leftarrow$) Conversely, let $\lambda(u) = \lambda(v) = (l_1, l_2, \ldots, l_k)$. Let the set of distinct letters of u and v be U and V respectively. Then there exist injections $\phi : U \longrightarrow \mathbb{N}$ and $\psi : V \longrightarrow \mathbb{N}$ so that all the distinct letters of u, v are mapped to letters. This implies that there is a bijection $\tau : U \longrightarrow V$ mapping each distinct letter of u to distinct letters of v so that $\tau\phi_{l_u} = \psi_{l_v}\tau$. Therefore the set of all rearrangements of u is isomorphic to the set of all rearrangements of v.

□

Definition 3.3.4, Definition 3.3.5 and Proposition 3.3.6 are three important concepts seen in [18] which will be used later on in describing important ideas in the following sections. More specifically, using these we will be introducing the idea of a histogram corresponding to a word.

Definition 3.3.4 ([18, Definition 1.2.]). *A diagram is a finite subset of* $\mathbb{N}^2$. *The elements of a diagram are called boxes.*

Definition 3.3.5 ([18, Definition 1.3.]). *Given a partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$, *the diagram associated to* λ *is*

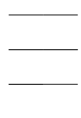
$$D(\lambda) = \{(i, j) | 1 \leqslant j \leqslant \lambda_i\},$$

where, by convention, $\lambda_i = 0$ *if* $i > k$.

Proposition 3.3.6 ([18, Proposition 1.4.]). *A diagram* D *is the diagram of some partition* λ *if and only if* D *is closed under coordinate-wise* $\leqslant$. *In other words,* $D = D(\lambda)$ *for some* λ *if and only if, for any* $(i, j) \in D$ *and any* (a, b) *with* $1 \leqslant a \leqslant i$ *and* $1 \leqslant b \leqslant j$, $(a, b) \in D$.

We say that a diagram is a *Young diagram* if it corresponds to some partition. For each word in the orbit of w_λ corresponding to the partition λ, we can think of a natural way to visualize it with the help of a diagram. The idea is that for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we have a diagram, the rows of which are corresponding to the distinct letters of the word w with the length of each row being λ_i, the multiplicities of each letter i. Therefore for each partition λ we have a diagram corresponding to it such that it has same number of rows as the length of λ and the i-th row in the diagram has length λ_i.

As an example if $\lambda = (2, 2, 1)$ then the corresponding diagram is as follows.



By definition a *histogram* is a diagram that represents the frequency of the data points in a data set. The choice of the word histogram here in this context is not exactly same as how histograms are perceived in statistics or data science. In our context we are concerned with the words. The numbers in a histogram indicate the positions of the letters in the corresponding word.

Definition 3.3.7 (Histogram). *Let* w *be a word considered as a function* $w : \{1, 2, 3, \ldots, n\} \to \mathbb{N}$. *Then histogram, denoted by* h_w *of the word* w *is the set* $\{w^{-1}(1), w^{-1}(2), w^{-1}(3), \ldots\}$ *of pre-images of its distinct letters.*

As an example, for $\lambda = (2, 2, 1)$ the canonical word $w_\lambda = 11223$ has letter 1 repeated in first and second positions, letter 2 in third and fourth positions and letter 3 is in fifth position. This gives us the histogram $h_{w_\lambda} = \{w_\lambda^{-1}(1), w_\lambda^{-1}(2), w_\lambda^{-1}(3)\} = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Whereas, ARRAY $\mapsto$ $12213 = w$ is a rearrangement of w_λ. It can be seen that in w, letter A $\mapsto 1$ appears in first and fourth positions, letter R $\mapsto 2$ in second and third positions and letter Y $\mapsto 1$ in fifth position. Then the set of distinct letters with its repetitions for w is $\{1^2, 2^2, 3^1\}$ and we find $h_w = \{\{1, 4\}, \{2, 3\}, \{5\}\}$. Therefore the histograms for w_λ and w are as shown in Figure 3.1.

1	1	2
2	3	4
3	5	

(a) $w_\lambda = 11223$ histogram

A $\to$ 1	1	4
R $\to$ 2	2	3
Y $\to$ 3	5	

(b) $w = 12213$ histogram for ARRAY

Figure 3.1: Histograms corresponding to words $w_\lambda = 11223$ and $w = 12213$

If w is some w_λ or any of its rearrangements then its histogram has the property that it is a sequence of subsets of $\{1, 2, \ldots, n\}$ decreasing in size, which is nicely visualized by a diagram that sometimes is also called a row tabloid, with rows filled by the numbers $1, 2, 3, \ldots$ which decide the positions of letters that occur in w as can be seen in Figure 3.1.

For a given histogram, one can recover the word w corresponding to it. Let the set of distinct letters of w is $\{1, 2, \ldots, l\}$. For every word w of length n we have a corresponding histogram, h_w, the rows of which are labelled by the distinct letters i of w. Each row $h_w(i)$ of

h_w has some elements which is a subset of $\{1, 2, \ldots, n\}$ and $\bigcup_{i=1}^{l} h_w(i) = \{1, 2, \ldots, n\}$. These subsets $h_w(i)$ describe which position in the word w the letter i occupies. Therefore, moving through all the distinct letters i by picking the subsets $h_w(i)$ and filling the corresponding positions of w by i accordingly until all of n positions are exhausted would give us back the word w from h_w.

As an example if we are given the diagram from Figure 3.1(b) then we can retrieve the word w corresponding to it using the letters. We have that in the histogram, $h_w(1) = \{1, 4\}$, $h_w(2) = \{2, 3\}$ and $h_w(3) = \{5\}$, which implies that in the word w letter 1 occupies the positions 1 and 4, letter 2 occupies the positions 2 and 3 and letter 3 occupies the position 5. Therefore, $w = 12213$ comes out naturally from the histogram.

Another aspect of this correspondence between rearrangements of w_λ and their histograms is that the action of S_n on this orbit can now be expressed in terms of histograms. A natural question that follows is in the complete set of all rearrangements of a word, how are the different rearrangements in this set ordered? The natural ordering that arises here is the lexicographic ordering.

If w is a word having shape λ and w' is a rearrangement of the letters of w, then we say that $w < w'$ lexicographically if the value of w is less that the value of w' i.e., w appears earlier than w' in the *lexicographic ordering* of the arrangements of w. This clearly implies that the canonical word w_λ appears first in the list of all rearrangements of it.

3.4 Tableaux and Pair of Words

We understand now the natural way of how diagrams are associated with a partition of a natural number n and each of the rows of these diagrams has a length. If we assign an empty box to each of these positions in each row, then we have a total of n empty boxes in the diagram. Now filling the n boxes of a diagram of shape λ with the numbers $\{1, 2, 3, \ldots, n\}$ gives us a Young tableau.

Definition 3.4.1 ([16, Definition 10.1.11: **Young Tableau**]). *Let $\lambda \vdash n$. We define a λ-tableau (or Young tableau of shape λ) to be an array t of integers obtained by placing $1, \ldots, n$ into the boxes of the Young diagram for λ. There are clearly $n!$ λ-tableaux.*

Definition 3.4.2 (**Standard Young Tableau**). *Let $\lambda \vdash n$. A tableau t of shape λ is said to be standard if the integers in the boxes are increasing along each row and each column.*

From now on we will use the abbreviation *SYT* for standard Young tableau. Among all the different arrangements of the numbers $\{1, 2, \dots, n\}$ in a λ-tableau, we can only find a handful of SYT of shape λ.

Definition 3.4.3 (Canonical Tableau). *The canonical tableau t_λ of shape λ is a SYT obtained by filling the boxes of a Young diagram row by row with non-repeated integers $1, 2, \dots, n$ in that sequence such that they are increasing to the right on each rows.*

In this process we automatically get an arrangement of the numbers in the boxes of the tableau such that the columns are increasing to the bottom. For example, if $\lambda = (2, 2, 1)$ then the corresponding canonical Young tableau is

$$t_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}.$$

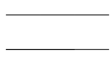
Symmetric group S_n acts from the right naturally on the numbering of the tableau by permuting them accordingly. Then by the right-action of S_n on the canonical tableau we can find all the possible numbering of the diagram. For example, $(1\ 2\ 3) \in S_5$ acts on the tableau mentioned above as follows

$$t_\lambda \cdot (1\ 2\ 3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \cdot (1\ 2\ 3) = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

giving us another numbering of the diagram of same shape. Hence, if the canonical tableau of some shape λ corresponds to the id $\in S_n$ then we can assign each tableau to a permutation in S_n described by the right-action of the permutation on the canonical tableau of the same shape.

Definition 3.4.4 ([16, Definition 10.1.5: **Conjugate Partition**]). *If $\lambda \vdash n$, then the conjugate partition λ^T of λ is the partition whose Young diagram is the transpose of λ. i.e., the Young diagram λ^T is obtained from the diagram of λ by exchanging rows and columns.*

For each partition λ of a number n, we have a corresponding word w and each w gives rise to a histogram. For each of the partitions $\lambda \vdash n$, we also have a corresponding conjugate partition λ^T. The diagram corresponding to the conjugate partition is the transpose of the diagram for λ. As an example, $\lambda = (2, 2, 1)$ then the diagram corresponding to the conjugate partition is



implying that $\lambda^T = (3,2)$.

Now as each of the partitions λ has a word corresponding to it, likewise we have a word corresponding to the conjugate partition λ^T. Therefore, the canonical word for $\lambda^T = (3,2)$ is $w_{\lambda^T} = 11122$. Likewise, there is a histogram associated to the word w_{λ^T} as shown below:

1	1	2	3
2	4	5	

Definition 3.4.5 (Intersection Histogram). *Let for two words* w_1 *and* w_2 *of same length* n*, their corresponding histograms be* h_{w_1} *and* h_{w_2} *respectively. We define the intersection histogram* $h(i,j) = h_{w_1}[i] \cap h_{w_2}[j]$ *where* $h_{w_1}[i]$*,* $h_{w_2}[j]$ *are subsets of* h_{w_1}*,* h_{w_2} *having indices* i *and* j *respectively.*

Such intersection histograms h may not always appear to be a histogram of a Young tableau. For the purpose of further build-up of the Specht modules by the construction of Specht matrices, we are interested in the intersection histograms that have a shape of a partition $\lambda \vdash n$, giving us a Young tableau of shape λ.

Example 3.4.6. Let two words of length $n = 5$ be $w_1 = 11223$ and $w_2 = 12113$. Then we have two histograms corresponding to these words, say h_{w_1} and h_{w_2} respectively. Then we have that $h_{w_1} = \{\{1,2\},\{3,4\},\{5\}\}$ and $h_{w_2} = \{\{1,3,4\},\{2\},\{5\}\}$ respectively. If we consider h to be the intersection histogram $h_{w_1} \cap h_{w_2}$, then we have

$$h(1,1) = \{1,2\} \cap \{1,3,4\} = \{1\}$$

$$h(1,2) = \{1,2\} \cap \{2\} = \{2\}$$

$$h(1,3) = \{3,4\} \cap \{5\} = \varnothing$$

$$h(2,1) = \{3,4\} \cap \{3,4\} = \{3,4\}$$

$$h(2,2) = \{3,4\} \cap \{2\} = \varnothing$$

$$h(2,3) = \{3,4\} \cap \{5\} = \varnothing$$

$$h(3,1) = \{5\} \cap \{1,3,4\} = \varnothing$$

$$h(3,2) = \{5\} \cap \{2\} = \varnothing$$

$$h(3,3) = \{5\} \cap \{5\} = \{5\}$$

and therefore we have the intersection histogram h as shown in Figure 3.2.

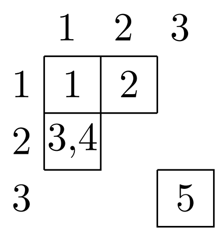


Figure 3.2: Intersection histogram for the pair of words $(11223, 12113)$

Clearly h does not give a shape that correspond to any partition of n implying that h is not a Young tableau. Also, we can notice that we have two entries in the box with coordinate $(2, 1)$ and the boxes above and before the box $(3, 3)$ corresponding to the coordinates $(1, 3), (2, 3), (3, 1), (3, 2), (2, 2)$ are all empty.

Definition 3.4.7 (Simultaneous histogram). *A simultaneous histogram h is a histogram in the intersection of two histograms corresponding to two different words u and v of same length n so that the following properties hold:*

1. *each box in h contains exactly one entry from $\{1, 2, \ldots, n\}$,*

2. *the shape of h is a diagram that represents a partition of n.*

Consider a pair of words (w_1, w_2) where w_1 and w_2 are both of length n and $\lambda(w_1), \lambda(w_2) \vdash n$. A simultaneous histogram is in the intersection of the histograms corresponding to words w_1 and w_2 such that it retains the shape of a Young diagram. The number of boxes along the rows and columns in the histogram correspond to the multiplicities of the distinct letters of the words w_1 and w_2 respectively. If we obtain the intersection histogram to be a Young tableau then we say that it is a simultaneous histogram. In order to achieve a tableau as the intersection histogram, we must have that the shapes of the associated words are conjugates to each other as shown in the following.

Proposition 3.4.8 ([18, Theorem 1.6])**.** *Let u and v be two words of same length, and h be the intersection histogram of their corresponding histograms, then it must be that $\lambda(u) = \lambda(v)^T$.*

Proposition 3.4.8 is an alternate to Theorem 1.6. in [18], however, the proofs of them are same.

Now let us add another layer of complexity by taking a pair of words (w_1, w_2) of same length n and consider the S_n-set

$$\{\text{rearrangements of } w_1\} \times \{\text{rearrangements of } w_2\}$$

where S_n acts diagonally on the two factors.

Example 3.4.9. Let for $n = 5$ and consider the words $w_1 = 11223$ and $w_2 = 11122$. Now if we take the pair (w_1, w_2) then we see that

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

does not have a free orbit as the column $\binom{1}{1}$ is repeated twice. To achieve a simultaneous histogram we must take a rearrangement of one of the words so that there is no repeated columns.

Let us consider $w_2' = 12121$. Now the pair

$$\begin{pmatrix} w_1 \\ w_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 2 & 1 \end{pmatrix}$$

does not have any repeated column, implies we can make the simultaneous histogram for it as shown in Figure 3.3.

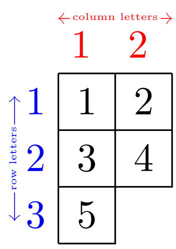


Figure 3.3: Simultaneous histogram of the words pair $(11223, 12121)$

It can be clearly seen that the shape of the simultaneous histogram is $\lambda(w_1) = (2, 2, 1)$ whereas the shape of the transpose of the histogram is $\lambda(w_2) = (3, 2) = \lambda(w_1)^T$. The distinct letters of the words w_1 and w_2' are corresponding to the number of boxes of the histogram along the rows and columns respectively as explained before.

$$\begin{array}{r|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline w_1 \rightarrow & 1 & 1 & 2 & 2 & 3 \\ w_2' \rightarrow & 1 & 2 & 1 & 2 & 1 \end{array}$$

The numbering of the boxes depends on the position of the pair of letters in each column of the words pair in the sequence of the columns. We can see in the illustration above that the number 1 corresponds to the letter pair $(1, 1)$ which is the coordinate of the box in the diagram. Similarly, 2 corresponds to the pair $(1, 2)$ and therefore it is the coordinate of the box containing 2 and so on. At some point in this way we will have exhausted all the integers $\{1, 2, \ldots, 5\}$ filled in the boxes of the histogram.

Now we can describe the S_n-action on the pairs of words. Let u, v be two words of length n and U, V be the sets of all their rearrangements respectively such that $U = u \cdot S_n$ and $V = v \cdot S_n$. Then in a similar way we can consider the S_n-action on the pair (u, v), which eventually will give us the set $U \times V$, all possible rearrangements of the columns of (u, v).

Definition 3.4.10 ([18, Definition 1.4]). *Two words w_1, w_2 of same length n are said to have complementary rearrangements if the diagonal action of S_n on the product*

$$\{\text{ rearrangements of } w_1 \} \times \{\text{ rearrangements of } w_2 \}$$

has a unique free orbit. If (w_1, w_2) is in the free orbit, then the words w_1 and w_2 are complementary.

That means, we obtain a unique free orbit if and only if in the pair of words there is no repeated columns. Also, w_1, w_2 have complementary rearrangements if and only if there exists a simultaneous histogram for this pair.

In other words, (w_1, w_2) has a simultaneous histogram if and only if $\lambda(w_1)^T = \lambda(w_2)$ and the elements in the i-th row of h_{w_1} has the same elements as in the i-th column of the h_{w_2}.

Theorem 3.4.11. *Let v and w be two words of length n. Then the action of S_n on the pairs (v', w'), where v' and w' are some rearrangements of v and w respectively, has a unique free orbit if and only if $\lambda(w)$ is the transposed partition of $\lambda(v)$ i.e., $\lambda(v)^T = \lambda(w)$.*

Proof. Let us assume that the canonical words v, w are of the form w_λ, w_μ for partitions $\lambda, \mu \vdash n$ respectively.

Let v', w' be some rearrangements of w_λ and w_μ respectively for which we have a simultaneous histogram h. This simultaneous histogram can be considered as a matrix with its rows labelled by the letters i in v', and columns labelled by the letters j in w', and entry $h(i, j)$ is $v'^{-1}(i)$ intersected with $w'^{-1}(j)$ where $v'^{-1}(i)$ and $w'^{-1}(j)$ are the histograms for single words v' and w' respectively. So, h is a collection of pairwise disjoint subsets of $\{1, 2, \ldots, n\}$ whose union is $\{1, 2, \ldots, n\}$.

Conversely, given a simultaneous histogram h for any rearrangements of w_λ and w_μ, the pair of words (v', w') can be recovered. The rows and columns of h are labelled by distinct letters i and j of v and w respectively. One can consider the row tabloid $\{h\}$ and column tabloid $[h]$ of h. Then we have that $\{h\} = v'^{-1}(i)$, the histogram for v' and $[h] = w'^{-1}(j)$, the histogram for w'. Therefore using the subsets corresponding to each of the letters $i \in \{h\}$, it is an easy exercise to find out the word v' as explained in Section 3.3 of this chapter. Similarly, using the subsets corresponding to the letters $j \in [h]$, word w' can be retrieved.

The problems of failures of finding the pair (v', w') are as follows:

(F_1) the subset at position (i, j) can have more than one element

(F_2) the subset at position (i, j) can be empty, while the one immediately to its right or below is nonempty

Let us now address the problems mentioned one by one.

Failure (F_1) clearly arises from a repeated column: the pair of letters (i, j) occurs more than once in the pair (v', w') and so the word pair does not lie in a free orbit. It can, therefore, be avoided by simply choosing the pair of words (v', w') in a way so that there is no repeated columns.

Failure (F_2) arises from the recursive property of the most frequent letter of word v' (i.e., letter 1) having to be paired up with every letter occurring in the word w', and ultimately relate to the uniqueness of free orbits. If the positions (i, j) of nonempty boxes give a diagram corresponding to a partition then the diagram is a Young diagram. If we obtain a Young diagram for the pair of words, then then we can avoid this failure.

If neither of (F_1) or (F_2) occurs, then each subset in the simultaneous histogram contains at most one element (no repeated columns), and the diagram formed by the coordinates (i, j)

of nonempty subsets in the simultaneous histogram of (v', w') is $\leqslant$-closed in the sense of Proposition 3.3.6 (and so it is a diagram of a partition λ), where each nonempty subset has exactly one element (and so the simultaneous histogram in this case is a Young tableau).

Therefore, as every distinct letter of w' pairs up with the most frequent letter 1 of v', followed by distinct letters of w' from the remaining letters pairs up with second most frequent letter 2 of v' and so on until all the letters are exhausted for each of the word, we can conclude that shape of w' is the transpose of shape of v'. Hence $\mu = \lambda^T \implies \lambda(w) = \lambda(v)^T$. □

Definition 3.4.12 (Canonical Pair). *Let (v, w) the corresponding pair of complementary words for the canonical tableau having shape of a partition λ. Then (v, w) is the canonical pair if we take v as it is but take w in such a way that all the distinct letters of w appear once each in strictly increasing order and then they repeat in the same way for the remaining letters until all their multiplicities are exhausted.*

It is clear that there can be only one canonical pair of words for each $\lambda \vdash n$. However, there are overall $n!$ ways to pair up two words of same length corresponding to a partition and its conjugate partition so that each of them correspond to a Young tableau. This can be seen in Example 3.4.9 how we have chosen the pair (w_1, w_2'), which is canonical. Simultaneous histograms are therefore, clearly, Young tableaux of shape $\lambda \vdash n$ with associated pair of words that define it. Different simultaneous histograms or tableaux can be achieved from the canonical pair of words as a direct result of the right action of S_n on the columns of the pair. There are as many boxes in such a histogram as the number of distinct, non-repeated columns in the pair of words. Hence, in such a way we will have all the possible $n!$ tableaux achieved. The following corollaries arise as direct implications of Theorem 3.4.11.

Corollary 3.4.13. *Let U and V be the set of all arrangements of complementary words u and v respectively. Then for $u' \in U$ and $v' \in V$, the pair (u', v') is in the free orbit if and only if the intersection histogram of u', v' is a tableau.*

Corollary 3.4.14. *Let $\lambda \vdash n$. The function $h : (v, w) \mapsto$ simultaneous histogram of (v, w) establishes a bijection between the elements of the unique free S_n-orbit on the set $\{$rearrangements of $w_\lambda\} \times \{$rearrangements of $w_{\lambda^T}\}$ and the set of tableaux of shape λ (with inverse function given by the recovery process described in Theorem 3.4.11).*

Now, we are going to look into an example where we may sometimes get a tableau at the intersection of two tableaux of same shape. One can also come across this concept in [14].

Let us consider two SYTs of shape $\lambda = (2, 2, 1)$,

$$t_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

respectively. Now, recalling the idea of single words and their histograms, for t_1 we have the row stabilizer subsets as $\{t_1\} = \{\{1, 4\}, \{2, 5\}, \{3\}\}$, and similarly for the transposed t_2 we have $\{t_2^T\} = \{\{1, 3, 5\}, \{2, 4\}\}$ (which is nothing but the column stabilizer of t_2, i.e., $[t_2]$). For two tableaux of same shape to have an intersection tableau, we consider the histogram of one tableau and the histogram of the transpose of the other tableau. Then, indices of the subsets in $\{t_1\}$ and $\{t_2^T\}$ decide the row index and the column index of the elements in the intersection tableau respectively. Let us denote the intersection tableau by t such that $t(i, j) = \{t_1\}[i] \cap \{t_2^T\}[j]$. Therefore we have

$$t(1, 1) = \{1, 4\} \cap \{1, 3, 5\} = \{1\}$$
$$t(1, 2) = \{1, 4\} \cap \{2, 4\} = \{4\}$$
$$t(2, 1) = \{2, 5\} \cap \{1, 3, 5\} = \{5\}$$
$$t(2, 2) = \{2, 5\} \cap \{2, 4\} = \{2\}$$
$$t(3, 1) = \{3\} \cap \{1, 3, 5\} = \{3\}$$
$$t(3, 2) = \{3\} \cap \{2, 4\} = \varnothing$$

which eventually gives us

$$t = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

We call such nonstandard tableau an ***inter-tableau*** as it is in the intersection of two SYT.

Now if we just change the order of the tableau and look into the row stabilizer of t_2 and the column stabilizer of t_1, then we find that $\{1, 2\} \cap \{1, 2, 3\} = \{1, 2\}$. Therefore, if some intersection tableaux s exists for $s(i, j) = \{t_2\}[i] \cap \{t_1^T\}[j]$, then $|ss(1, 1)| = |\{1, 2\}| \neq 1$. Hence no intersection tableau exists for such a pair.

Therefore an interesting fact that can be noticed in the last two examples, and is applicable in general, is that if there exists a inter-tableau in the intersection of two Young tableaux t_1 and t_2, then the order may matter. Hence, the following proposition.

Proposition 3.4.15. *Let* t_1 *and* t_2 *be two tableaux of shape* λ. *Then there exists an inter-tableau* s *such that* $s(i,j) = \{t_1\}[i] \cap \{t_2^T\}[j]$ *if and only if for any* (i,j) *we have that* $|s_{(i,j)}| = 1$ *where* (i,j) *is a position of a box in the Young tableau of shape* λ.

Later in Section 3.5 we will find that in the process of constructing the irreducible representation corresponding to any partition λ, words play a very important role.

3.5 Specht Objects

In this section we build the irreducible representation corresponding to a partition in a new way. In order to make things simple and free-flowing, we are going to introduce a different way to look at the Young tableau.

Definition 3.5.1 (Flat-tableau). *Let* t *be a Young tableau of shape* λ *where* $\lambda \vdash n$. *We say that flat-tableau* $t^\flat$ *is a list filled by numbers picked from the tableau* t *row by row in that order.*

Therefore a flat-tableau $t^\flat$ is nothing but a list consisting of just the numbers $\{1, 2, \ldots, n\}$ in the order how they are seen in the boxes of tableau t if read from left-to-right of each row, one after another. As an example, if we have the tableau

$$t = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 5 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

then the flat-tableau of t is $t^\flat = [1, 4, 5, 2, 3]$.

Flat-tableau corresponding to the canonical tableau of shape λ is $t^\flat_\lambda = [1, 2, \ldots, n]$. Let $t^\flat$ be any flat-tableau of shape λ. Then by comparing the two flat-tableaux $t^\flat_\lambda$ and $t^\flat$ we can retrieve the permutation $\sigma \in S_n$ such that $t^\flat_\lambda \cdot \sigma = t^\flat$. If we put the list of numbers $t^\flat_\lambda$ in the first row of a matrix and $t^\flat$ in the second row of the matrix, we retrieve a two-line notation of the permutation σ.

Example 3.5.2. For $\lambda = (2, 2, 1) \vdash 5$ if $t^\flat = [1, 4, 5, 2, 3]$ be a flat-tableau of same shape, then we have

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix} = (2\,4)(3\,5)$$

such that $[1, 2, 3, 4, 5] \cdot (2\,4)(3\,5) = [1, 4, 5, 2, 3]$.

Therefore, flat-tableaux are very useful to understand the permutation action of the symmetric group on a tableau. As flat-tableau is simply a list of numbers $\{1, 2, \ldots, n\}$ in some order, a permutation action on a tableau changes the ordering of the numbers in the set.

Definition 3.5.3. *Let* t *be a tableau of shape* $\lambda \vdash n$ *and* t_λ *be the canonical tableau. Let* $t^\flat$ *and* $t^\flat_\lambda$ *be the flat-tableaux corresponding to* t *and* t_λ *respectively. If for* $\sigma \in S_n$ *we have* $t^\flat_\lambda \cdot \sigma = t^\flat$ *then we define* $\mathrm{sgn}(t^\flat) = \mathrm{sgn}(\sigma)$.

Definition 3.5.4 (Young character). *Let* $t^\flat_\lambda$ *be the canonical flat-tableau for the tableau* t *of shape* $\lambda \vdash n$*. For* $\sigma \in S_n$ *if we have another flat-tableau of same shape* $t^\flat = t^\flat_\lambda \cdot \sigma$*, then we define the Young character denoted by* $Y(t)$ *as* $Y(t) = \mathrm{sgn}(t^\flat)$.

Therefore, for each of the flat-tableau $t^\flat$ we have a corresponding tableau t, and for each t we have a pair of words. Depending on the permutation σ that changes (w_1, w_2) to (w'_1, w'_2) corresponding to $t^\flat_\lambda$ and $t^\flat$ respectively, the Young character gives value 1 when σ is even and -1 when σ is odd.

Definition 3.5.5 ([18, Definition 1.11])**.** *For any* n*, the representation* $\varepsilon : S_n \to GL(V)$ *is the one dimensional vector space* V *on which* S_n *acts by sign. This means, for* $v \in V$, $g \cdot v = v$ *if* g *is an even permutation and* $g \cdot v = -v$ *if* g *is an odd permutation.*

So we can consider an alternate notation of the tableau $t = (w'_1, w'_2)$ to indicate the pair of words connected to the tableau. Then Young character can be written as $Y(w'_1, w'_2)$ to indicate the underlying pair of words. We must also decide on a convention of what we get as a Young character when two words w_1 and w_2 are not complementary.

Definition 3.5.6. *If a word pair* (w'_1, w'_2) *is in free orbit then the Young character* $Y(w'_1, w'_2) = Y(t)$ *is either* 1 *or* -1 *depending on the underlying permutation, else* $Y(w'_1, w'_2) = 0$.

For any partition λ and its conjugate partition λ^T, if the words are w_1 and w_2 respectively then we can create a Specht matrix $\varphi(\lambda)$ of which the rows are labelled by all the rearrangements of the word w_1 and columns by all the rearrangements of the word w_2. We call the word w_1 as *row word* and the word w_2 as *column word* of the Specht matrix $\varphi(\lambda)$.

Definition 3.5.7 ([18, Definition 2.4.5: **Specht Matrix**])**.** *Let* $\lambda \vdash n$*. Let* $w_1 = w_\lambda$ *and* $w_2 = w_{\lambda^T}$ *be the complementary canonical words of shapes* λ *and* λ^T *respectively. Let* A *and* B *be the sets of all rearrangements of* w_1 *and* w_2 *respectively. Then we define the Specht* *matrix* $\varphi(\lambda) : A \times B \to \{0, \pm 1\}$ *defined by* $(w_1', w_2') \mapsto Y(w_1', w_2')$ *where* $Y(w_1', w_2') = Y(t)$ *only if* (w_1', w_2') *corresponds to the tableau* t.

Then we have that the entries in the matrix belong to the set $\{-1, 0, 1\}$. Each entry corresponds to a pair of a row word and a column word. There are $|\{\text{arrangements of } w_1\}| \times |\{\text{arrangements of } w_2\}|$ elements in total in the matrix but only for the nonzero entries we have a tableau corresponding to it. This is the case due to repeated columns occurring in the pair of row and column words resulting in no unique free orbit and so the respective element in the matrix is set to be 0. Therefore we have exactly $n!$ nonzero elements in the matrix.

Now let us have a look at an example of a Specht matrix for an S_n-group.

Example 3.5.8. Let us consider for $n = 4$, partition $\lambda = (2, 1, 1)$ and its conjugate partition $\lambda^T = (3, 1)$. Then canonical words are $u_\lambda = 1123$ and $v_{\lambda^T} = 1112$ respectively. It can be seen that the pair $(u_\lambda, v_{\lambda^T})$ is not a canonical pair and therefore we choose the row and column words $u_\lambda = 1123$ and $v' = 1211$ from the canonical pair, which corresponds to the canonical simultaneous histogram of shape λ.

Let $w_1 = u_\lambda$ and $w_2 = v'$. The Specht matrix $\varphi(\lambda)$ is as shown in Table 3.2. It can be clearly noticed that the rows of the Specht matrix is labelled by different rearrangements of the row word 1112 and columns are labelled by different rearrangements of the column word 1211, in lexicographical order.

	1112	1121	1211	2111
1123	0	0	1	-1
1132	0	0	-1	1
1213	0	-1	0	1
1231	1	0	0	-1
1312	0	1	0	-1
1321	-1	0	0	1
2113	0	1	-1	0
2131	1	0	1	0
2311	1	-1	0	0
3112	0	-1	1	0
3121	1	0	-1	0
3211	-1	1	0	0

Table 3.2: Specht matrix $\varphi((2, 1, 1))$

Let us denote the column space of the Specht matrix $\varphi(\lambda)$ by S^λ for $\lambda \vdash n$. We claim that S^λ is an irreducible module of S_n. We are going to prove this in a series of steps as follows.

Let A be the set of all the rearrangements of w_λ. Then for each partition λ of n, we define a permutation module M^λ. A basis of this module M^λ is described by the set A.

Definition 3.5.9 (Permutation Module). *Let $\lambda \vdash n$ and w_λ be the canonical word of shape λ. We define the permutation module*

$$M^\lambda = \langle q_w : w \in A \rangle_{\mathbb{C}}$$

as a vector space over $\mathbb{C}$, where A is the set of all rearrangements of the word w_λ.

Let ν be a vector in M^λ. Then we have that $\nu = \sum_{w \in A} \alpha_w q_w$ for $\alpha_w \in \mathbb{C}$. Let $\sigma \in S_n$. Then we have that

$$\nu \cdot \sigma = (\sigma_{w \in A} \alpha_w q_w) \cdot \sigma = \sigma_{w \in A} (\alpha_w q_w) \cdot \sigma = \sum_{w \cdot \sigma \in A} \alpha_{w \cdot \sigma} q_{w \cdot \sigma} = \nu_\sigma.$$

It shows that M^λ is a S_n-module. Therefore we have it as follows.

Proposition 3.5.10. *Let $\lambda \vdash n$ and M^λ be a permutation module. Then M^λ is a S_n-module.*

Theorem 3.5.11 ([18, Theorem 1.19])**.** *Let S^λ be the column-span of the Specht matrix (as a subspace of M^λ). Then the module S^λ is an irreducible module of S_n.*

Let t be a tableau of shape $\lambda \vdash n$ and $\mathbb{C}[S_n]$ be group ring. Let $R(t)$ and $C(t)$ be respectively the row and column stabilizers of the tableau t. Let us define the formal sums

$$p_t = \sum_{p \in R(t)} p, \; c_t = \sum_{c \in C(t)} \overline{c}$$

where $\overline{c} = \mathrm{sgn}(c) \cdot c$. Then product $p_t c_t$ is an element of $\mathbb{C}[S_n]$. Now let $\{t\}$ be the row tabloid of t. Then we have

$$\{t\} = \sum_{\sigma \in R(t)} t \cdot \sigma = t \cdot \sum_{\sigma \in R(t)} \sigma = t \cdot p_t.$$

In the sense of a Specht matrix $\varphi(\lambda)$, for any tableau t of shape λ, which as a simultaneous histogram is connected to a nonzero element in the matrix, we have that p_t is simply the sum of all the tableaux in the same row as t. Likewise, c_t is the signed sum of all the tableaux in the same column as that of t, i.e., by adding all the nonzero tableaux in the column of t with the corresponding matrix elements being their coefficients we get c_t.

In terms of $\{t\}$, it is nothing but all the tableaux obtained by rearranging the elements in the rows of the tableau t, which has no effect on the positions of the distinct letters in the corresponding row word, say w_1'. In $\varphi(\lambda)$ all entries in the same row of t are corresponding to the row tabloid $\{t\}$, and therefore, in simpler term we can indicate that for any row word $w_1' \in A$ we have a corresponding $\{t\}$. We can now safely say that all the entries corresponding to the row of t in $\varphi(\lambda)$ can be denoted by $Y(w_1', w_2 \cdot \sigma)$ for all $\sigma \in R(t)$.

Following [6] we define a polytabloid for a tableau t given by

$$\nu_t = \{t\} \cdot c_t.$$

We have $\nu_t = t \cdot p_t c_t$. If $t = t_\lambda$ then $\nu_\lambda = p_\lambda c_\lambda$ where $\nu_\lambda, p_\lambda, c_\lambda$ are ν_t, p_t, c_t corresponding to t_λ. By a permutation action of the column stabilizer on $\{t\}$ we obtain the elements in the same column of the tableau t rearranged and so we obtain the new row tabloids $\{t \cdot \pi\}$ such that $\pi \in C(t)$.

Let t be of shape λ. The span of the polytabloids ν_t is called *Specht module* corresponding to λ. Clearly it is a submodule of M^λ. This definition of Specht module can be found in any literature on symmetric group representations, for instance in Section 7.2 of [6].

Definition 3.5.12. *Let w_1, w_2 be two words having complementary rearrangements of shape λ and corresponding to a Specht matrix $\varphi(\lambda)$. We define the elements*

$$\nu_{w_2} = \sum_{w_1'} Y(w_1', w_2) q_{w_1'}$$

where the sum runs over the rearrangements w_1' of w_1.

If we take the row tabloid $\{t\}$ with its corresponding element as its coefficient and let the column stabilizer act on the histogram by permuting all the elements in the same column and consider those histograms with their signs, then we obtain ν_t. In a simpler way, if we consider ν_t to be a vector in the permutation module M^λ, then in terms of Specht matrix we can simply consider it by recording all the coefficients of the tabloids in a sequence. Therefore, ν_t is up to a sign same as ν_{w_2} which is the column corresponding to t with the column label w_2 in the Specht matrix.

Definition 3.5.13. *For any partition $\lambda \vdash n$ we define the module S^λ as the column space of the Specht matrix $\varphi(\lambda)$ to be the subspace of M^λ spanned by ν_{w_2}.*

Proposition 3.5.14. *For a partition λ of n we have that $v_{w_2} = \mathrm{sgn}(t^\flat) \cdot v_t$.*

Proof. Let t be a tableau of shape λ, then we have

$$v_t = \{t\} \cdot c_t = \{t\} \cdot \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma)\sigma = \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma)\{t \cdot \sigma\} = \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma)\{t'\}.$$

From the discussion above we can consider a column of the matrix $\varphi(\lambda)$ by its column label w_2. If tableau t corresponds to the word pair (w_1, w_2) then $t' = t \cdot \sigma$ for $\sigma \in C(t) = \mathrm{Stab}(w_2)$ we have that t' corresponds to the pair (w_1', w_2).

By definition we have $Y(w_1', w_2) \neq 0$ if and only if the pair (w_1', w_2) corresponds to a tableau t'. As $(t \cdot \sigma)^\flat = t^\flat \cdot \sigma$, we have that

$$Y(w_1', w_2) = \mathrm{sgn}(t^{\flat'}) = \mathrm{sgn}((t \cdot \sigma)^\flat) = \mathrm{sgn}(t^\flat \cdot \sigma) = \mathrm{sgn}(t^\flat) \cdot \mathrm{sgn}(\sigma).$$

Therefore we have that

$$\begin{aligned}
v_{w_2} &= \sum_{w_1'} Y(w_1', w_2) q_{w_1'} \\
&= \sum_{w_1'} \mathrm{sgn}(t^\flat) \cdot \mathrm{sgn}(\sigma) q_{w_1'} \\
&= \mathrm{sgn}(t^\flat) \cdot \sum_{w_1'} \mathrm{sgn}(\sigma) q_{w_1'} \\
&= \mathrm{sgn}(t^\flat) \cdot \sum_{w_1'} \mathrm{sgn}(\sigma)\{t'\} \\
&= \mathrm{sgn}(t^\flat) \cdot \sum_{w_1'} \mathrm{sgn}(\sigma)\{t \cdot \sigma\} \\
&= \mathrm{sgn}(t^\flat) \cdot v_t.
\end{aligned}$$

□

Hence, by Proposition 3.5.14 we get a well established connection between the polytabloids and the columns of the Specht matrix. This also gives us that Definition 3.5.13 is equivalent to the definition of a Specht module. Therefore we call the module S^λ to be the Specht module for λ as a subspace of M^λ. This implies that a basis of S^λ corresponds to a subset of A.

Lemma 3.5.15 ([6, Section 7.2, Exercise 1]). *1. For $\sigma \in R(t)$ and $\pi \in C(t)$, we have that $p_t \cdot \sigma = \sigma \cdot p_t = p_t$ and $c_t \cdot \pi = \pi \cdot c_t = \mathrm{sgn}(\pi) c_t$.*

2. *We have that* $p_t \cdot p_t = |R(t)|p_t$ *and* $c_t \cdot c_t = |C(t)|c_t$.

Proof. 1. For any $\sigma \in R(t)$ we have that

$$p_t \cdot \sigma = (\sum_{p \in R(t)} p) \cdot \sigma = \sum_{p \in R(t)} (p \cdot \sigma) = \sum_{p' \in R(t)} p' = p_t.$$

We can also have that

$$\sigma \cdot p_t = \sigma \cdot (\sum_{p \in R(t)} p) = \sum_{p \in R(t)} (\sigma \cdot p) = \sum_{p'' \in R(t)} p'' = p_t.$$

Therefore, $p_t \cdot \sigma = \sigma \cdot p_t = p_t$. Similarly $c_t \cdot \pi = \pi \cdot c_t = \text{sgn}(\pi)c_t$ can be proved.

Now for any $\pi \in C(t)$ we have that

$$c_t \cdot \pi = (\sum_{c \in C(t)} \overline{c}) \cdot \pi = \sum_{c \in C(t)} (\text{sgn}(\pi)\overline{c} \cdot \pi) = \text{sgn}(\pi) \sum_{c \in C(t)} (\overline{c} \cdot \pi) = \text{sgn}(\pi) \sum_{c' \in C(t)} \overline{c'} = c_t.$$

Also, we have that

$$\pi \cdot c_t = \pi \cdot (\sum_{c \in C(t)} \overline{c}) = \sum_{c \in C(t)} (\text{sgn}(\pi)\pi \cdot \overline{c}) = \text{sgn}(\pi) \sum_{c \in C(t)} (\pi \cdot \overline{c}) = \text{sgn}(\pi) \sum_{c'' \in C(t)} \overline{c''} = c_t.$$

2.

$$p_t \cdot p_t = \sum_{\substack{\sigma_i \in R(t) \\ i \in \{1,\dots,|R(t)|\}}} p_t \sigma_i = \underbrace{p_t + \cdots + p_t}_{|R(t)| \text{ times}} = |R(t)|p_t.$$

From 1, we have that $c_t \cdot \pi = \text{sgn}(\pi)c_t \implies \text{sgn}(\pi)c_t \cdot \pi = c_t$ for any $\pi \in C(t)$. Then

$$c_t \cdot c_t = c_t \cdot (\sum_{\pi \in C(t)} \text{sgn}(\pi) \cdot \pi) = \sum_{\substack{\pi_i \in C(t) \\ i \in \{1,\dots,|C(t)|\}}} (\text{sgn}(\pi_i)c_t \cdot \pi_i) = \underbrace{c_t + \cdots + c_t}_{|C(t)| \text{ times}} = |C(t)|c_t.$$

□

Lemma 3.5.16 ([6, Section 7.2])**.** *For all* t *and all* $\sigma \in S_n$ *we have that* $v_t \cdot \sigma = v_{t \cdot \sigma}$.

Proof. For any $\sigma \in S_n$, we have $c_{t \cdot \sigma} = \sum_{c \in C(t \cdot \sigma)} \overline{c} = \sigma^{-1} c_t \sigma$. Therefore, we have that

$$v_t \cdot \sigma = \{t\}c_t \cdot \sigma = \{t\} \cdot \sigma c_{t \cdot \sigma} = t p_t \cdot \sigma c_{t \cdot \sigma} = t \cdot \sigma p_{t \cdot \sigma} c_{t \cdot \sigma} = \{t \cdot \sigma\} \cdot c_{t \cdot \sigma} = v_{t \cdot \sigma}.$$

□

Therefore, from Lemma 3.5.16 it is clear that S^λ is closed under S_n-action.

Lemma 3.5.17 ([6, Section 7.2, Lemma 2]). *If $\lambda \geqslant \lambda'$ and t, t' are their corresponding tableaux respectively, then for a pair of integers in the same row of t' and the same column of t, we have $\{t'\} \cdot c_t = 0$, and if no such pair exists then we have $\{t'\} \cdot c_t = \pm v_t$.*

Lemma 3.5.18 ([6, Section 7.1, Corollary]). *For two standard tableaux t and t' with the property that $t' > t$, we find a pair of integers in the same row of t' and same column of t.*

Corollary 3.5.19 ([6, Section 7.2, Corollary]). *For two standard Young tableaux t and t' with the property that $t' > t$ we have $\{t'\} \cdot c_t = 0$.*

Proof. It is given that $t' > t$. Then Lemma 3.5.18 implies that there exists a pair of integers that is present in the same row of t' and same column of t. Let σ be the transposition that permutes the two integers. Then we have that

$$c_t \cdot \sigma = -c_t, \qquad \text{since } \sigma \in C(t)$$
$$\text{and } \{t'\} \cdot \sigma = \{t'\}, \qquad \text{since } \sigma \in R(t').$$

So we have,

$$\{t'\} \cdot c_t = (\{t'\} \cdot \sigma)c_t = \{t'\}(\sigma \cdot c_t) = \{t'\}(c_t \cdot \sigma) = \{t'\}(-c_t) = -\{t'\} \cdot c_t.$$

Therefore, we have that $\{t'\} \cdot c_t = -\{t'\} \cdot c_t \implies \{t'\} \cdot c_t = 0$. □

Now from Lemma 3.5.15, Lemma 3.5.17 and Corollary 3.5.19 we have the following.

Theorem 3.5.20 ([6, Section 7.2]). *For a tableau t of shape λ we have*

$$M^\lambda \cdot c_t = S^\lambda \cdot c_t = \mathbb{C} \cdot v_t \neq 0; \tag{3.1}$$
$$M^{\lambda'} \cdot c_t = S^{\lambda'} \cdot c_t = 0 \text{ if } \lambda' > \lambda. \tag{3.2}$$

Proof. For any tableau t of shape λ we have $S^\lambda = \langle v_t \rangle$. As v_t are linear combinations of tabloids, we have that S^λ is a $\mathbb{C}[S_n]$-submodule of M^λ. Then $v_t \in S^\lambda \implies v_t \cdot c_t \in S^\lambda \cdot c_t$ for $c_t \in C(t)$.

We know that $v_t = \{t\} \cdot c_t$, then

$$v_t \cdot c_t = \{t\} \cdot c_t \cdot c_t = \{t\} \cdot |C(t)| c_t = |C(t)| \cdot \{t\} c_t = |C(t)| \cdot v_t.$$

So we have that $v_t c_t \in \mathbb{C} \cdot v_t$. As all t are nonzero, we have that $M^\lambda \cdot c_t = S^\lambda \cdot c_t = \mathbb{C} \cdot v_t \neq 0$.

Now for any $\lambda' > \lambda$ we have that $M^{\lambda'} = \langle \{t'\} \rangle$ where t' are tableaux corresponding to λ'. Then we have that

$$M^{\lambda'} \cdot c_t = \langle \{t'\} \cdot c_t \rangle = 0.$$

As $S^{\lambda'} \cdot c_t \leqslant M^{\lambda'} \cdot c_t$, we can conclude that $M^{\lambda'} \cdot c_t = S^{\lambda'} \cdot c_t = 0$. □

It is implicit from the equations in Theorem 3.5.20 that each S^λ is irreducible for λ ranging over all partitions of n. A well-known fact is that irreducibility is same as indecomposability over characteristic zero. If $S^\lambda = V \oplus W$, then $\mathbb{C} \cdot v_t = S^\lambda \cdot c_t = V \cdot c_t \oplus W \cdot c_t$ implies that one of V or W must contain v_t. Without loss of generality, say $V \cdot c_t$ contains v_t, then it must be true that V contains v_t. Now, we know that $\mathbb{C} \cdot v_t$ is a one-dimensional space. We have that $c_t \in \mathbb{C}[S_n]$ and V is a S_n-module, implying that $V \cdot c_t \in V$. Therefore, if V contains v_t then $S^\lambda = \mathbb{C}[S_n] \cdot v_t = V$.

As for each partition of n we have an irreducible representation, we can say that over the complex numbers there are as many irreducible representations as the number of partitions of n. From (3.2) we find that c_t acts as 0 on $S^{\lambda'}$ if $\lambda' > \lambda$ and therefore for two distinct partitions λ and λ' their corresponding Specht modules S^λ and $S^{\lambda'}$ are non-isomorphic.

Theorem 3.5.21. *The set of Specht modules $\{S^\lambda$ for all $\lambda \vdash n\}$ gives the complete set (up to isomorphism) of pairwise non-isomorphic irreducible representations of S_n.*

Proof. Through this novel construction, we find that there are as many irreducible representations of S_n as the number of partitions of n. For each $\lambda \vdash n$, we have S^λ is an irreducible representation of S_n. Every irreducible representation of S_n is isomorphic to exactly one S^λ, and hence we get that any pair of two irreducible representations are non-isomorphic. Hence, $\{S^\lambda$ for all $\lambda \vdash n\}$ produces the complete set of irreducible representations of S_n up to isomorphism. □

Let S^λ be the Specht module spanned by the columns of $\varphi(\lambda)$. We need to show that the columns containing the SYT entries are linearly independent and span the module S^λ, i.e., the

$\dim(S^\lambda)$ is the number of SYTs of shape λ. As the first step to show the linear independence, we need to prove that all the SYT elements have different row and column labels.

Lemma 3.5.22. *The standard Young tableaux entries in the Specht matrix belong to distinct rows and columns.*

Proof. Let t_1, t_2 be two λ-tableaux. If the entries in the first i rows of t_2 are a rearrangement of the elements in the first i rows of t_1, then we say that t_1 and t_2 are equivalent. All such Young tableaux from the same equivalence class of the shape-λ. In other words, their corresponding tabloids are same.

If two distinct λ-tableaux generated by different rearrangements of words creating simultaneous histograms belong to the same equivalence class of shape λ, then the row-word arrangements remain the same for them and the column word varies depending on the permutation applied on its arrangement.

For such pair of words we will find that the elements corresponding to them in the Specht matrix sit in the same row. That is, the elements in each row of the Specht matrix correspond to the same row tabloid of shape λ and they all are in the same equivalence class.

Now, among all these λ-tableaux in each row equivalence class, we can find at most one standard Young tableau as the SYT can be obtained if and only if the labelling of the boxes are in increasing order of left-to-right and top-to-bottom. Therefore in one equivalence class of a Young tableaux of shape λ there is only one SYT.

By the above discussion we can say that the other rows of the same Specht matrix are generated by the equivalence class of different labelling of rows of the Young tableaux of shape λ. If any SYT belongs to any other equivalence class then the corresponding element in Specht matrix will be in a distinct row than the other ones. From these arguments we can conclude that no two SYT-entries can be in the same row. A similar argument can be devised for the columns and their equivalence classes leading to all the SYT-entries belong to distinct columns.

Hence, we conclude that the standard Young tableaux entries in the Specht matrix belong to distinct rows and columns.

□

The next theorem shows that the SYT elements in each respective rows and columns come as the first nonzero elements.

Theorem 3.5.23. *Let $\lambda \vdash n$. Then in the Specht matrix of λ the elements corresponding to the standard tableaux are the first nonzero elements in the respective rows and columns.*

Proof. Let n be a natural number and $\lambda \vdash n$. Let us consider $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. Then we consider $\mu = \lambda^T = (\mu_1, \mu_2, \ldots, \mu_k)$ be the conjugate partition of λ.

Let t be a column-standard tableau of shape λ, i.e., the entries in each column of t are strictly increasing from top to bottom. Let v and w be the words corresponding to the partitions λ and μ respectively for which we have a Specht matrix. Then in the Specht matrix $\varphi(v, w)$ the rows are labelled by the rearrangements of the row word v and the columns are labelled by the rearrangements of the column word w.

For each column of $\varphi(v, w)$, if we look through the respective tableaux for all nonzero entries, we find only one tableau with all its columns sorted in an increasing order, i.e., for each column in the matrix we have only one fully column-standard tableau.

Let $C = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_k}$ be a subgroup of S_n that stabilizes the columns of the tableau t. Then the tableaux appearing in a column of $\varphi(v, w)$ belong to the same column equivalence class of t, i.e., the column tabloid $[t]$.

By the construction of simultaneous histogram for a pair of words, it is clear that the distinct letters of the word v label the rows of t. As t is column standard, the smaller entries in a column have lower row-index than the larger entries in the same column. Then for each column of t we have $v_{(i,j)} \leqslant v_{(i+1,j)}$ where $v_{(i,j)}$ indicates the letter in v corresponding to the box (i, j) in t.

For any $\pi \in C$, we have π acts on t by permuting the entries in the columns of t. Call the permuted tableau t_π. Then t_π is column non-standard as t, t_π both belong to the same $[t]$ and there can be only one column standard tableau in each $[t]$. That means in t_π we have at least one of the columns with $t_{\pi_{(i,j)}} > t_{\pi_{(i+1,j)}}$. Now consider π acting on the respective row word such that $v \mapsto \pi \cdot v$, v'. Therefore in v' we have $v'_{(i,j)} \geqslant v'_{(i+1,j)}$ implying that $v < v'$ lexicographically.

Hence if we label the rows of the $\varphi(v, w)$ in the lexicographic order of the row words then we have the column standard tableaux entries appearing on the top of the respective columns making it the first nonzero entry.

Likewise we can argue with respect to the lexicographic ordering of the column word w and the row stabilizer $R = S_{\lambda_1} \times S_{\lambda_2} \times \cdots S_{\lambda_l}$ acting on the tableaux corresponding to it and establish the fact that the first non-zero entries in the respective rows indeed turn out to be the ones corresponding to the row standard tableaux if the columns are labelled lexicographically.

Whenever we have a tableau t being both row and column standard then we say that t is standard. Hence we deduce from above that the standard tableaux entries in the Specht matrix $\varphi(\nu, w)$ are the first nonzero entries in respective rows and columns given that the rows and columns are labelled lexicographically.

□

By extracting just the SYT rows and columns from the Specht matrix, we find that the elements appearing before the SYT elements in each row are mostly 0 making it into a lower-triangular matrix. Therefore, in the Specht matrix the SYT rows and columns are linearly independent.

In Example 3.5.8, we can think of the Specht matrix $\varphi((2,1,1))$ as shown in Figure 3.4 where the elements in the matrix are now given as the simultaneous histograms themselves for the corresponding pair of row and column words rather than the Young characters of the pairs.

Now if we concentrate on just the SYT of shape $(2,1,1)$ and extract the respective rows and columns they belong to, we obtain the submatrix as shown in Figure 3.5. Converting the SYTs in this submatrix back to their Young character values, we find the submatrix from Table 3.3 of the original Specht matrix.

	1112	1121	1211
1123	0	0	1
1213	0	-1	0
1231	1	0	0

Table 3.3: SYT-extracted submatrix of Specht matrix $\varphi((2,1,1))$

We see that the submatrix takes a nice form with its nonzero entries on the non-prime diagonal. Another rearrangement of the column words can be used to convert the matrix to a diagonal matrix with respect to the main diagonal. Therefore, evidently, we establish that the rows (respectively columns) containing the SYT elements in the Specht matrix are linearly independent.

We can easily compute Specht matrices for any small $n \geqslant 2$. By computing Specht matrices for $n \geqslant 5$, we found that for certain partitions of n the extracted Specht submatrices do not

In Figure 3.4, it can be seen that on the row labels, any two row words are connected by an edge and the number seen on these edges indicate the product of Coxeter generators σ whose action changes row word w to $w \cdot \sigma$. By the right-action of same permutation σ on t_w corresponding to w, the tableau $t_{w \cdot \sigma}$ corresponding to $w \cdot \sigma$ can be obtained. Similarly for the column labels.

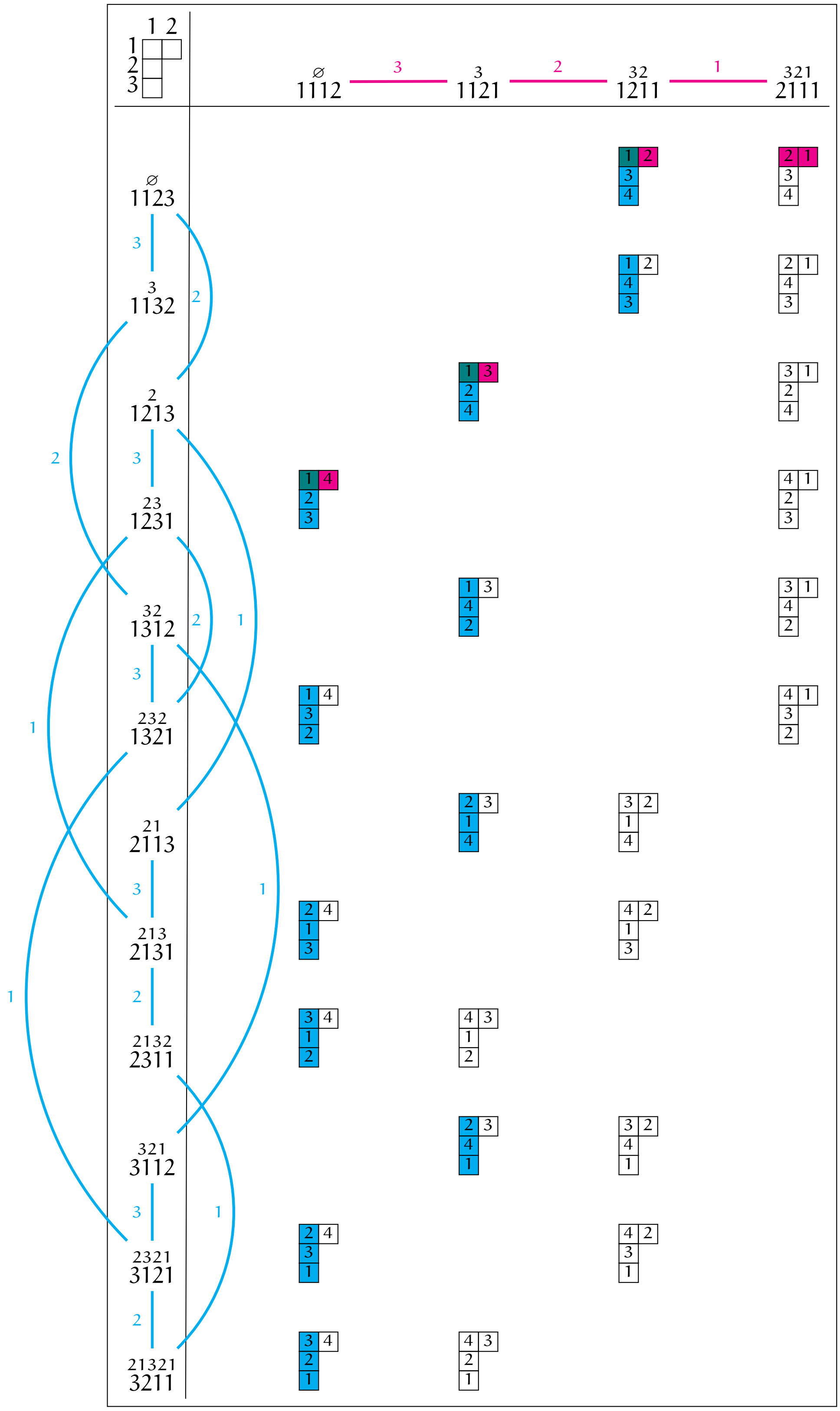


Figure 3.4: Specht matrix $\varphi((2,1,1))$

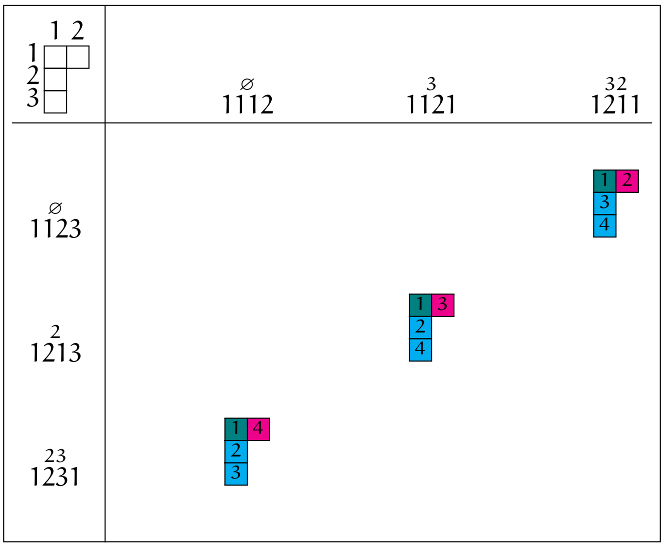


Figure 3.5: Truncated Specht submatrix $\varphi((2,1,1))$

have the form of a diagonal matrix as can be seen in Table 3.3, rather they take the form of a lower-triangular matrix with respect to the main or anti-diagonal. We can consider the Example 3.5.24 to demonstrate this.

Example 3.5.24. Let $n = 5$ and $\lambda = (2,2,1)$. Then the conjugate partition is $\lambda^T = (3,2)$. Then for the row and column words being $w_1 = 11223$ and $w_2 = 11122$ respectively, we find the extracted Specht submatrix as shown in Table 3.4.

	12121	11221	12112	11212	11122
11223	1	0	0	0	0
12123	0	-1	0	0	0
11232	0	0	-1	0	0
12132	0	0	0	1	0
12312	1	0	0	0	-1

Table 3.4: SYT-extracted submatrix of Specht matrix $\varphi((2,1,1))$

An element 1 corresponding to the pair of words $(12312, 12121)$ can be seen here. It happens due to the existence of inter-tableau as mentioned before. Whenever a pair of integers is present in the same row of one SYT and same column of another SYT, in the intersection of these two SYT we find an inter-tableau. This phenomenon turns the SYT-extracted Specht

In the submatrix in Table 3.4 it can be noticed that the row and column labels are not in lexicographic order. These labellings are arranged in such a way that helps us to bring the shape of the submatrix into a lower-triangular form.

submatrix into a lower-triangular matrix rather than just diagonal. Such inter-tableaux appears in Specht matrices for partitions of $n \geqslant 5$.

Now we must show that the SYT columns in the Specht matrix span the module S^λ. As we put all the rearrangements of w_1 vertically at the left of the matrix in our construction, we have that the length of each column of the matrix is same as the number of the rearrangements of w_1. Hence the choice of columns for the basis of the module S^λ.

Using arguments from [6], [15] and [18], we give a global argument that the linearly independent columns of the Specht matrix do indeed span the Specht module. In a combinatorial sense, hook length formula gives us the number of SYT for each partition of n.

Definition 3.5.25 ([5])**.** *If* $b = (i, j)$ *is a box in the diagram of* λ*, then it has a* ***hook***

$$\Xi_b = \Xi_{i,j} = \{(i, j')|j' \geqslant j\} \cup \{(i', j)|i' \geqslant i\}$$

with corresponding ***hooklength***

$$\xi_b = \xi_{i,j} = |\Xi_{i,j}|$$

Let us define f^λ to be the dimension of Specht module as found in literature [6]. Then the hook formula is as follows:

Lemma 3.5.26 ([5])**.** *For each* $\lambda \vdash n$ *we have* f^λ *to be the number of standard tableaux of shape* λ *decided by the hook length formula*

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} \xi_{i,j}}.$$

From regular representation of S_n and combinatorial description by *Robinson-Schensted-Knuth (RSK) correspondence* [4], we know that

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

Now, as the SYT columns in the Specht matrix are linearly independent, there are precisely f^λ independent columns in each matrix. Hypothetically, if there exists one more column that is linearly independent, then the RSK correspondence fails. This happens due to the fact that the number of SYT would be less than the number of independent columns. Therefore, it is not possible to have anymore linearly independent vectors in the space than f^λ.

A basis consists only of the linearly independent vectors. Hence we showed that the span of the set of SYT-columns in the Specht matrix for λ is indeed the Specht module S^λ.

We know that each S^λ is irreducible and Specht modules associated with different partitions are non-isomorphic. So there are as many Specht modules as the number of partitions λ of n. This forms the complete set of irreducible representations.

From the RSK-correspondence we find that the sum of the squares of the degrees of irreducible representations is same as the order of the group S_n.

Let for our construction $d^\lambda = \dim(S^\lambda)$. The RSK-correspondence implies that for each λ we have $d^\lambda \geqslant f^\lambda$. Now if $|t_\lambda|$ be the number of SYT of shape λ, we have

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = \sum_{\lambda \vdash n} |t_\lambda|^2 = n!.$$

This implies it must be true that $d^\lambda = f^\lambda$. Hence, dimension of each S^λ is equal to the number of the SYT of shape λ.

We have now established that the vectors in the set of SYT columns of the Specht matrix are linearly independent and they span the whole module S^λ. Therefore, we have the following theorem.

Theorem 3.5.27. *If $\lambda \vdash n$ then the set of SYT-columns in $\varphi(\lambda)$ forms a basis of S^λ.*

It is well-known that the set of polytabloids v_t for all standard tableaux t of shape λ is a basis of the Specht module S^λ. Due to the well-established connection between v_t and v_{w_2} (columns of Specht matrices) for w_2 being a column word in the set B of all its rearrangements, we have proved that $\langle v_{w_2} \rangle = S^\lambda$. With help of the Specht matrices we have now bridged all the gaps between the existing literature and this new construction. Therefore, it is now well connected to the existing theory.

3.6 Representing Matrices

Naturally the question that follows next is how do we represent the group using this construction of the Specht matrices. To answer that let us now get to the construction of representing matrices for the group elements. The mechanism to build these matrices uses simple ideas from linear algebra such as solving linear equations.

We pick a σ from the finite symmetric group A_{n-1}. Let X be the set of all rearrangements of the row word w_1 and $|X| = k$. By the action of σ on the set X, we obtain a permuted list X' of the same set, i.e., $X' = \{i^\sigma | i \in X\}$. Let $\pi \in S_k$ be the permutation such that $X \cdot \pi = X'$. In the Specht matrix sense it means that the elements in each column of the matrix $\varphi(\lambda)$ is now rearranged by π. Let C be the set of columns of $\varphi(\lambda)$, and $C' = \{c^\pi | c \in C\}$ which is another rearrangement of C itself. Let $\mathcal{B}$ be the set of basis vectors, the SYT-columns in C and let $\mathcal{V}$ be the set of rearranged SYT-columns in C'. Now, we express each $v \in \mathcal{V}$ as a linear combination of the vectors $\mathfrak{b} \in \mathcal{B}$. Then, we extract the coefficients of these linear combinations and put them in the rows of a matrix, denoted by $M_\sigma \in GL_k(\mathbb{C})$ for $k = |\mathcal{B}|$, which is the *representing matrix* corresponding to $\sigma \in A_{n-1}$.

Let for $\lambda \vdash n$ there be k number of SYT. Then the dimension of the Specht module S^λ is k. Let us have

$$\mathcal{B} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathfrak{b}_1 & \mathfrak{b}_2 & \cdots & \mathfrak{b}_k \\ \downarrow & \downarrow & & \downarrow \end{bmatrix},$$

the matrix made up of the basis vectors (SYT-columns in C). Now for $\sigma \in S_n$, we have that $C' = C \cdot \sigma$. Let $\mathcal{V} = \{v_1, v_2, \ldots, v_k\} \subset C'$ be the set of rearranged SYT-columns from $\mathcal{B}$. Let $X_j = \{x_{j1}, x_{j2}, \ldots, x_{jk}\} \subset \mathbb{C}$ be the sets of coefficients for $j \in \{1, 2, \ldots, k\}$ corresponding to each v_j such that

$$v_j = \sum_{i=1}^{k} x_{ji} \mathfrak{b}_i \implies \mathcal{B} \cdot X_j = v_j.$$

Then we have that

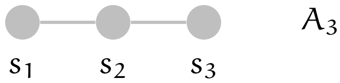
$$X_j = \mathcal{B}^{-1} \cdot v_j.$$

Now, if we collect these sets of coefficients X_j for all the v_j in a set then we obtain the representing matrix corresponding to σ as

$$M_\sigma = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kk} \end{bmatrix}.$$

These representing matrices give us the natural irreducible representations of symmetric group. As can be seen, the mechanism used are basic linear algebraic techniques of solving system of linear equations. The following example shows explicitly the compatibility of this construction with the group elements.

Example 3.6.1. For $n = 4$, let us consider the group A_3.



Then generator set of A_3 is

$$S = \{s_1, s_2, s_3\} = \{(1\ 2), (2\ 3), (3\ 4)\}.$$

We have the partitions of n as

$$(1, 1, 1, 1), (2, 1, 1), (2, 2), (3, 1) \text{ and } (4).$$

Now let us pick each of these generators, $s \in S$ and for them we find the corresponding representing matrix M_s for each $\lambda \vdash 4$.

1. For the partition $\lambda = (1, 1, 1, 1)$ we have $M_s^\lambda = \begin{bmatrix} -1 \end{bmatrix}$ for all $s \in S$.
2. For $\lambda = (2, 1, 1)$ we find:

   - if $s = (1\ 2)$ then
   $$M^\lambda_{(1\ 2)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
   - if $s = (2\ 3)$ then
   $$M^\lambda_{(2\ 3)} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
   - if $s = (3\ 4)$ then
   $$M^\lambda_{(3\ 4)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

In the group, we have that $((1\,2)\cdot(2\,3))^3 = ((2\,3)\cdot(3\,4))^3 = e_{A_3}$. Here we get to see that

$$(M^\lambda_{(1\ 2)} \cdot M^\lambda_{(2\ 3)})^3 = (M^\lambda_{(2\ 3)} \cdot M^\lambda_{(3\ 4)})^3 = (M^\lambda_{(1\ 2)} \cdot M^\lambda_{(3\ 4)})^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. For $\lambda = (2,2)$ we find:

- if $s = (1\,2)$ then
$$M^\lambda_{(1\ 2)} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
- if $s = (2\,3)$ then
$$M^\lambda_{(2\ 3)} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
- if $s = (3\,4)$ then
$$M^\lambda_{(3\ 4)} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Note that, $M^\lambda_{(1\ 2)(3\ 4)} = M^\lambda_{(1\ 2)} M^\lambda_{(3\ 4)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In fact, for any double transposition σ, we have that $M^\lambda_\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Likewise, see that

$$(M^\lambda_{(1\ 2)} \cdot M^\lambda_{(2\ 3)})^3 = (M^\lambda_{(2\ 3)} \cdot M^\lambda_{(3\ 4)})^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. For $\lambda = (3,1)$ we find:

- if $s = (1\ 2)$ then

$$M^{\lambda}_{(1\ 2)} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

- if $s = (2\ 3)$ then

$$M^{\lambda}_{(2\ 3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

- if $s = (3\ 4)$ then

$$M^{\lambda}_{(3\ 4)} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Likewise, see that

$$(M^{\lambda}_{(1\ 2)} \cdot M^{\lambda}_{(2\ 3)})^3 = (M^{\lambda}_{(2\ 3)} \cdot M^{\lambda}_{(3\ 4)})^3 = (M^{\lambda}_{(1\ 2)} \cdot M^{\lambda}_{(3\ 4)})^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Lastly, for $\lambda = (4)$ we have $M_s = \begin{bmatrix} 1 \end{bmatrix}$ for all $s \in S$.

Therefore, it can be noticed that for all the partitions above, our construction works efficiently. Another interesting point to note is the dimension of these representing matrices M^{λ}_{σ}. Each of these matrices M^{λ}_{σ} is square and its dimension is the square of number of SYT for the partition λ. To think of it with the help of a concrete example we can consider all the representing matrices for $\sigma = (1\ 2)$ for all the partitions $\lambda \vdash 4$ and put them in the diagonal of a block matrix, the dimension of which will be the sum of the dimensions of each of S^{λ}, and the squares of these dimensions add up to the order of S_4 i.e., $4!$.

$$\begin{pmatrix} -1 & & & & & & & & & \\ & 1 & 1 & 1 & & & & & & \\ & 0 & -1 & 0 & & & & & & \\ & 0 & 0 & -1 & & & & & & \\ & & & & 1 & 1 & & & & \\ & & & & 0 & -1 & & & & \\ & & & & & & 1 & 0 & -1 & \\ & & & & & & 0 & 1 & 1 & \\ & & & & & & 0 & 0 & -1 & \\ & & & & & & & & & 1 \end{pmatrix}$$

3.7 Characters and Character Tables

The significance of group characters in the representation theory of finite groups cannot be disregarded. It helps us to verify the construction in terms of traces of the representing matrices.

For each partition $\lambda \vdash n$, we can pick the set of the conjugacy class representatives and find the character for each of the set elements, which are simply the traces of their representing matrices for the particular λ.

To demonstrate this we need the conjugacy class representatives, the classes and their corresponding diagrams. Here we work with the symmetric group S_4 or the Coxeter group A_3 to demonstrate this.

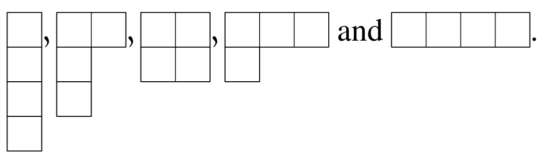
Let the set of conjugacy class representatives of S_4 be

$$\text{Class}(S_4) = \{\text{id}, (1\ 2), (1\ 2)(3\ 4), (1\ 2\ 3), (1\ 2\ 3\ 4)\}$$

which respectively correspond to the partitions

$$(1,1,1,1), (2,1,1), (2,2), (3,1) \text{ and } (4).$$

Then we have their corresponding Young diagrams respectively as



Now that we have all the prerequisites, we can start with an example.

Example 3.7.1. For all partitions of $n = 4$, we list out the characters for each of the class representatives as follows:

1. For $\lambda = (1,1,1,1)$ we find:
   - $M^{\lambda}_{\text{id}} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi(\text{id}) = 1$
   - $M^{\lambda}_{(1\ 2)} = \begin{bmatrix} -1 \end{bmatrix}$ implies that $\chi((1\ 2)) = -1$
   - $M^{\lambda}_{(1\ 2)(3\ 4)} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi((1\ 2)(3\ 4)) = 1$
   - $M^{\lambda}_{(1\ 2\ 3)} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi((1\ 2\ 3) = 1$
   - $M^{\lambda}_{(1\ 2\ 3\ 4)} = \begin{bmatrix} -1 \end{bmatrix}$ implies that $\chi((1\ 2\ 3\ 4)) = -1$

2. For $\lambda = (2,1,1)$ we find:
   - $M^{\lambda}_{\text{id}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ implies that $\chi(\text{id}) = 3$
   - $M^{\lambda}_{(1\ 2)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ implies that $\chi((1\ 2)) = -1$
   - $M^{\lambda}_{(1\ 2)(3\ 4)} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ implies that $\chi((1\ 2)(3\ 4)) = -1$
   - $M^{\lambda}_{(1\ 2\ 3)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ implies that $\chi((1\ 2\ 3)) = 0$
   - $M^{\lambda}_{(1\ 2\ 3\ 4)} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ implies that $\chi((1\ 2\ 3\ 4)) = 1$

3. For $\lambda = (2,2)$ we find:
   - $M^{\lambda}_{\text{id}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ implies that $\chi(\text{id}) = 2$
   - $M^{\lambda}_{(1\ 2)} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ implies that $\chi((1\ 2)) = 0$

- $M^\lambda_{(1\ 2)(3\ 4)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ implies that $\chi((1\,2)(3\,4)) = 2$
- $M^\lambda_{(1\ 2\ 3)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ implies that $\chi((1\,2\,3)) = -1$
- $M^\lambda_{(1\ 2\ 3\ 4)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ implies that $\chi((1\,2\,3\,4)) = 0$

4. For $\lambda = (3, 1)$ we find:

- $M^\lambda_{\text{id}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ implies that $\chi(\text{id}) = 3$
- $M^\lambda_{(1\ 2)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ implies that $\chi((1\,2)) = 1$
- $M^\lambda_{(1\ 2)(3\ 4)} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ implies that $\chi((1\,2)(3\,4)) = -1$
- $M^\lambda_{(1\ 2\ 3)} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$ implies that $\chi((1\,2\,3)) = 0$
- $M^\lambda_{(1\ 2\ 3\ 4)} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$ implies that $\chi((1\,2\,3\,4)) = -1$

5. For $\lambda = (4)$ we find:

- $M^\lambda_{\text{id}} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi(\text{id}) = 1$
- $M^\lambda_{(1\ 2)} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi((1\,2)) = 1$
- $M^\lambda_{(1\ 2)(3\ 4)} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi((1\,2)(3\,4)) = 1$
- $M^\lambda_{(1\ 2\ 3)} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi((1\,2\,3)) = 1$
- $M^\lambda_{(1\ 2\ 3\ 4)} = \begin{bmatrix} 1 \end{bmatrix}$ implies that $\chi((1\,2\,3\,4)) = 1$

Now summing it all up altogether in Table 3.5, we find that it is the same as the character table in GAP.

	id	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$
$(1,1,1,1)$	1	-1	1	1	-1
$(2,1,1)$	3	-1	-1	0	1
$(2,2)$	2	0	2	-1	0
$(3,1)$	3	1	-1	0	-1
(4)	1	1	1	1	1

Table 3.5: Character table of Coxeter group A_3

Chapter 4

Monomial Groups

We now know the construction for the very basic type of Coxeter groups, and so we can construct the Specht modules for some of their generalisations in this chapter. This construction is going to be analogous to that of symmetric groups but with added layers of complexities.

Let us consider a complex vector space V of dimension n. Then a non-trivial element ρ of $GL(V)$ which acts trivially on a hyperplane called the reflecting hyperplane of ρ, is a complex reflection of $GL(V)$. If we then take G to be a finite subgroup of $GL(V)$ generated by complex reflections, we can call the pair (V, G) to be a complex reflection group. Therefore, (V, G) is a finite group of transformations of a complex vector space generated by complex reflections, i.e., transformations that fix some hyperplanes.

Any finite Coxeter group can naturally be regarded as a complex reflection group by complexifying the vector space on which its reflection representation acts.

4.1 Monomial Groups, Generators and Class Representatives

A root of unity is a complex number that yields 1 when raised to the power r where r is some positive integer. Therefore, an r-*th root of unity* is a complex number z satisfying the equation $z^r = 1$.

For any positive integer r, the r-*th cyclotomic polynomial* is the unique polynomial which is irreducible over $\mathbb{Q}$ with integer coefficients that is a factor of $z^r - 1$ and is not a factor of $z^k - 1$ for any $k < r$. All roots of this polynomial are all *primitive roots of unity* $\exp(\frac{2k\pi i}{r})$, where k runs over positive integers not greater than r and coprime to r. In other words, the r-th

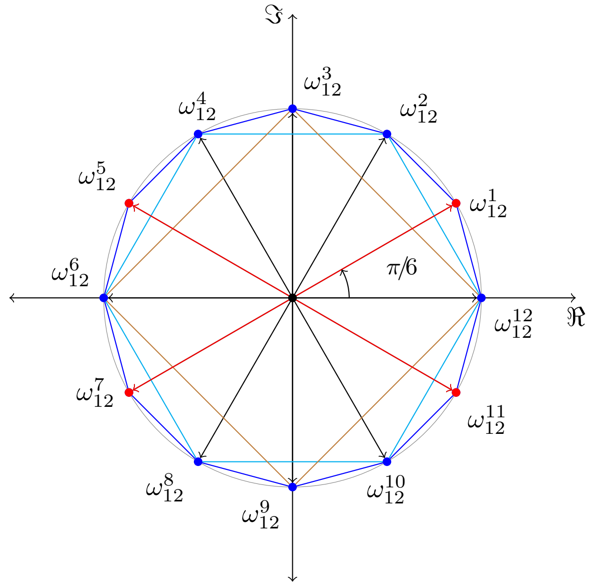


Figure 4.1: Twelfth roots of unity

cyclotomic polynomial is equal to

$$\Phi_r(z) = \prod_{\substack{1 \leqslant k \leqslant r \\ \gcd(k,r)=1}} (z - \exp(\frac{2k\pi i}{r})).$$

Therefore, we can say that the zeros of the polynomial $p(z) = z^r - 1$ are precisely the r-th roots of unity ω_r, each with multiplicity 1. For example, the twelfth roots of unity are shown in Figure 4.1.

Definition 4.1.1 (Monomial Matrix). *A matrix $M \in GL_n(\mathbb{C})$ that has the same nonzero pattern as a permutation matrix with exactly one nonzero entry ω in each row and each column is called a monomial matrix.*

Monomial matrices are also known as generalised permutation matrices. Clearly, an invertible matrix M is a monomial matrix if and only if it can be written as a product of an invertible diagonal matrix D and a permutation matrix P, i.e., $M = DP$. Now let us have a look at Example 4.1.2 that shows this decomposition of the monomial matrices.

Example 4.1.2. Let the monomial matrix be

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 \\ \omega_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

with third roots of unity embedded in $GL_7(\mathbb{C})$. Then $A = DP$ where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_3^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Definition 4.1.3 (**Monomial Group**). *Let* $n \in \mathbb{N}$ *and* r *be the number of roots of unity. A monomial group* $G_{n,r}$*, as a subgroup of* $GL_n(\mathbb{C})$*, is a group of* $n \times n$ *monomial matrices with entries being the* r*-th roots of unity.*

It is now time to see it from the perspective of the complex reflection groups. Before we start with complex reflection groups, we give a formal definition of complex reflections.

Definition 4.1.4 ([8, Definition 5.2.9.: **Complex Reflections**]). *Let* V *be a finite-dimensional vector space over* $\mathbb{C}$*. Given any nonzero* $v \in V$ *and an element* $g \in GL(V)$ *of finite order, we say that* g *is a complex reflection with root* v *if* v *is an eigenvector for* g*.*

Let G be a finite subgroup of $GL(V)$ consisting of the set R of complex reflections. Then we say that G is a *complex reflection group* with respect to R if G is generated by R. A definition of complex reflection groups fitting our purpose for the chapter is given as follows.

Definition 4.1.5 ([13, Section 2.1]). *Let $r, p, n \in \mathbb{N}$ such that n is divisible by p and let $\omega_r = \exp(\frac{2\pi i}{r})$ where $i = \sqrt{-1}$. The groups $G(r, p, n)$ are the subgroups of $GL_n(\mathbb{C})$ consisting of matrices such that*

- *the entries are either 0 or power of ω_r,*
- *there is exactly one nonzero entry in each row and column, and*
- *the (r/p)-th power of the product of all nonzero entries is 1.*

We are mostly interested in these groups with respect to the roots of unity. For that reason in this chapter we put a restriction on p by fixing $p = 1$. Then the groups turn out to be $G(r, 1, n)$ with r-th power of the product of all nonzero entries is 1. Therefore, the order of the group is $|G(r, 1, n)| = r^n \cdot n!$.

As $G(r, 1, n)$ now fits the Definition 4.1.3, from now on let us use the general notation $G_{n,r} = G(r, 1, n)$ for the monomial groups. We know that $G_{n,r} \subset GL_n(\mathbb{C})$ and each $g \in G_{n,r}$ is of the form $M = DP$ for some diagonal matrix D and permutation matrix P.

Let $\Delta_n \subseteq G_{n,r}$ be the subgroup of all nonsingular diagonal matrices D. Clearly, Δ_n is an abelian group and it is a normal subgroup of $G_{n,r}$. We denote it by $\Delta_n \trianglelefteq G_{n,r}$. Since each diagonal entry can be an r-th root of unity ω_r^k, we have that $|\Delta_n| = r^n$.

Let $\mathfrak{P}_n \subseteq G_{n,r}$ be the subgroup of all permutation matrices P. Clearly, $|\mathfrak{P}_n| = n!$. Naturally, we have that $\mathfrak{P}_n \cong S_n$.

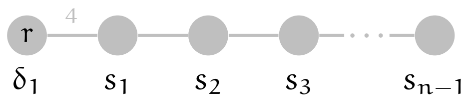
As $\Delta_n \cap S_n = \{1\}$, we have $|\Delta_n \cdot S_n| = |\Delta_n||S_n|/|\Delta_n \cap S_n| = r^n n!$. Then we have that $G_{n,r} = \Delta_n \rtimes S_n$. From here we can also draw that $G_{n,r}/\Delta_n$ is canonically isomorphic to S_n.

Let $\delta_j \in G_{n,r}$ for $1 \leqslant j \leqslant n$ be the diagonal matrix whose j-th diagonal entry is ω_r and all the other entries are 1. Then Δ_n is generated by $\delta_1, \ldots, \delta_n$.

Then we get that the monomial groups $G_{n,r}$ are generated by the set $\langle \delta_j, s : \delta_j^r = s^2 = 1 \rangle$ for $1 \leqslant j \leqslant n$. Here $s = s_i = (i, i+1)$ for $i \in \{1, \cdots, n-1\}$ be the transpositions that generate S_n. Let I_n be an identity matrix of dimension $n \times n$. For each element $g \in G_{n,r}$, we have the corresponding generalised permutation matrices. The following proposition describes these matrices.

Proposition 4.1.6. *Let $G_{n,r}$ be a monomial group generated by the set $\langle \delta_j, s : \delta_j^r = s^2 = 1 \rangle$ for $1 \leqslant j \leqslant n$. Then the corresponding matrix δ_1 is a diagonal matrix with its diagonal entries in the order $\{\omega_r, 1, 1, \ldots, 1\}$ where $\omega_r = \exp(\frac{2\pi i}{r})$, and the corresponding matrix s is obtained by the action of s on I_n by permuting its rows.*

Now we look into the Coxeter-like diagram of the monomial groups and thereafter we will understand this group from a different aspect. The diagram is as follows:



$G(r, 1, n)$ $(n \geqslant 2)$

In the diagram, we can see the generators indicated by the nodes are $\delta_1, s_1, s_2, \ldots, s_{n-1}$ such that $\delta_1^r = s_i^2 = 1$. We also have that $s_1\delta_1 s_1\delta_1 = \delta_1 s_1\delta_1 s_1$ and $(s_i\delta_1)^4 = 1$ only when $\delta_1^2 = s_i^2 = 1$.

4.2 Conjugacy Classes in $G_{n,r}$

The groups $G_{n,r}$ are isomorphic to the wreath product of the cyclic group of order r with the symmetric group S_n. Hence the conjugacy classes and characters are parameterized by r-tuples of partitions, such that the total sum of their parts equal n.

Definition 4.2.1 (Multipartition). *Let* $n = \sum_{i=0}^{r-1} a_i$ *and* $n, a_i \in \mathbb{N}$ *for* $i = 0, 1, 2, \ldots, r-1$. *Let* $\lambda_i \vdash a_i$. *Then the tuple* $\lambda = (\lambda_0, \ldots, \lambda_{r-1})$ *is called a multipartition of* n *denoted by* $\lambda \overset{r}{\vdash} n$.

As an example, let us consider $n = 10$. Now for $r = 3$ and $a_0 = 4, a_1 = 4, a_2 = 2$ we have $a_0 + a_1 + a_2 = n$. If $\lambda_0 = (3, 1) \vdash a_0, \lambda_1 = (2, 2) \vdash a_1$ and $\lambda_2 = (1, 1) \vdash a_2$ then we have $\lambda = (\lambda_0, \lambda_1, \lambda_2) = ((3, 1), (2, 2), (1, 1)) \overset{3}{\vdash} 10$.

From Chapter 8 in [12], we know that the conjugacy classes are described by the cycle structures of the elements in the groups. Therefore it is important that we discuss the cycle structures for the elements in the monomial groups $G_{n,r}$ as a subgroup of $GL_n(\mathbb{C})$.

Definition 4.2.2. *Cycle structure of an element in* $G_{n,r}$ *is defined to be the tuple of cycle lengths of each cycle that contributes to the group element.*

We already know that each element $M \in G_{n,r}$ can be written as $M = DP$ where D is a diagonal matrix and P is a permutation matrix. It is sufficient to use the notations $D = \text{diag}(d_1, d_2, \ldots, d_n)$ where each d_i is an r-th roots of unity sitting on the main diagonal of the matrix D, and $P = \pi$ where $\pi \in S_n$ is the permutation corresponding to the matrix P. The conjugacy classes of $G_{n,r}$ are decided by the cycle structure of matrices $M \in G_{n,r}$ in the way as follows.

Any permutation $\pi = \pi_1\pi_2\cdots\pi_k \in S_n$ is a product of disjoint cycles $\pi_1, \pi_2, \dots, \pi_k$, and so π has a cycle structure associated to it. Let $1 \leqslant i \leqslant n$ be a point and so i belongs to some permutation π_i. For each i we also have a corresponding root of unity $d_i = \omega_{r_i}$ at the i-th position in the list of diagonal elements $[d_1, d_2, \dots, d_n]$.

Let $\Gamma = [1, \omega_r, \omega_r^2, \dots, \omega_r^{r-1}]$ be the list of all r-th roots of unity. Let us denote the element at the k-th position in the list Γ by $\Gamma[k]$ for $0 \leqslant k < r$. Now for all $i \in \pi$, if $\prod_i \omega_{r_i} = \Gamma[k]$ then π_i contributes to the cycle structure corresponding to the partition λ_k. Now running i through all the points $1, 2, \dots, n$, we find the cycle structure for each λ_k for $0 \leqslant k < r$, and therefore eventually for multipartition λ.

By using Example 4.1.2 we demonstrate the cycle structure and the conjugacy classes as described above. If we use the shorter notations then we can express the matrices D and P respectively as

$$D = (1, \omega_3^2, \omega_3, \omega_3, \omega_3, \omega_3, 1), \quad \text{and} \quad P = (1,5,7,4)(2,6,3).$$

Let us write $P = \pi_1\pi_2$ where $\pi_1 = (1,5,7,4)$ and $\pi_2 = (2,6,3)$. Let $R = \{1, \omega_3, \omega_3^2\}$ be the set of third roots of unity. For each of the points in π_i we have a corresponding root of unity in D, which will together make a subset of D. Then for π_1 we have the set $D_1 = \{1, \omega_3, 1, \omega_3\}$ and similarly for π_2 we have the set $D_2 = \{\omega_3^2, \omega_3, \omega_3\}$. The product of the elements in the set D_1 is ω_3^2 and it implies that π_1 contributes to the partition λ_2. Likewise, the product of the elements in the set D_2 is $\omega_3^4 = \omega_3^3 \cdot \omega_3 = \omega_3$ implying that π_2 contributes to the partition λ_1. Therefore the cycle structure of the group element A is $(\varnothing, (3), (4))$.

Theorem 4.2.3 ([12, Theorem 4.2.8])**.** $\sigma, \rho \in G_{n,r}$ *have the same cycle type if and only if* σ *is a conjugate to* ρ.

Conjugacy classes of the complex reflection groups are parameterized by the multipartitions of the natural number n. We will learn more about multipartitions in the next section.

Therefore, we have established similar facts for monomial groups analogous to the symmetric groups that two elements are in the same conjugacy class if they have the same cycle structure.

Now before moving to the next ideas, let us have a look at an example of the monomial matrices that generate the monomial group with third roots of unity as a subgroup of $GL_5(\mathbb{C})$.

In the group $G_{5,3}$ we have the generators $\{\omega_3, (1,2), (2,3), (3,4), (4,5)\}$. Then the corresponding generalized permutation matrices are as follows.

$$\omega_3 \longmapsto \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(1,2) \longmapsto \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(2,3) \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(3,4) \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(4,5) \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Likewise for any other elements $\sigma \in G_{n,r}$ we have the matrices obtained by the action of σ on I_n. Therefore, we have the whole group $G_{n,r}$ embedded inside $GL_n(\mathbb{C})$.

4.3 Construction of Irreducible Characters of $G_{n,r}$

The conjugacy classes of $G_{n,r}$ correspond to the multipartitions of n. In [12] we find that each multipartition corresponds to an irreducible character of $G_{n,r}$. These characters can be constructed by Clifford theory [8, Section 5.5].

Let $\underline{n} = (n_0, \dots, n_{r-1})$ such that $n = \sum_{k=0}^{r-1} n_k$. Let $\lambda = (\lambda_0, \dots, \lambda_{r-1})$ be a multipartition of $n = n_0 + \cdots + n_{r-1}$ such that $\lambda_k \vdash n_k$ for $0 \leqslant k \leqslant r-1$. The construction of the irreducible characters for $G_{n,r}$ corresponding to each λ is a complex process and therefore we are going to start with a special case.

Let $n = n_k$ for some k where $0 \leqslant k \leqslant r-1$. Now, let λ be a multipartition with all component partitions being empty except for λ_k. Then we have that $\lambda_k \vdash n$. In this case the group is $G_{n_k,r} = G_{n,r}$ and so we denote the corresponding characters by χ_{λ_k} labelled by $\lambda_k \vdash n$.

Let all the δ_j act as ω_r^k where ω_r^k is an r-th root of unity. Then it gives a one-dimensional character $\eta_{n_j}^{(k)}$ of Δ_n denoted by $\eta_{n_k}^{(k)}(\delta_j) = \omega_r^k$ for all $j = 1, \dots, n$. This character can be extended to all $G_{n,r}$ by setting $\overline{\eta}_{n_k}^{(k)}(\sigma) = 1$ for all $\sigma \in S_n$.

Now, let $\mathfrak{P}_{n_k} \subseteq G_{n_k,r}$ be the natural subgroup isomorphic to S_{n_k}. Let $\mathrm{Irr}(S_{n_k})$ be the set of irreducible characters of S_{n_k}. Then the Specht module S^{λ_k} has character $\chi_{\lambda_k} \in \mathrm{Irr}(S_{n_k})$. By composition with the canonical projection $G_{n_k,r} \to S_{n_k}$, let $\tilde{\chi}_{\lambda_k} \in \mathrm{Irr}(G_{n_k,r})$ be the irreducible character corresponding to χ_{λ_k}. Then the inner tensor product $\overline{\eta}_{n_k}^{(k)} \otimes \tilde{\chi}_{\lambda_k}$ is an irreducible character of $G_{n_k,r} = G_{n,r}$.

For each n_k we have an irreducible character by the above construction, which gives us, in general for $\underline{n} = (n_0, \dots, n_{r-1})$ an irreducible character of $G_{n_0,r} \times \cdots \times G_{n_{r-1},r}$. We now proceed with the construction of the irreducible characters for the $G_{n,r}$ group.

Let us define a linear character $\eta_{\underline{n}}$ of $G_{n,r}$ by $\eta_{\underline{n}} = \boxtimes_{k=o}^{r-1} \eta_{n_k}^{(k)}$ where $\boxtimes$ is the outer tensor product of $\eta_{n_k}^{(k)}$. Then the characters $\eta_{\underline{n}}$ gives a complete set representatives for $\mathrm{Irr}(\Delta_n)$ under the induced action of $G_{n,r}$ since it acts on Δ_n by permutation of δ_k. So, by Clifford's Theorem [11, Theorem (6.5)] we find that

$$\mathrm{Irr}(G_{n,r}) = \bigsqcup_{\sum_{i=0}^{r-1} n_i = n} \mathrm{Irr}(G_{n,r} \mid \eta_{\underline{n}}) \tag{4.1}$$

where $\mathrm{Irr}(G_{n,r} \mid \eta_{\underline{n}})$ is the irreducible characters of $G_{n,r}$ that contains $\eta_{\underline{n}}$.

Again, the stabilizer of $\eta_{\underline{n}}$ in $G_{n,r}$ is the subgroup $\prod_{k=0}^{r-1} G_{n_k,r} = G_{n_0,r} \times \cdots \times G_{n_{r-1},r}$, since $G_{n,r}$ acts on Δ_n by permutation of δ_k. Let $\mathrm{Irr}\big(\prod_{k=0}^{r-1} G_{n_k,r} \mid \eta_{\underline{n}}\big)$ be the set of all $\psi \in \mathrm{Irr}\big(\prod_{k=0}^{r-1} G_{n_k,r}\big)$ whose restriction to Δ_n contains $\eta_{\underline{n}}$. We then have the description

$$\mathrm{Irr}\left(\prod_{k=0}^{r-1} G_{n_k,r} \mid \eta_{\underline{n}}\right) = \left\{ \boxtimes_{k=0}^{r-1} (\eta_{n_k}^{(k)} \otimes \tilde{\chi}_{\lambda_k}) \mid \lambda_k \vdash n_k \right\}. \tag{4.2}$$

Then, by [11, Theorem (6.5)], the induction $\mathrm{Ind}_{\prod_{k=0}^{r-1} G_{n_k,r}}^{G_{n,r}}$ defines a bijection

$$\mathrm{Irr}(\prod_{k=0}^{r-1} G_{n_k,r} \mid \eta_{(\underline{n})}) \xrightarrow{\sim} \mathrm{Irr}(G_{n,r} \mid \eta_{\underline{n}}), \quad \boxtimes_{k=0}^{r-1}(\eta_{n_k}^{j} \otimes \tilde{\chi}_{\lambda_k}) \longmapsto \chi_{\lambda}. \tag{4.3}$$

Therefore, combination of (4.1) and (4.3) gives us a natural parametrization of $\mathrm{Irr}(G_{n,r})$ by the multipartitions $\lambda = (\lambda_0, \ldots, \lambda_{r-1})$ such that $\sum_{k=0}^{r-1} |\lambda_k| = n$.

4.4 Multipartitions and Words

Some notions of tableaux to be used throughout this chapter are going to be defined in this section. The groups $G_{n,r}$ are isomorphic to the wreath product of the cyclic group of order r with the symmetric group S_n. From the discussions in Section 4.2 and Section 4.3 we find that the classes and characters are parameterized by r-tuples of partitions, such that the total sum of their parts equal n.

In order to construct the Specht modules with the idea of Specht matrices the first requirement is to decide on the types of words which depends on the types of letters. To establish a general case of this construction, we are going to introduce different indices to denote which letter belongs to which partition in the multipartition.

Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1})$ be a multipartition of n. Here we make the convention of using the indices $i = 0, 1, \ldots, r-1$ to indicate the letters corresponding to the partition λ_i. Therefore the letters would be of the form l_i.

For a multipartition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1}) \overset{r}{\vdash} n$, we construct a ***canonical word*** w_λ. For each λ_i we have the corresponding canonical words w_{λ_i} made up of the letters in the same way as we have seen for the S_n words, except for each letter now has an index $i \in \{0, 1, \ldots, r-1\}$. Then the canonical word corresponding to the multipartition λ is $w_\lambda = w_{\lambda_0} w_{\lambda_1} \cdots w_{\lambda_{r-1}}$.

As an example if we consider the multipartition $\lambda = (\lambda_0, \lambda_1, \lambda_2) = ((3,1),(2,2),(1,1))$ then the corresponding component canonical words are $w_{\lambda_0} = 1_0 1_0 1_0 2_0, w_{\lambda_1} = 1_1 1_1 2_2 2_2$

and $w_{\lambda_2} = 1_2 2_2$. The canonical word corresponding to λ is therefore $w_\lambda = w_{\lambda_0} w_{\lambda_1} w_{\lambda_2} = 1_0 1_0 1_0 2_0 1_1 1_1 2_1 2_1 1_2 2_2$ where the letters indexed $0, 1, 2$ correspond to the partitions λ_0, λ_1 and λ_2 respectively, and the multiplicities of the letters are defined as seen for the symmetric groups.

Now, for $\lambda \overset{r}{\vdash} n$ we have the corresponding canonical word w_λ. As there are n letters in w_λ, the symmetric group acts naturally from the right by rearranging the letters. As a result of this S_n-action, we find an orbit as a set consisting of all the rearrangements of w_λ. Therefore, the action on a single orbit yields a transitive action of S_n. These permutation actions on words are the same as those in in Chapter 3.

If we denote the stabilizer subgroup of S_n corresponding to the word w_λ by Stab(w_λ) then we have that Stab$(w_\lambda) = S_{\lambda_0} \times \cdots \times S_{\lambda_{r-1}}$ where S_{λ_k} denotes the stabilizer subgroup of the symmetric group S_{n_k}. Each S_{λ_i} stabilizes the corresponding word component w_{λ_i}. As $\prod_{k=0}^{r-1} S_{n_k} \leqslant S_n$, we have that $\prod_{k=0}^{r-1} S_{\lambda_k} \leqslant S_n$.

Evidently, as each component partition λ_i counts the multiplicities of letters indexed i, the multipartition counts the multiplicities of all the letters in a word. Since different multipartitions belong to different conjugacy classes of Young subgroups, it can be said that the isomorphism type of a set of rearrangements of a word is a multipartition of n.

Therefore, for any given word w, one can associate a shape, which is a multipartition. This can be achieved by counting the number of times each letter with index i appears and then by assigning it to a partition λ_i, the i-th component partition of a multipartition λ. We use the same notation as of symmetric groups to denote the associated shape by $\lambda(w)$ for a word w.

Proposition 4.4.1. *As permutation action of $\prod_{k=0}^{r-1} S_{n_k}$, the rearrangements of a word w_1 of length n and the rearrangements of a word w_2 of length n are isomorphic if and only if $\lambda(w_1) = \lambda(w_2)$.*

Proof. This proof follows from the proof of the Proposition 3.3.3 in Chapter 3. The only difference here is that the letters in this case are indexed rather than just letters in the symmetric groups. □

The following definitions and propositions are built up in a similar fashion to the definitions of a diagram and related properties as discussed in [18] and Chapter 3.

Definition 4.4.2 (Multidiagram). *A multidiagram is a finite subset of product of r copies of $\mathbb{N}^2$. The elements of a multidiagram are called diagrams and the elements of each diagram are called boxes.*

Definition 4.4.3. *Given a multipartition* $\lambda = (\lambda_0, \lambda_2, \dots, \lambda_{r-1})$*, the multidiagram associated to* λ *is*

$$\mathbb{D}(\lambda) = \{D(\lambda_i) | i = 0, 1, \dots, r-1\}$$

where $D(\lambda_i)$ *is the diagram corresponding to* λ_i*.*

Proposition 4.4.4. *A multidiagram* $\mathbb{D}(\lambda) = [D(\lambda_0), \dots, D(\lambda_{r-1})]$ *is the multidiagram of a multipartition* $\lambda = (\lambda_0, \dots, \lambda_{r-1})$ *if and only if each* $D(\lambda_i)$ *for* $0 \leqslant i < r$ *is individually closed under coordinate-wise* $\leqslant$*.*

Proposition 4.4.4 is a generalised version of Proposition 3.3.6. Therefore it follows from the arguments seen in [18].

To construct the Specht modules, we are interested in multidiagrams where each component diagram is corresponding to a partition. Naturally, as a visual representation of each word in the orbit of w_λ, we can associate a multihistogram to it. This idea of a multihistogram follows the same model for histograms. There is a $\lambda(w) = (\lambda_0, \lambda_1, \dots, \lambda_{r-1})$ for any word w. For each λ_i we have a histogram $h_{w_{\lambda_i}}$. Therefore, for any word w, we can find a multihistogram $(h_{w_{\lambda_0}}, \dots, h_{w_{\lambda_{r-1}}})$ as a tuple of histograms.

Definition 4.4.5 (Multihistogram). *Let* w *be a word considered as a function* $w : \{1, 2, 3, \dots, n\} \to \{1_0, 2_0, \dots, 1_1, 2_1, \dots, 1_{r-1}, 2_{r-1}, \dots\}$*. Then the multihistogram, denoted by* $\mathbb{H}_w$ *of the word* w *is the set* $\{\{w^{-1}(1_0), w^{-1}(2_0), \dots\}, \{w^{-1}(1_1), w^{-1}(2_1), \dots\}, \dots, \{w^{-1}(1_{r-1}), w^{-1}(2_{r-1}), \dots\}\}$ *of pre-images of its distinct letters.*

As an example, let us consider $\lambda = ((3,1), (2,2), (1,1))$ and $w_\lambda = 1_0 1_0 1_0 2_0 1_1 1_1 2_1 2_1 1_2 2_2$. Then the rows of each histogram corresponding to each partition is labelled by distinctly indexed letters. Let $\sigma = (1\ 8\ 5\ 7\ 3\ 4\ 6\ 10)(2\ 9) \in S_{10}$. We obtain a rearrangement of $w_\lambda \cdot \sigma = 2_1 1_2 2_0 1_1 2_1 2_2 1_0 1_1 1_0 1_0$ by the right action of σ on w_λ. Their respective multihistograms are as shown in Figure 4.2.

Therefore, for each component histogram we get a sequence of subsets of $\{1, 2, \dots, n\}$ decreasing in size. The union of the collection of all subsets for all the component histograms give us the set $\{1, 2, \dots, n\}$ back. From these subsets, we can recover the word corresponding to a given multihistogram. This recovery process is an extension of the same for the symmetric groups.

Proposition 4.4.6. *For any given multihistogram* $\mathbb{H}_w$ *there is a unique word* w *corresponding to it. The word* w *can be recovered from* $\mathbb{H}_w$*.*

$$\begin{array}{c|ccc} 1_0 & 1 & 2 & 3 \\ 2_0 & 4 & & \end{array}, \quad \begin{array}{c|cc} 1_1 & 5 & 6 \\ 2_1 & 7 & 8 \end{array}, \quad \begin{array}{c|c} 1_2 & 9 \\ 2_2 & 10 \end{array}$$

(a) w_λ multihistogram

$$\begin{array}{c|ccc} 1_0 & 7 & 9 & 10 \\ 2_0 & 3 & & \end{array}, \quad \begin{array}{c|cc} 1_1 & 4 & 8 \\ 2_1 & 1 & 5 \end{array}, \quad \begin{array}{c|c} 1_2 & 2 \\ 2_2 & 6 \end{array}$$

(b) $w_\lambda \cdot \sigma$ multihistogram

Figure 4.2: Multihistograms for w_λ and $w_\lambda \cdot \sigma$

Let w be a word with letters $\{1_0, 2_0, 3_0, \ldots, 1_1, 2_1, 3_1, \ldots, 1_2, 2_2, 3_3, \ldots\ l_k\}$. For each of the indices we have a few letters. Consider a multihistogram $\mathbb{H}_w = \{h_{w_{\lambda_0}}, \ldots, h_{w_{\lambda_{r-1}}}\}$. Now, each row of $h_{w_{\lambda_i}}$ has some elements, giving us a subset of $\{1, 2, \ldots, n\}$ corresponding to a row tabloid. Each row of $h_{w_{\lambda_i}}$ is labelled by a letter with index i. Let us denote the row subsets by $h_{w_{\lambda_i}}[l_i]$ where l_i is the letter labelling the respective rows. Then we have that $\bigcup_{i=0}^{r-1} h_{w_{\lambda_i}}[l_i] = \{1, 2, \ldots, n\}$. These subsets $h_{w_{\lambda_i}}[l_i]$ describe the position of the letter l_i in the word w. Therefore, running through all $i \in \{0, 1, \ldots, r-1\}$ and for all letters l_i, we obtain the n-lettered word w corresponding to $\mathbb{H}_w$.

For example, consider the multihistogram in Figure 4.2(b). Then the corresponding set of distinct letters $l_i \in \{1_0, 2_0, 1_1, 2_1, 1_2, 2_2\}$ with respective set of subsets $h_{w_{\lambda_i}}[l_i] \in \{\{7, 9, 10\}, \{3\}, \{4, 8\}, \{1, 5\}, \{2\}, \{6\}\}$ describes the positions of each l_i in the word. Therefore, for all $i \in \{0, 1, 2\}$, we obtain the word corresponding to this multihistogram is $w = 2_1 1_2 2_0 1_1 2_1 2_2 1_0 1_1 1_0 1_0$.

The symmetric group S_n acts on the words w of length n by giving all the possible rearrangements of the word. The same can be understood with the help of their respective multihistograms. Naturally, the question that follows is how these rearrangements of a single word are ordered. For the symmetric groups we had the natural lexicographic ordering, which was defined by the ordering of the natural numbers. As we have indexed letters now, we can not extend the same natural ordering here. Therefore, we define a lexicographic ordering on the words for this group.

Definition 4.4.7 ([17, **Lexicographic Order**]). *Let us set the convention of the ordering of the letters* $1_0 < 2_0 < 3_0 < \cdots < 1_1 < 2_1 < 3_1 < \cdots < 1_{r-1} < 2_{r-1} < 3_{r-1} < \cdots$.

Given the ordering among the letters in Definition 4.4.7, we can compare any two words made up of these letters. Consider two words $w = w_1 w_2 \dots w_n$ and $v = v_1 v_2 \dots v_n$ of same length n. Then we say that $w = v$ if $w_i = v_i$, $w < v$ or w appears earlier than v in the lexicographic order if we have $w_i < v_i$, and $w > v$ or w appears later than v in the lexicographic order if some $w_i > v_i$ for any $0 \leqslant i < r$.

4.5 Multitableaux and Pair of Words

We have an analogue of a Young diagram for the case of monomial groups similar to the idea in Chapter 3. As there is a multidiagram corresponding to every multipartition of a natural number n, each row of the corresponding component diagram has a length. Assigning an empty box to each of these positions in the rows of the diagrams, we convert the diagrams into Young diagrams. Therefore, the multidiagram of which each component diagram is a Young diagram, is called a *Young multidiagram*. Clearly there are n boxes in a Young multidiagram. Now, filling the n boxes with numbers $\{1, 2, \dots, n\}$ will give us an analogue of the Young tableaux that are used to represent S_n.

Definition 4.5.1 (Young Multitableau). *If $\lambda \overset{r}{\vdash} n$, then a λ-multitableau (or Young multitableau of shape λ) is a 3-dimensional array t of integers obtained by placing $\{1, 2, \dots, n\}$ in the boxes of r Young diagrams corresponding to each λ_i respectively, i.e., $t = (t_0, \dots, t_{r-1})$ is a Young multitableau where t_i is a Young tableau of shape λ_i for $i = 0, 1, \dots, r-1$.*

As an example, we have $\lambda = (\lambda_0, \lambda_1, \lambda_2) = ((3,1),(2,2),(1,1)) \overset{3}{\vdash} 10$ and therefore one of the corresponding multitableaux is as follows

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 6 & 2 \\ \hline 7 & 10 \\ \hline\end{array}, \begin{array}{|c|}\hline 8 \\ \hline 9 \\ \hline\end{array}.$$

Clearly, by the right-action of S_{10}, we will have a corresponding multitableau for each element in S_{10}. This implies there will be a total of $10!$ Young multitableaux of shape $((3,1),(2,2),(1,1))$.

As a generalisation, for any multipartition of n, we have a total of $n!$ Young multitableaux. The analogue of standard tableau and canonical tableau are as follows.

Definition 4.5.2 (Standard Multitableau). *A Young multitableau is a standard multitableau when the numbers in the boxes of each t_i in t are increasing to the right and to the bottom for each row and each column respectively.*

From now on we will use the abbreviation *SYMT* for standard Young multitableau.

Definition 4.5.3 (Canonical Multitableau). *Canonical multitableau of shape λ is a SYMT denoted by $t_\lambda = (t_0, \dots, t_{r-1})$ can be obtained by filling out the boxes of t_0 row by row with non-repeated integers $\{1, \dots, a_0\}$ so that t_0 is standard, followed by filling the boxes of t_1 with integers $\{a_0 + 1, a_0 + 2, \dots, a_0 + a_1\}$ and so on until all the tableaux t_i for $0 \leqslant i < r$ are filled with integers $\{1, \dots, n\}$.*

As an example, for $\lambda = ((3,1),(2,2),(1,1)) \overset{3}{\vdash} 10$ we have the standard canonical λ-multitableau as

$$t_\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & 8 \\ \hline\end{array},\ \begin{array}{|c|}\hline 9 \\ \hline 10 \\ \hline\end{array}.$$

Therefore we have the natural action of symmetric group on the labelling of the boxes of the multitableau. As an example, $(1,3)(4,10)(6,8,9) \in S_{10}$ acts from the right on the multitableau mentioned above as follows

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & 8 \\ \hline\end{array},\ \begin{array}{|c|}\hline 9 \\ \hline 10 \\ \hline\end{array} \cdot (1,3)(4,10)(6,8,9) = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 10 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 5 & 9 \\ \hline 7 & 6 \\ \hline\end{array},\ \begin{array}{|c|}\hline 8 \\ \hline 4 \\ \hline\end{array}$$

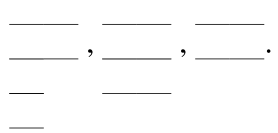
by preserving the shape of the multitableau but with another numbering of the diagram. For each multitableau we have a corresponding element in S_n. Let the canonical SYMT be the multitableau that corresponds to the id $\in S_n$. Then, by applying the group action on the canonical SYMT we can generate all the possible numberings of the multitableau.

For any multipartition we also need to define the corresponding conjugate multipartition which will give us another tableau which we will use later on.

Definition 4.5.4 (Conjugate Multipartition). *Let $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r-1}) \overset{r}{\vdash} n$. Then the conjugate multipartition of λ is $\mu = (\mu_0, \mu_1, \dots, \mu_{r-1}) \overset{r}{\vdash} n$ such that $\mu_i = \lambda_i^T$ for all i. Therefore we can also write $\mu = \lambda^T$ to denote the conjugate multipartition of λ analogous to the symmetric group case.*

Therefore, for the conjugate multipartitions λ^T we also have corresponding multidiagram of which every component diagram is the transpose diagram of λ_i in λ.

For example, if $\lambda = ((3,1),(2,2),(1,1))$ then we have $\lambda^T = ((2,1,1),(2,2),(2))$ which is obtained from the multidiagram



The canonical word for $\lambda^T = ((2,1,1),(2,2),(2))$ is therefore $w_{\lambda^T} = 1_0 1_0 2_0 3_0 1_1 1_1 2_1 2_1 1_2 1_2$. So we have the multihistogram associated to w_{λ^T} as:

$$\begin{array}{ll} 1_0 & 1 \;\; 2 \\ 2_0 & 3 \\ 3_0 & 4 \end{array}, \quad \begin{array}{ll} 1_1 & 5 \;\; 6 \\ 2_1 & 7 \;\; 8 \end{array}, \quad \begin{array}{ll} 1_2 & 9 \;\; 10 \end{array}$$

If we take any rearrangement of the integers in each row of the histograms in the multihistogram, which describe the positions of the respective row labelling letters in the corresponding words, then it has no effect on the word itself and remains itself due to the fact that rearranging a letter in its corresponding positions makes no visual change to the word. Therefore, the subsets of $\{1,2,\dots,n\}$ respective to the row labels fix the word. Therefore it gives us an equivalence class for the multihistograms which is analog to the equivalence class of the Young tableau in S_n, known as tabloids.

Definition 4.5.5 (Multitabloid). *Two λ-multitableaux $t = (t_0, \dots, t_{r-1})$ and $s = (s_0, \dots, s_{r-1})$ are row-equivalent, $t \sim s$, if corresponding Young tableaux t_i and s_i are row-equivalent to each other for all $i = 0, \dots, r-1$. A multitabloid of shape λ, or λ-multitabloid is then*

$$\{t\} = \{s | s \sim t\}$$

where t is of shape λ.

In a similar fashion we can define the column-equivalence for two λ-multitableaux. Therefore, we can say that all the λ-multitableuax that are row equivalent to t belong to the same row equivalence class. Likewise, we also define the column equivalence class $[t]$ for t.

Definition 4.5.6 (Intersection Multihistogram). *Let for two words u and v of same length n, their corresponding multihistograms be $\mathbb{H}_u = \{h_{u_0}, h_{u_1}, \dots, h_{u_{r-1}}\}$ and $\mathbb{H}_v = \{h_{v_0}, h_{v_1}, \dots, h_{v_{r-1}}\}$ respectively. We define the intersection multihistogram $\mathbb{H} = \mathbb{H}_u \cap \mathbb{H}_v$ such that $\mathbb{H} = (h_{u_0} \cap h_{v_0}, h_{u_1} \cap h_{v_1}, \dots, h_{u_{r-1}} \cap h_{v_{r-1}})$, i.e., each component histogram h_i of $\mathbb{H}$ obtained by the intersection of the corresponding histograms h_{u_i} and h_{v_i} of $\mathbb{H}_u$ and $\mathbb{H}_v$ respectively for $i \in \{0,1,\dots,r-1\}$.*

We have seen previously in S_n how two histograms intersect with each other and that results in a histogram of a Young tableau. Similarly, as all the component histograms h_i of the multihistogram $\mathbb{H}$ are obtained by the intersection of two corresponding histograms, they may not always turn out to be tableaux. We are only interested in intersection multihistograms that have shapes $\lambda \overset{r}{\vdash} n$.

Definition 4.5.7 (Simultaneous Multihistogram). *Let (u, v) be a pair of words of same length n. Then there exists a simultaneous multihistogram $\mathbb{H} = (h_0, h_1, \ldots, h_{r-1})$ for the words pair (u, v) obtained by the intersection of two multihistograms corresponding to u and v respectively given that the following criteria hold*

1. *each component histogram h_i for $i \in \{0, 1, \ldots, r-1\}$ is a simultaneous histogram,*
2. *the shape of $\mathbb{H}$ is a multidiagram that represents a multipartition of n.*

Similar to the ideas discussed in Chapter 3, we consider a pair of n-lettered words (w_1, w_2) where $\lambda(w_1), \lambda(w_2) \overset{r}{\vdash} n$. As the $G_{n,r}$ word letters are now indexed unlike in the symmetric groups, we need one extra criterion to be satisfied for a pair of words to be fit in a simultaneous multihistogram, more on that later. Moreover, a simultaneous multihistogram is a Young multitableau such that each of the component tableaux has its rows and columns labelled by the distinct letters in the row and column word respectively and the pair (w_1, w_2) has complementary arrangements. Also, we can see that for the intersection multihistogram to become a multitableau, the component histograms for words have to be tableaux themselves. Therefore, the shapes of histograms of the corresponding words that intersect each other need to be conjugates of each other.

Proposition 4.5.8. *Let u and v be two words of same length, and h be the intersection multihistogram of their corresponding multihistograms, then the shape of u must be the transpose of the shape of v.*

Example 4.5.9. Let $n = 10$. Consider $w_1 = 1_0 1_0 1_0 2_0 1_1 1_1 2_2 2_2 1_2 2_2$ and $w_2 = 1_0 1_0 2_0 3_0 1_1 1_1 2_1 2_1 1_2 1_2$. As clearly seen if we use the pair

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1_0 & 1_0 & 1_0 & 2_0 & 1_1 & 1_1 & 2_1 & 2_1 & 1_2 & 2_2 \\ 1_0 & 1_0 & 2_0 & 3_0 & 1_1 & 1_1 & 2_1 & 2_1 & 1_2 & 1_2 \end{pmatrix}$$

then they fail to satisfy criterion 1 in Definition 4.5.7 and therefore cannot be fitted in a simultaneous multihistogram. Instead if we consider the word $w_1 = 1_0 1_0 1_0 2_0 1_1 1_1 2_2 2_2 1_2 2_2$ as it is and take a rearrangement of w_2 as $w_2' = 1_0 2_0 3_0 1_0 1_1 2_1 1_1 2_1 1_2 1_2$, we can achieve a unique free orbit. Now we have the pair

$$\begin{pmatrix} w_1 \\ w_2' \end{pmatrix} = \begin{pmatrix} 1_0 & 1_0 & 1_0 & 2_0 & 1_1 & 1_1 & 2_1 & 2_1 & 1_2 & 2_2 \\ 1_0 & 2_0 & 3_0 & 1_0 & 1_1 & 2_1 & 1_1 & 2_1 & 1_2 & 1_2 \end{pmatrix}$$

fully satisfies all the criteria to be fitted in a simultaneous multihistogram as shown in Figure 4.3.

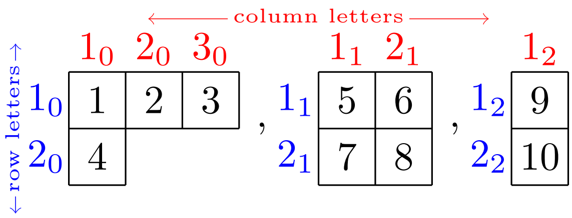


Figure 4.3: Simultaneous multihistogram of the pair (w_1, w_2')

We can see that $\lambda(w_1) = ((3,1),(2,2),(1,1))$ and $\lambda(w_2) = ((2,1,1),(2,2),(2)) = \lambda(w_1)^T$. Now in the pair (w_1, w_2'), each column $(l_i^{w_1}, l_i^{w_2'})$ is corresponding to a numbering in the boxes of tableaux in the multihistogram. The respective numbers in the boxes therefore denote the position of the column in the sequence of columns in the word pair.

$$\begin{array}{r|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline w_1 \rightarrow & 1_0 & 1_0 & 1_0 & 2_0 & 1_1 & 1_1 & 2_1 & 2_1 & 1_2 & 1_2 \\ w_2' \rightarrow & 1_0 & 2_0 & 3_0 & 1_0 & 1_1 & 2_1 & 1_1 & 2_1 & 1_2 & 1_2 \end{array}$$

Therefore, by right-action of symmetric group on the pair of word we get a different ordering of the columns and this in return give us all the possible numbering of the boxes in the simultaneous multihistogram.

We see that the multitableau for the pair of words (w_1, w_2) in Example 4.5.9 is a simultaneous multihistogram. An interesting point to notice is that the letters in any column of the pair of words have same index. The rows and columns of the i-th histogram in the simultaneous histogram is labeled by the letters indexed by i in the words w_1 and w_2 respectively.

For two multihistograms to intersect each other to give a simultaneous multihistogram, we must have the i-th component histogram of one word intersects the i-th component histogram of the other word, i.e., in the pair of words (w_1, w_2) the letters in any column must have same index.

So in the same way, these simultaneous multihistograms can be achieved by the intersection of two multihistograms for corresponding words such that each component histogram for index i of one word is intersecting the component histogram with index i of the other word. Hence, (w_1, w_2) is a simultaneous multihistogram if and only if $\lambda(w_1)^T = \lambda(w_2)$ and the elements in the j-th row of the i-th histogram of w_1-multihistogram has same elements as in the j-th column of the i-th histogram of w_2-multihistogram.

Again, S_n acts on each of the words by rearranging its letters. For two words u, v of same length n, we get the sets of rearrangements U, V by the S_n action on u, v respectively. Therefore, S_n action on the pair (u, v) gives us the set $U \times V$, i.e., all possible rearrangements of the columns in the pair (u, v). Now let us restrict our focus to the pairs of words that in each column we have both letters with same index.

Definition 4.5.10. *Let $\lambda \overset{r}{\vdash} n$. Let $u = w_\lambda$ and $v = w_{\lambda^T}$ be the canonical words corresponding to λ and λ^T respectively. Let U and V be the sets of all rearrangements of u and v respectively. We define $(U \times V)^\#$ to be the subset of $U \times V$ consisting of all pair of words such that each pair has letters with matching indices in every column. Then $(U \times V)^\#$ is a S_n-set that has a unique free orbit characterised by a pair of words that has no repeated columns.*

Theorem 4.5.11. *Let v and w be two words of length n. Then the action of S_n on the pair (v', w'), where v' and w' are some rearrangements of v and w respectively, has a unique free orbit if and only if $\lambda(w)$ is the transposed multipartition of $\lambda(v)$.*

Proof. The proof of this theorem is straightforward and uses the same idea of the proof of Theorem 3.4.11.

Let v, w be two words with of same length n. Let $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r-1}) \overset{r}{\vdash} n$ and $\mu = (\mu_0, \mu_1, \dots, \mu_{r-1}) \overset{r}{\vdash} n$ such that $\lambda(v) = \lambda$ and $\lambda(w) = \mu$. Now, let v', w' be some rearrangement of the words v, w respectively for which we have a simultaneous multihistogram $H = (h_0, h_1, \dots, h_{r-1})$. Therefore, H can be considered as a matrix of which the columns are made of letters indexed by $i \in \{0, 1, \dots, r-1\}$ and corresponding to each of the words respectively.

Now, for each of the words v' and w' we have a corresponding multihistogram. We also know that multihistograms are set of subsets of $\{1, 2, \ldots, n\}$. Then for a pair of words (v', w'), we have two multihistograms and two sets of subsets of $\{1, 2, \ldots, n\}$ corresponding to each word v' and w' respectively.

The simultaneous multihistogram H therefore is simply the intersection of the two multihistograms corresponding to v' and w' such that each tableau $h_i \in H$ is obtained by the intersection of the corresponding i-th histograms of each word respectively. Now, for each of these component tableau h_i in H, we can apply the recovery process explained in Theorem 3.4.11 to avoid the failures $(F1)$ and $(F2)$.

Therefore, for each index i we find that the every distinct letters indexed i of w' pair up with the most frequent letter 1_i of v', followed by distinct i-indexed letters of w' from the remaining letters pair up with the second most frequent letter 2_i of v' and so on until all the letters indexed i are over. Iterating through all the indices, we find that for each i, the shape of i-th histogram in the multihistogram for w' is the transpose of the i-th histogram in the multihistogram of v'. Hence, $\mu_i = \lambda_i^T \implies \lambda(w) = \lambda(v)^T$.

□

Of all the possible rearrangements of the numbering of the multitableau, we have a canonical multitableau which corresponds to a pair of word. We can call such a pair of words a canonical pair. A canonical pair of word for the monimial groups can be chosen by following the same guidelines of choosing the canonical pair of words for symmetric groups, except that we now also have to carefully consider the indices of the letters.

Definition 4.5.12 (Canonical Pair). *A pair (u, v) of words of same length is the canonical pair if we take u as it is but consider v in such a way that all the distinct letters of index 0 appear first once each in the increasing order and then they repeat the pattern until their multiplicities are exhausted, then repeat the same for letters with index 1, and so on until all the letters of v are exhausted.*

There can be only one canonical pair of such words, however, there are overall $r^n \cdot n!$ ways to pair up words of same length corresponding to a multipartition and its conjugate multipartition so that each of them correspond to a Young multitableau. Therefore, as in Example 4.5.9 we can see that all the criteria for satisfying a pair to be canonical hold, we can say that the chosen pair (w_1, w_2') is canonical. Quite clearly, canonical multitableau always corresponds to canonical pair of words and that is why in terms of the symmetric group actions we consider the canonical pair/multitableau to be a representative of the the id $\in S_n$. Symmetric group acts on the canonical multitableau from the right by giving us all possible $n!$ possible multitableaux of the same shape.

Therefore, we have the following corollary as a direct implication of Definition 4.5.10 and Theorem 4.5.11.

Corollary 4.5.13. *Let $\lambda \overset{r}{\vdash} n$. The function $\mathbb{H} : (\nu, w) \mapsto$ simultaneous multihistogram of (ν, w) establishes a bijection between the elements of the unique free S_n-orbit on the set $\{$rearrangements of $w_\lambda\} \times \{$rearrangements of $w_{\lambda^T}\}$ and the set of multitableaux of shape λ.*

At the intersection of two standard multitableaux of same shape sometimes we may find an ***inter-multitableau***, which of course is nonstandard. Let us understand it with the help of the following example.

Example 4.5.14. Let us consider two SYMT of shape $\lambda = ((2, 2, 1), (1, 1), (2))$,

$$t_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 8 & 9 \\ \hline\end{array} \quad \text{and} \quad t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 8 & 9 \\ \hline\end{array}$$

respectively. Then for t_1 we have the row stabilizer subset as $\{t_1\} = \{\{\{1, 4\}, \{2, 5\}, \{3\}\}, \{\{6\}, \{7\}\}, \{8, 9\}\}$ and similarly for the transposed t_2 we have $\{t_2^T\} = \{\{\{1, 3, 5\}, \{2, 4\}\}, \{\{6, 7\}\}, \{8\}, \{9\}\}$ (which is the column stabilizer of t_2). Therefore, extending the same idea of a inter-tableau for a pair of same-shaped tableaux for the cases of multitableaux t_1, t_2, we obtain a inter-multitableau t such that $t_{k_{(i,j)}} = \{t_1\}_{k_i} \cap \{t_2^T\}_{k_j}$. Therefore, we have

$$\begin{aligned}
t_{1_{(1,1)}} &= \{1,4\} \cap \{1,3,5\} = \{1\} \\
t_{1_{(1,2)}} &= \{1,4\} \cap \{2,4\} = \{4\} \\
t_{1_{(2,1)}} &= \{2,5\} \cap \{1,3,5\} = \{5\} \\
t_{1_{(2,2)}} &= \{2,5\} \cap \{2,4\} = \{2\} \\
t_{1_{(3,1)}} &= \{3\} \cap \{1,3,5\} = \{3\} \\
t_{1_{(3,2)}} &= \{3\} \cap \{2,4\} = \varnothing \\
t_{2_{(1,1)}} &= \{6\} \cap \{6,7\} = \{6\} \\
t_{2_{(2,1)}} &= \{7\} \cap \{6,7\} = \{7\} \\
t_{3_{(1,1)}} &= \{8,9\} \cap \{8\} = \{8\} \\
t_{3_{(1,2)}} &= \{8,9\} \cap \{9\} = \{9\}
\end{aligned}$$

implying that we have

$$t = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline\end{array}\,,\ \begin{array}{|c|c|}\hline 8 & 9 \\ \hline\end{array}\,.$$

Again, if we change the order of the multitableaux t_1 and t_2 and consider the row stabilizers $\{t_2\}$ and $\{t_1^T\}$, then we find that in the intersection tableau s (say), we have

$$|s_{1_{(1,1)}}| = |\{1,2\} \cap \{1,2,3\}| = |\{1,2\}| \neq 1.$$

Therefore intersection multitableau, in this case, only exists for (t_1, t_2) and not for (t_2, t_1).

It is quite clear that for any two multitableaux of same shape, we have an inter-multitableau by the extension of Proposition 3.4.15 for each of the corresponding component tableaux.

Proposition 4.5.15. *Let* $t_1 = (t_{1_0}, t_{1_1}, \dots, t_{1_{r-1}})$ *and* $t_2 = (t_{2_0}, t_{2_1}, \dots, t_{2_{r-1}})$ *be two multitableaux of shape* $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r-1})$. *Then there exists an inter-multitableau* t *such that* $t_{k_{(i,j)}} = \{t_1\}_{k_i} \cap \{t_2^T\}_{k_j}$ *for* $k \in \{0, 1, \dots, r-1\}$ *if and only if for any* (i,j) *we have that* $|t_{k_{(i,j)}}| = 1$ *where* (i,j) *is a position of a box in the Young tableau* t_k *of shape* λ_k.

For $n \geqslant 5$, we get to see inter-multitableau occur in the intersection of two SYMT of same shape given that the above criteria hold. Again, two words are complementary if an only if the diagonal action of the symmetric group on the product of the sets of rearrangements of the words, gives a unique free orbit. Therefore, complementary words correspond to a multipartition and its conjugate multipartition respectively to be fitted in a simultaneous multihistogram.

4.6 Specht Objects

Now let us construct the Specht modules for these monomial groups. To make things simpler, we make an analog of flat-tableaux that was seen for the symmetric groups.

Definition 4.6.1 (Flat-multitableau). *Let* $t = (t_0, t_1, \dots, t_{r-1})$ *be a Young multitableau of shape* $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r-1})$ *where* λ *is a multipartition of* $n = a_0 + a_1 + \cdots + a_{r-1}$ *such that* $\lambda_i \vdash a_i$ *for* $i = 0, 1, \dots, r-1$. *Then we say that flat-multitableau* $t^\flat$ *is the flat list of the flat-tableaux* t_i.

As an example, if we have

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 8 & 9 \\ \hline\end{array}$$

then the flat-multitableau $t^\flat = [1, 2, 3, 4, 5, 6, 7, 8, 9]$.

Again, for each multitableau we have a flat-multitableau and by the right action of symmetric group on the canonical flat-tableau $t^\flat_\lambda$, we obtain all possible flat-multitableau on n points. We can therefore find the permutation acting on the canonical flat-multitableau that gives us all the other flat-multitableau by using the same process as described in Chapter 3.

Definition 4.6.2 (Young Character for Multitableau). *Let* $t^\flat_\lambda$ *be the canonical flat-multitableau corresponding to multitableau* t *of shape* $\lambda \overset{r}{\vdash} n$. *If we obtain another flat-multitableau* $t^\flat = t^\flat_\lambda \cdot \sigma$ *of same shape* λ *for some* $\sigma \in S_n$, *then we define the Young character denoted by* $Y(t)$ *as* $Y(t) = \mathrm{sgn}(t^\flat)$.

Hence, the Young character of a multitableau t is decided by the sign of the permutation σ that converts the canonical multitableau t_λ to t. Therefore, Definition 3.5.6 still holds for this case due to the similarities in the definitions of Young characters for S_n and $G_{n,r}$.

Then by considering the alternate notation of the multitableau $t = (w_1^\lambda, w_2^\lambda)$ where the pair of words corresponds to the multitableau t of shape $\lambda \overset{r}{\vdash}$, we write $Y(t) = Y(w'_1, w'_2)$. Therefore, we obtain the Specht matrix for a pair of complementary words w_1, w_2 as follows.

Definition 4.6.3 (Specht Matrix for $G_{n,r}$ Group). *Let $\lambda \overset{r}{\vdash} n$. Let $w_1 = w_\lambda$ and $w_2 = w_{\lambda^T}$ be the complementary canonical words of length n and of shapes λ and λ^T respectively. Let A and B be the sets of all rearrangements of w_1 and w_2 respectively. Then we define the Specht matrix $\varphi(\lambda) : A \times B \to \{0, \pm 1\}$ defined by $(w'_1, w'_2) \mapsto Y(w'_1, w'_2)$ where $Y(w'_1, w'_2) = Y(t)$ only if (w'_1, w'_2) corresponds to the multitableau t.*

The only extra criteria now added are in terms of how these simultaneous multihistograms are built considering the indices of the letters. Clearly, there will be $n!$ nonzero entries in the Specht matrix $\varphi(w_1, w_2)$.

For a concrete understanding , let us have a look at an example on how we build the Specht matrix for monomial groups.

Example 4.6.4. Let for $n = 4$ and we consider the multipartition $\lambda = ((1,1),(1),(1))$. Then the conjugate multipartition is $\lambda^T = ((2),(1),(1))$. Therefore, we have the corresponding canonical row and column words $u_\lambda = 1_0 2_0 1_1 1_2$ and $v_{\lambda^T} = 1_0 1_0 1_1 1_2$ respectively. If we consider the words as a pair then we have

$$\begin{pmatrix} u_\lambda \\ v_{\lambda^T} \end{pmatrix} = \begin{pmatrix} 1_0 & 2_0 & 1_1 & 1_2 \\ 1_0 & 1_0 & 1_1 & 1_2 \end{pmatrix}$$

and clearly there is no column repeated in the pair. We already have obtained the free orbit for this pair and we can also see that in any column of the pair the letters have same index, implying that this pair satisfies the criteria for a simultaneous multihistogram. Also it can be noticed that the pair is corresponding to the canonical multitableau of shape λ, therefore we can obtain the Specht matrix.

Let $w_1 = u_\lambda$ and $w_2 = v_{\lambda^T}$. Then, the Specht matrix $\varphi(\lambda)$ is shown in Table 4.1. It can be seen that each row in the matrix is labelled by some rearrangement of the row word $1_0 2_0 1_1 1_2$ and each column is labelled by some rearrangement of the column word $1_0 1_0 1_1 1_2$.

In Chapter 3, we proved that the column space of the Specht matrices are the Specht modules associated with a partition. Referring to it, we claim that for the monomial groups the column space of the Specht matrix $\varphi(\lambda)$ for $\lambda \overset{r}{\vdash} n$ is an irreducible representation.

Let $\lambda \overset{r}{\vdash} n$. We define a generalised permutation module M^λ. Let w_λ be a canonical word corresponding to λ and A be the set of all the rearrangements of w_λ. A basis of this module M^λ is described by the set A.

	$1_01_01_11_2$	$1_01_01_21_1$	$1_01_11_01_2$	$1_01_11_21_0$	$1_01_21_01_1$	$1_01_21_11_0$	$1_11_01_01_2$	$1_11_01_21_0$	$1_11_21_01_0$	$1_21_01_01_1$	$1_21_01_11_0$	$1_21_11_01_0$
$1_02_01_11_2$	1	·	·	·	·	·	·	·	·	·	·	·
$1_02_01_21_1$	·	−1	·	·	·	·	·	·	·	·	·	·
$1_01_12_01_2$	·	·	−1	·	·	·	·	·	·	·	·	·
$1_01_11_22_0$	·	·	·	1	·	·	·	·	·	·	·	·
$1_01_22_01_1$	·	·	·	·	1	·	·	·	·	·	·	·
$1_01_21_12_0$	·	·	·	·	·	−1	·	·	·	·	·	·
$2_01_01_11_2$	−1	·	·	·	·	·	·	·	·	·	·	·
$2_01_01_21_1$	·	1	·	·	·	·	·	·	·	·	·	·
$2_01_11_01_2$	·	·	1	·	·	·	·	·	·	·	·	·
$2_01_11_21_0$	·	·	·	−1	·	·	·	·	·	·	·	·
$2_01_21_01_1$	·	·	·	·	−1	·	·	·	·	·	·	·
$2_01_21_11_0$	·	·	·	·	·	1	·	·	·	·	·	·
$1_11_02_01_2$	·	·	·	·	·	·	1	·	·	·	·	·
$1_11_01_22_0$	·	·	·	·	·	·	·	−1	·	·	·	·
$1_12_01_01_2$	·	·	·	·	·	·	−1	·	·	·	·	·
$1_12_01_21_0$	·	·	·	·	·	·	·	1	·	·	·	·
$1_11_21_02_0$	·	·	·	·	·	·	·	·	1	·	·	·
$1_11_22_01_0$	·	·	·	·	·	·	·	·	−1	·	·	·
$1_21_02_01_1$	·	·	·	·	·	·	·	·	·	−1	·	·
$1_21_01_12_0$	·	·	·	·	·	·	·	·	·	·	1	·
$1_22_01_01_1$	·	·	·	·	·	·	·	·	·	1	·	·
$1_22_01_11_0$	·	·	·	·	·	·	·	·	·	·	−1	·
$1_21_11_02_0$	·	·	·	·	·	·	·	·	·	·	·	−1
$1_21_12_01_0$	·	·	·	·	·	·	·	·	·	·	·	1

Table 4.1: Specht matrix $\varphi(((1,1),(1),(1)))$

Definition 4.6.5 (Generalised Permutation Module). *Let* $\lambda \overset{r}{\vdash}$ *and* w_λ *be the canonical word. We define the generalised permutation module*

$$M^\lambda = \langle q_w : w \in A\rangle_{\mathbb{C}}$$

as a vector space over $\mathbb{C}$*, where* A *is the set of all rearrangements of the word* w_λ*.*

Let $w \in A$ be a word and $w = ((l_1)_{a_1} \cdots (l_j)_{a_j} \cdots (l_n)_{a_n})$ where $(l_j)_{a_j}$ denotes that the letter l_j has index a_j. Then for $\delta_j \in G_{n,r}$ we have $q_w \cdot \delta_j = \omega_r^{a_j} \cdot q_w$. Then we define the action of $G_{n,r}$ on the set A as follows.

Definition 4.6.6. *For* $g \in G_{n,r}$ *and* $w \in A$*, we have* $G_{n,r}$ *acting on* M^λ *defined by*

$$q_w \cdot g = \begin{cases} q_{w \cdot g}, & \text{if } g \in S_n \\ \omega_r^{a_j} \cdot q_w, & \text{if } g = \delta_j. \end{cases}$$

Now we have that M^λ is a S_λ-module, and so it is a Δ_n-module. Therefore, as $q_{w_\lambda} \cdot \delta_k = \omega_r^{a_k} \cdot q_{w_\lambda} \in M^\lambda$, we have the following proposition.

Proposition 4.6.7. *Let* $\lambda \overset{r}{\vdash} n$ *and* M^λ *be a generalised permutation module. Then* M^λ *is a* $G_{n,r}$*-module.*

Let $R(t)$ and $C(t)$ in $\mathfrak{P}_n \cong S_n$ be respectively the row and column stabilizers of the multitableau t. Then we have

$$p_t = \sum_{p \in R(t)} p, \; c_t = \sum_{c \in C(t)} \bar{c}$$

where $\bar{c} = \operatorname{sgn}(c) \cdot c$. Then product $p_t c_t$ is an element of $\mathbb{C}[G_{n,r}]$. If $\{t\}$ is the row multitabloid of t then we have $\{t\} = t \cdot p_t$ as before.

Definition 4.6.8. *We define a polymultitabloid ν_t for a multitableau t given by $\nu_t = \{t\} \cdot c_t$.*

Similar to the symmetric groups, we have that $\nu_t = t \cdot p_t c_t$, and for $t = t_\lambda$ we then find $\nu_\lambda = p_\lambda c_\lambda$. Then we obtain the elements in the same column of the multitableau t rearranged by a permutation action of $C(t)$ on $\{t\}$ and so for $\pi \in C(t)$ we get the new row multitabloids $\{t \cdot \pi\}$.

Definition 4.6.9. *Let w_1, w_2 be two words having complementary rearrangements of shape $\lambda \overset{r}{\vdash} n$ and corresponding to a Specht matrix $\varphi(\lambda)$. Then we define the elements*

$$\nu_{w_2} = \sum_{w_1'} Y(w_1', w_2) q_{w_1'}$$

where $Y(w_1', w_2)$ is the Young character corresponding to the pair (w_1', w_2) and $q_{w_1'}$ is the vector corresponding to w_1' in M^λ.

Again, letting $C(t)$ act on the multitabloid $\{t\}$ with its corresponding element as its coefficient, and considering these new multitabloids with their signs, we obtain ν_t. In $\varphi(\lambda)$ we simply record all the coefficients of these multitabloids in a sequence, which is the column corresponding to t with the column label w_2 in $\varphi(\lambda)$.

In terms of the Specht matrix, as we are only considering the S_n-action on the multitableaux permuting the elements in the columns of $\varphi(\lambda)$, we have that Proposition 3.5.14 is true for the $G_{n,r}$ groups as well. Then we have it as follows.

Proposition 4.6.10. *For a multipartition λ of n we have that $\nu_{w_2} = \operatorname{sgn}(t^\flat) \cdot \nu_t$.*

Proof. This proof follows the same notions as the proof of Proposition 3.5.14 except for t is now a multitableau of shape $\lambda \overset{r}{\vdash} n$. □

Proposition 4.6.10 well establishes a connection between the polymultitabloids and the columns of $\varphi(\lambda)$. Then we define the module S^λ spanned by the columns of $\varphi(\lambda)$ to be the Specht module for λ as a subspace of M^λ.

Definition 4.6.11 (Specht Module for $G_{n,r}$ Group). *For a multipartition $\lambda \overset{r}{\vdash} n$, the Specht module S^λ is the subspace of M^λ spanned by all v_t.*

In other words, the column space of the Specht matrix $\varphi(\lambda)$ is the Specht module S^λ spanned by v_{w_2} where w_2 is the column word in $\varphi(\lambda)$.

Following the same theme from Chapter 3, we have following lemma as the analog of Lemma 3.5.15 for monomial groups.

Lemma 4.6.12. *Let t be a λ-multitableau for $\lambda \overset{r}{\vdash} n$. Then the following hold.*

1. *For $\sigma \in R(t)$ and $\pi \in C(t)$, we have that $p_t \cdot \sigma = \sigma \cdot p_t = p_t$ and $c_t \cdot \pi = \pi \cdot c_t = \mathrm{sgn}(\pi)c_t$.*

2. *We have that $p_t \cdot p_t = |R(t)|p_t$ and $c_t \cdot c_t = |C(t)|c_t$.*

Proof. The proof follows from Lemma 3.5.15. □

Lemma 4.6.13. *For all multitableau t of shape $\lambda \overset{r}{\vdash} n$ and all $g \in G_{n,r}$ we have that $v_t \cdot g \in S^\lambda$.*

Proof. Here we have two cases for $g \in S_n$ and for $g \in \Delta_n$ where $G_{n,r} = \Delta_n \rtimes S_n$.

1. If $g \in S_n$, then from Lemma 3.5.16 we have that $v_t \cdot g = v_{t \cdot g}$. Therefore, we find that $v_t \cdot g \in S^\lambda$.

2. If $g = \delta_j$ for some $\delta_j \in \Delta_n$ where $1 \leqslant j \leqslant n$, then we have it as follows.

   Let w be a column word of $\varphi(\lambda)$ and $w = ((l_1)_{a_1} \cdots (l_j)_{a_j} \cdots (l_n)_{a_n})$ where $(l_j)_{a_j}$ denotes that the letter l_j has index a_j. Proposition 4.6.10 implies that $v_t = \mathrm{sgn}(t^\flat) \cdot v_w$ and so $\langle v_t \rangle = \langle v_w \rangle$. Now, as $S^\lambda \leqslant M^\lambda$, we must have that $\langle v_w \rangle \leqslant \langle q_w \rangle$. It implies that $v_w \in \langle q_w \rangle$. Then from Definition 4.6.6, for $g = \delta_j$ we have that $v_w \cdot g = \omega_r^{a_j} \cdot v_w \in \langle v_w \rangle$. Therefore, $v_w \cdot g \in S^\lambda$.

□

Therefore, Lemma 4.6.13 clearly implies that S^λ is closed under $G_{n,r}$-action.

Proposition 4.6.14. *S^λ is a $G_{n,r}$-module for $\lambda \overset{r}{\vdash} n$.*

Now if we show that the Specht module S^{λ} (as constructed from the Specht matrix) has character χ_{λ} (as constructed by Clifford theory in Section 4.3) then we have that for a pair of distinct multipartitions λ and λ' of n, S^{λ} and $S^{\lambda'}$ are non-isomorphic.

Definition 4.6.6 gives us the action of δ_j on the permutation module M^{λ} for $1 \leqslant j \leqslant n$. It contains the character $\eta_{\underline{n}}$. We have that $S^{\lambda} \leqslant M^{\lambda}$. Therefore, from Proposition 4.6.10 and Definition 4.6.11 we have the same action of δ_j on the Specht module S^{λ} spanned by v_{w_2}, contains $\eta_{\underline{n}}$. For $0 \leqslant k < r$, we have that δ_j acts as the k-th power of root of unity if k is the index of the j-th letter in the word w_2. By construction, considering only the action of δ_j we have that the restriction of S^{λ} to Δ_n contains $\eta_{\underline{n}}$.

Let $n = (n_0, \dots, n_{r-1})$ and $\lambda = (\lambda_0, \dots, \lambda_{r-1})$ be a multipartition of n such that $\lambda_k \vdash n_k$. We know that $S_n \cong \mathfrak{P}_n$ and $S_{n_k} \cong \mathfrak{P}_{n_k}$. Now if we consider the canonical word pair $(w_{1_\lambda}, w'_{2_{\lambda^T}})$, then we have the indices of the letters sorted for both row and column words. The stabilizer subgroup S_{λ_0} acts on the pair by fixing the index 0 and so we obtain a block matrix M_0 inside the Specht matrix $\varphi(\lambda)$. The space spanned by the columns of the matrix M_0 is isomorphic to Specht module S^{λ_0}. If the irreducible character of S^{λ_0} is $\tilde{\chi}_{\lambda_0}$, then we have $\mathrm{tr}(M_0) = \tilde{\chi}_{\lambda_0}$ where $\mathrm{tr}(M_0)$ is the trace of M_0. Therefore, the inner tensor product $\overline{\eta}^{(0)}_{n_0} \otimes \tilde{\chi}_{\lambda_0}$ is an irreducible character of G_{n_0}. This is true for all λ_k for $0 \leqslant k < r$. Then the restriction to $\mathfrak{P}_n$ containing $\eta_{\underline{n}}$ is induced from the product of the symmetric group characters.

Then we have an induction from $\prod_{k=0}^{r-1} S_{n_k}$ to S_n in the same way as the induction from $\prod_{k=0}^{r-1} G_{n_k,r}$ to $G_{n,r}$ as described in Section 4.3. Therefore, S^{λ} has character χ_{λ}. Then for $\lambda \overset{r}{\vdash} n$, we have that S^{λ} is an irreducible representation of $G_{n,r}$ and for any two multipartitions λ, λ', we have S^{λ} is non-isomorphic to $S^{\lambda'}$ whenever $\lambda \neq \lambda'$.

Theorem 4.6.15. *The set of Specht modules* $\{S^{\lambda}$ *for all* $\lambda \overset{r}{\vdash} n\}$ *produces the complete set (up to isomorphism) of pairwise non-isomorphic irreducible representations of* $G_{n,r}$.

Proof. By this construction we find that the number of irreducible representations of $G_{n,r}$ is same as the number of multipartitions of n. S^{λ} corresponding to each $\lambda \overset{r}{\vdash} n$ is an irreducible representation of $G_{n,r}$. As every irreducible representation of $G_{n,r}$ is isomorphic to exactly one S^{λ}, and hence any pair of two irreducible representations are non-isomorphic. Therefore, $\{S^{\lambda}$ for all $\lambda \overset{r}{\vdash} n\}$ exhausts the complete set of irreducible representations of $G_{n,r}$ up to isomorphism. □

To make this concrete, we will now prove that the columns corresponding to the SYMT create a basis for the module S^λ. The goal is to show that the columns of interest are linearly independent and span S^λ.

Lemma 4.6.16. *The standard Young multitableaux entries in the Specht matrix belong to distinct rows and columns.*

Proof. In the Specht matrix all the entries sitting in any row belong to the multitableaux of same row-equivalence class or row multitabloid. If t is a standard λ-multitableau then in the row multitabloid $\{t\}$ there can be only one SYMT, i.e., t itself.

This implies that every SYMT belong to different row-equivalence classes and therefore they all sit in different rows.

Likewise, every SYMT belong to different column-equivalence classes and therefore they all sit in different columns.

Hence the standard Young multitableaux entries in the Specht matrix belong to distinct rows and columns. □

To prove the linear independence of rows and columns with SYMT entries, we only need to show that the SYMT entries are the first nonzero entries in their respective rows and columns. We already have defined a natural lexicographic ordering on the set of all such words. We use that ordering to achieve this result.

Theorem 4.6.17. *In the Specht matrix corresponding to the multipartition $\lambda \overset{r}{\vdash} n$, we have the standard multitableaux elements as the first nonzero elements in their respective rows and columns.*

Proof. As usual in the Specht matrix the rows are labelled by the rearrangements of the row word u and columns are labelled by the rearrangements of the column word v where $\lambda(u) = \lambda$ and $\lambda(v) = \mu$ such that $\mu = (\mu_0, \dots, \mu_{r-1}) = (\lambda_0^T, \dots, \lambda_{r-1}^T) = \lambda^T$.

In each column we have a column-equivalence class, i.e. a column multitabloid. Then each of the particular columns consisting the SYMT entries has exactly one such element with simultaneous multihistogram of which all the columns are standard.

By the construction of simultaneous multihistogram for a pair of words, it is clear that the distinct letters of the word u appear in positions corresponding to the entries in the first column of each of the component tableau t_k and they have repetitions in respective positions corresponding to the further columns in each t_k for $k \in \{0,\dots,r-1\}$. As each t_k is column-standard, if the smaller entry in a column has corresponding row word letter l and larger entry in the same column has corresponding row word letter l' then $l < l'$.

Now if we label the boxes of each t_k by (i,j) to denote its position as i-th row and j-th column then for the column standard t_k we have that $t_{k_{(i,j)}} < t_{k_{(i+1,j)}}$. For any $\pi \in C = S_{\mu_0} \times S_{\mu_1} \times \cdots \times S_{\mu_{r-1}}$, column stabilizer, and for $t_k \cdot \pi = t_k^\pi$ we have $t^\pi_{k_{(i,j)}} > t^\pi_{k_{(i+1,j)}}$ implying that at least one of the columns in $t_{k_{(i+1,j)}} \cdot \pi$ is non-standard. Therefore, for $u_\pi = u \cdot \pi$ we have $u_{(i,j)} > u_{\pi_{(i+1,j)}}$ implying that $u < u_\pi$, i.e., u_π comes later than u in lexicographical order. In that case if in the Specht matrix $\varphi(u,v)$ we label the row word lexicographically then the column standard multitableaux entries appearing on top of the respective columns making it the nonzero elements.

Likewise we can establish a similar argument that the row standard multitableaux entries are the first nonzero elements in their respective rows.

Hence, for t being both row and column standard, i.e., t being a SYMT we have that the corresponding element in the respective rows and columns are the first nonzero entries.

□

Hence we establish by combining Lemma 4.6.16 and Theorem 4.6.17 together that the rows and columns containing SYMT entries in the Specht matrix are linearly independent.

Now, from the example shown in Table 4.1 if we extract only the rows and columns containing the SYMT entries then we obtain the submatrix shown in Table 4.2.

	$1_01_01_11_2$	$1_01_01_21_1$	$1_01_11_01_2$	$1_01_11_21_0$	$1_01_21_01_1$	$1_01_21_11_0$	$1_11_01_01_2$	$1_11_01_21_0$	$1_11_21_01_0$	$1_21_01_01_1$	$1_21_01_11_0$	$1_21_11_01_0$
$1_02_01_11_2$	1	·	·	·	·	·	·	·	·	·	·	·
$1_01_12_01_2$	·	·	−1	·	·	·	·	·	·	·	·	·
$1_11_02_01_2$	·	·	·	·	·	·	1	·	·	·	·	·
$1_02_01_21_1$	·	−1	·	·	·	·	·	·	·	·	·	·
$1_01_22_01_1$	·	·	·	·	1	·	·	·	·	·	·	·
$1_21_02_01_1$	·	·	·	·	·	·	·	·	·	−1	·	·
$1_01_11_22_0$	·	·	·	1	·	·	·	·	·	·	·	·
$1_11_01_22_0$	·	·	·	·	·	·	·	−1	·	·	·	·
$1_01_21_12_0$	·	·	·	·	·	−1	·	·	·	·	·	·
$1_21_01_12_0$	·	·	·	·	·	·	·	·	·	·	1	·
$1_11_21_02_0$	·	·	·	·	·	·	·	·	1	·	·	·
$1_21_11_02_0$	·	·	·	·	·	·	·	·	·	·	·	−1

Table 4.2: SYMT-extracted submatrix of Specht matrix $\varphi(((1,1),(1),(1)))$

It can be seen that each row and column has exactly one nonzero entry. If we rearrange the row words in a different way then we obtain the matrix in Table 4.3. Similar to the case of the symmetric group, we can see that the submatrix takes a nice form with its nonzero entries on its main diagonal.

	$1_01_01_11_2$	$1_01_01_21_1$	$1_01_11_01_2$	$1_01_11_21_0$	$1_01_21_01_1$	$1_01_21_11_0$	$1_11_01_01_2$	$1_11_01_21_0$	$1_11_21_01_0$	$1_21_01_01_1$	$1_21_01_11_0$	$1_21_11_01_0$
$1_02_01_11_2$	1	·	·	·	·	·	·	·	·	·	·	·
$1_02_01_21_1$	·	−1	·	·	·	·	·	·	·	·	·	·
$1_01_12_01_2$	·	·	−1	·	·	·	·	·	·	·	·	·
$1_01_11_22_0$	·	·	·	1	·	·	·	·	·	·	·	·
$1_01_22_01_1$	·	·	·	·	1	·	·	·	·	·	·	·
$1_01_21_12_0$	·	·	·	·	·	−1	·	·	·	·	·	·
$1_11_02_01_2$	·	·	·	·	·	·	1	·	·	·	·	·
$1_11_01_22_0$	·	·	·	·	·	·	·	−1	·	·	·	·
$1_11_21_02_0$	·	·	·	·	·	·	·	·	1	·	·	·
$1_21_02_01_1$	·	·	·	·	·	·	·	·	·	−1	·	·
$1_21_01_12_0$	·	·	·	·	·	·	·	·	·	·	1	·
$1_21_11_02_0$	·	·	·	·	·	·	·	·	·	·	·	−1

Table 4.3: Rearranged SYMT-extracted submatrix of Specht matrix $\varphi(((1,1),(1),(1)))$

Therefore, it is evident that the rows (respectively columns) containing the SYMT elements in the Specht matrix are linearly independent.

Similar to the case for symmetric groups, we have noticed that for $n \geqslant 5$ we find the inter-multitableau for some of the multipartitions of n.

Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1}) \overset{r}{\vdash} n = \sum_{k=0}^{r-1} a_k$. Straightforwardly, if in the truncated submatrix of the Specht matrix corresponding to at least one λ_k for symmetric group S_{n_k} we find an inter-tableau, then Specht matrix for λ containing λ_k has inter-multitableau. Let us understand it using Example 4.6.18.

Example 4.6.18. Let $n = 7$ and $r = 3$. Let us consider $\lambda = ((2,2,1),(1,1),\varnothing)$ and the corresponding conjugate multipartition $\lambda^T = ((3,2),(2),\varnothing)$. The respective row and column words are $1_01_02_02_03_01_12_1$ and $1_01_01_02_02_01_11_1$. If we now take the truncated version of the Specht matrix $\varphi(\lambda)$ then we find a lower-triangular matrix with all the diagonal entries being the SYMT entries.

Now if we take the SYMT entries with the corresponding row and column simultaneous multihistograms being

$$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 6\\\hline 7\\\hline\end{array}, \varnothing \quad \text{and} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 6\\\hline 7\\\hline\end{array}, \varnothing$$

then we have another entry at the intersection of these row and column, giving us a inter-multitableau

It can be seen that in Example 4.6.18, the multipartitions λ and λ^T both have component partitions which are empty. In general, an empty component partition $\varnothing$ in a multipartition implies that there is no tableau associated to it. An empty partition is a partition of zero, and so the number of boxes in the tableau is zero. Therefore, an empty partition $\varnothing$ does not form the shape of a visual tableau.

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline \end{array}, \varnothing.$$

The reason behind the existence of this inter-multitableau is similar to that of the symmetric groups. We can notice that the integer pair $1, 2$ is present in the first column of the first component tableau of one of the multitableau and in the first row of the first component tableau of the other one. Interestingly, the partition $(2, 2, 1)$ which is a component of the multipartition λ, was noticed to have inter-tableau in corresponding the Specht matrix for the symmetric groups.

Now to prove for the basis of S^λ what is left to be shown is the spanning property of the set of SYMT columns. Again, due to our interest in the columns of the Specht matrix, we put a similar argument shown for the symmetric group, for monomial groups case. Similarly rows can be considered to be a basis of the modules S^{λ^T}.

Now we introduce the generalised hooklength formula which is analogous to the hooklength formula for the symmetric groups. This is going play a crucial role in proving the spanning set with a global argument.

For n being a non-negative integer, we have the multinomial coefficient $\binom{n}{n_0, n_1 \ldots, n_{r-1}} = \frac{n!}{n_0! n_1! \cdots n_{r-1}!}$. Now if f^λ be the dimension of S^λ, we define generalised Hooklength formula.

Definition 4.6.19 (Generalised Hooklength Formula). *Let* $n_0 + \cdots + n_{r-1} = n$ *and* $\lambda = (\lambda_0, \cdots, \lambda_{r-1})$ *where each* $\lambda_k \vdash n_k$*. Then by using the original hooklength formula*

$$f^{\lambda_k} = \frac{n_k!}{\prod\limits_{i,j \in \lambda_k} h_{i,j}^{\lambda_k}},$$

and the multinomial coefficient, we define the ***generalised hooklength***

$$f^\lambda = \binom{n}{n_0, \ldots, n_{r-1}} f^{\lambda_0} \cdots f^{\lambda_{r-1}}.$$

Lemma 4.6.20. *The generalised hooklength formula can be realised as*

$$f^\lambda = \frac{n!}{\prod\limits_{k=0}^{r-1} \prod\limits_{i,j \in \lambda_k} h_{i,j}^{\lambda_k}}.$$

i.e., $n!$ *divided by all the hooklengths in the multi-diagram of shape* λ.

Proof. We have that $f^\lambda = \binom{n}{n_0, \ldots, n_{r-1}} f^{\lambda_0} \cdots f^{\lambda_{r-1}}$. Then

$$\begin{aligned}\binom{n}{n_0,\ldots,n_{r-1}} f^{\lambda_0}\cdots f^{\lambda_{r-1}} &= \frac{n!}{n_0!\cdots n_{r-1}!}\cdot f^{\lambda_0}\cdots f^{\lambda_{r-1}} \\ &= \frac{n!}{n_0!\cdots n_{r-1}!}\cdot \frac{n_0!}{\prod\limits_{i,j\in\lambda_0} h_{i,j}^{\lambda_0}}\cdots \frac{n_{r-1}!}{\prod\limits_{i,j\in\lambda_{r-1}} h_{i,j}^{\lambda_{r-1}}} \\ &= \frac{n!}{\left(\prod\limits_{i,j\in\lambda_0} h_{i,j}^{\lambda_0}\right)\cdots\left(\prod\limits_{i,j\in\lambda_{r-1}} h_{i,j}^{\lambda_{r-1}}\right)} \\ &= \frac{n!}{\prod\limits_{k=0}^{r-1}\prod\limits_{i,j\in\lambda_k} h_{i,j}^{\lambda_k}}.\end{aligned}$$

Hence we conclude the proof. □

Theorem 4.6.21. *The number of standard Young multitableaux for the monomial groups corresponding to a multipartition* λ *of* n *is given by the generalised hooklength formula* f^{λ}.

Proof. Let $\lambda \overset{r}{\vdash} n$ and each $\lambda_k \vdash n_k$ where $\sum_{k=0}^{r-1} n_k = n$. If $t = (t_0, t_1, \ldots, t_{r-1})$ be the multitableau corresponding to λ then for each of the λ_k we have a tableau t_k. Now each t_k can be standard in f^{λ_k} many ways.

The number of ways to fill out the boxes of the tableaux t_k in the multitableau t with the numbers $\{1, \ldots, n\}$ without repetition is given by the multinomial coefficient $\binom{n}{n_0,\ldots,n_{r-1}}$.

Now, if we want to fill the boxes in the multitableau t in such a manner that each of the Young tableau t_k is standard, then we can do that in $\binom{n}{n_0,\ldots,n_{r-1}} f^{\lambda_0}\cdots f^{\lambda_{r-1}}$ many ways.

Now the formula in Lemma 4.6.20 gives us the number of ways we can fill all the boxes in the multitableau t such that no integer is repeated and each of the t_k is standard. Therefore, it gives us the total number of possible standard Young multitableaux of shape λ.

Hence, the number of SYMT of shape λ is same as the generalised hooklengths for monomial groups. □

Now in the same spirit of the RSK correspondence [4] for the symmetric groups, we build the same for the monomial groups and call it a general RSK correspondence as seen in Theorem 4.6.22.

Theorem 4.6.22. *The general RSK correspondence for monomial groups is*

$$\sum_{\lambda \overset{r}{\vdash} n} (f^{\lambda})^2 = r^n n!.$$

Proof. For a proof of the generalized RSK correspondence, i.e., the fact that the squares of all these f^λ add up to $r^n n!$, one can look at the construction in Section 4.3. In Section 5.5.4 of [8], one can find the same for a special case when $r = 2$. The representation λ is built as an induced representation from the subgroup $J = G_{n_0,r} \times \cdots \times G_{n_{r-1},r}$ of $G_{n,r}$.

Here, $G_{n_k,r}$ is a semidirect product of Δ_{n_k} and the symmetric group S_{n_k}. Letting the generators of Δ_n act as multiplication with the r-th root of unity ω_r, and the symmetric group as Specht module S^{λ_k}, we get a representation of dimension f^{λ_k} for each factor $G_{n_k,r}$, and by putting them together, we get a representation of dimension $(f^{\lambda_0} \cdots f^{\lambda_{r-1}})$ for the subgroup J. This corresponds to assigning the numbers $1, \ldots, n_1$ to the first diagram, then $n_1+1, \ldots, n_1+n_2$ to the second, and so on.

Now the index of the subgroup J in $G_{n,r}$ is same as the index of $S_{n_0} \times \cdots \times S_{n_{r-1}}$ in S_n, namely $\binom{n}{n_0,\ldots,n_{r-1}}$, so inducing the representation from the subgroup J to $G_{n,r}$ gives a representation of dimension $f^\lambda = \binom{n}{n_0,\ldots,n_{r-1}} f^{\lambda_0} \cdots f^{\lambda_{r-1}}$.

If we do this for all the representations of J, i.e., for this choice of $n_0, \ldots, n_{r-1}$, and add up the corresponding $(f^\lambda)^2$, we get

$$\begin{aligned}\binom{n}{n_0,\ldots,n_{r-1}}^2 \sum_{\lambda_k \vdash n_k} ((f^{\lambda_0})^2 \cdots (f^{\lambda_{r-1}})^2) &= \binom{n}{n_0,\ldots,n_{r-1}}^2 \Big(\sum_{\lambda_0 \vdash n_1} (f^{\lambda_0})^2\Big) \cdots \Big(\sum_{\lambda_{r-1} \vdash n_{r-1}} (f^{\lambda_{r-1}})^2\Big) \\ &= \binom{n}{n_0,\ldots,n_{r-1}}^2 n_0! \cdots n_{r-1}! \\ &= \binom{n}{n_0,\ldots,n_{r-1}} n!\end{aligned} \tag{4.4}$$

by the original hook formula.

Finally, the Multinomial Theorem implies that

$$\sum_{n_0+\cdots+n_{r-1}=n} \binom{n}{n_0,\ldots,n_{r-1}} = r^n, \tag{4.5}$$

and therefore in total we get

$$\sum_{\lambda \overset{r}{\vdash} n} (f^\lambda)^2 = r^n n!, \tag{4.6}$$

which is the order of $G_{n,r}$. □

It is now safe to say that the correspondence shown in Theorem 4.6.22 is analogous to the RSK correspondence seen in Chapter 3. Now we establish a similar global argument to that of the symmetric group case to prove that the set of SYMT-columns in $\varphi(\lambda)$ is a spanning set for S^λ. We have noticed already that the SYMT columns in the Specht matrix are linearly independent and there are precisely f^λ of them. There cannot be anymore linearly independent columns as otherwise it would fail the general RSK correspondence. Therefore, there are exactly as many linearly independent columns as the number of t_λ, where t_λ is a SYMT of shape λ.

From [12, Section 4.4] we know that each S^λ is irreducible, and for any two multipartitions $\lambda, \mu \overset{r}{\vdash} n$ given $\lambda \neq \mu$, we have that S^λ and S^μ are non-isomorphic. This clearly implies that there are as many Specht modules as the number of multipartitions λ of n. This forms the complete set of irreducible representations. Then the general RSK-correspondence gives that the sum of the squares of the degrees of irreducible representations is same as the order of the group $G_{n,r}$.

Let for our construction $d^\lambda = \dim(S^\lambda)$. Then for each λ we have that $d^\lambda \geqslant f^\lambda$ and so from Theorem 4.6.22 we have that

$$\sum_{\lambda \overset{r}{\vdash} n} (f^\lambda)^2 = \sum_{\lambda \overset{r}{\vdash} n} |t_\lambda|^2 = r^n n!$$

where $|t_\lambda|$ be the number of SYMT of shape λ. Then it implies that $d^\lambda = f^\lambda$ must be true. Hence, dimension of each S^λ is equal to the number of the SYMT of shape λ.

As we have all the components now, we have established that the set of SYMT columns in $\varphi(\lambda)$ for $\lambda \overset{r}{\vdash} n$ spans the full module S^λ. Therefore we have the following.

Theorem 4.6.23. *If $\lambda \overset{r}{\vdash} n$ then the set of SYMT-columns in $\varphi(\lambda)$ forms a basis of S^λ.*

By definition the module spanned by columns of $\varphi(\lambda)$ is the Specht module S^λ and $S^\lambda = \langle v_t \rangle$ for t being a standard Young multitableau of shape λ. We have established the connection between v_t and v_{w_2} which implies that $\langle v_{w_2} \rangle = S^\lambda$. Therefore we have established the theory in a similar way to that of the symmetric groups.

4.7 Representing Matrices

It is now time for us to understand how this construction works in terms of producing representing matrices.

Let $G_{n,r} \leqslant GL_n(\mathbb{C})$ be the monomial group and we pick an element $A \in G_{n,r}$. By using decomposition of the monomial matrices, we generate the representing matrices. Let N be the list of nonzero elements from each row of A. We take L to be the list of the positions of the nonzero elements in each row of the matrix A.

Consider σ to be the permutation described by the list L such that any point i under σ moves to the point sitting at the i-th position in L. Now let X be the set of all rearrangements of the row word w_1 and $|X| = k$. By the action of σ on each word in X, we obtain a permuted list $X' = \{u^\sigma | u \in X\}$. It gives us a map from S_n to S_k.

Let $\pi \in S_k$ such that $X \cdot \pi = X'$. It means that the elements in each column of the matrix $\varphi(\lambda)$ is now rearranged by π. Let C be the set of columns of $\varphi(u, v)$, and $C' = \{c^\pi | c \in C\}$. Clearly, C' is another rearrangement of C itself. Now pick the column vectors with SYMT in C' and then express them as a linear combination of the SYMT column vectors from C. We now extract the coefficients of these linear combination and put them in the rows of a matrix, say P.

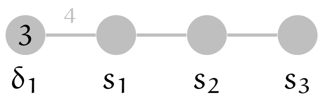
We know that $|N| = n$. Let $N = [\zeta_1, \zeta_2, \ldots, \zeta_n]$ such that $\zeta_i \in \mathbb{C}$ for $i \in \{1, 2, \ldots, n\}$. Let us consider that each word $u \in X$ is of the form $u = ((l_1)_{j_1}(l_2)_{j_2} \cdots (l_n)_{j_n})$ where each $(l_i) \in \{1, 2, \ldots, n\}$ and $0 \leqslant j_i < r$. Then for each word $u \in X$, we can find a corresponding element $x = \prod_{i=1}^{n} (\zeta_i)^{j_i} \in \mathbb{C}$. More specifically, each of these x is an r-th root of unity.

Now, consider the subset $R \subset X$ consisting of only those row words that are corresponding to the SYMT. For each word $u \in R$, now we keep adding its corresponding element x to a list $\mathbb{I}$. Then we obtain that $\mathbb{I} = \left[\prod_{i=1}^{n} (m_i)^{j(u)_i} | u \in R \right]$ where $j(u) = [j_1, j_2, \ldots, j_n]$ and $j(u)_i = j_i$. We define a diagonal matrix D of which the main diagonal is the list $\mathbb{I}$, i.e., $D = \mathrm{diag}(\mathbb{I})$.

Each of the matrices P and D is clearly of dimension $f^\lambda \times f^\lambda$. We now take the product of them in the order that D is multiplied by P from the right, and call it M_A, i.e., $M_A = DP$. Therefore, the matrix M_A is the corresponding representing matrix for the group element A.

This method of finding the matrix P fits the same description in terms of linear algebraic formulations of representing matrices found in the Chapter 3. The order of the matrices in the product DP is to indicate the right action of the symmetric group that we have used as a convention throughout the discussions. The following example shows explicitly the compatibility of this construction with the monomial group elements.

Example 4.7.1. Let us consider that $n = 4$ and $r = 3$. Therefore the group is $G_{4,3}$.



$G(3,1,4)$

The generator set of $G_{4,3}$ is therefore $S = \{\delta_1, s_1, s_2, s_3\}$ where

$$\delta_1 = \begin{bmatrix} \omega_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We have the a total of 51 multipartitions of 4 for $r = 3$. As showing the representing matrices for all of them would be way too much, we choose the following multipartitions for demonstration:

$$((1,1,1,1), \varnothing, \varnothing), \quad ((2),(1),(1)), \quad ((1), \varnothing, (3)), \quad (\varnothing, \varnothing, (4)).$$

Now let us pick each of these generators, $\sigma \in S$, and for each of them we find the corresponding representing matrix M_σ.

1. For $\lambda = ((1,1,1,1), \varnothing, \varnothing)$, we find:

   (a) if $\sigma = \delta_1$ then
   $$D_{\delta_1} = \begin{bmatrix} 1 \end{bmatrix}, P_{\delta_1} = \begin{bmatrix} 1 \end{bmatrix} \text{ and } M_{\delta_1} = \begin{bmatrix} 1 \end{bmatrix}$$

   (b) if $\sigma = s_i$ for $i \in \{1,2,3\}$ then
   $$D_{s_i} = \begin{bmatrix} 1 \end{bmatrix}, P_{s_i} = \begin{bmatrix} -1 \end{bmatrix} \text{ and } M_{s_i} = D_{s_i} P_{s_i} = \begin{bmatrix} -1 \end{bmatrix}$$

2. For $\lambda = ((2),(1),(1))$, we find:

   (a) if $\sigma = \delta_1$ then
   $$D_{\delta_1} = \text{diag}(\{1, 1, \omega_3, 1, 1, \omega_3^2, 1, \omega_3, 1, \omega_3^2, \omega_3, \omega_3^2\}), P_{\delta_1} = I_{12}$$

   and therefore

   $$M_{\delta_1} = D_{\delta_1} P_{\delta_1} = \text{diag}(\{1, 1, \omega_3, 1, 1, \omega_3^2, 1, \omega_3, 1, \omega_3^2, \omega_3, \omega_3^2\})$$

(b) if $\sigma = s_1$ then

$$D_{s_1} = I_{12},$$

$$P_{s_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and therefore

$$M_{s_1} = D_{s_1} P_{s_1} = P_{s_1}$$

(c) if $\sigma = s_2$ then

$$D_{s_2} = I_{12},$$

$$P_{s_2} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

and therefore

$$M_{s_2} = D_{s_2} P_{s_2} = P_{s_2}$$

(d) if $\sigma = s_3$ then

$$D_{s_3} = I_{12},$$

$$P_{s_3} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and therefore

$$M_{s_3} = D_{s_3} P_{s_3} = P_{s_3}$$

3. For $\lambda = ((1), \varnothing, (3))$, we find:

(a) if $\sigma = \delta_1$ then

$$D_{\delta_1} = \mathrm{diag}(\{1, \omega_3^2, 1, \omega_3^2, \omega_3^2\}), P_{\delta_1} = I_4$$

and therefore

$$M_{\delta_1} = D_{\delta_1} P_{\delta_1} = \mathrm{diag}(\{1, \omega_3^2, 1, \omega_3^2, \omega_3^2\})$$

(b) if $\sigma = s_1$ then

$$D_{s_1} = I_4, \quad P_{s_1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and therefore

$$M_{s_1} = D_{s_1} P_{s_1} = P_{s_1}$$

(c) if $\sigma = s_2$ then

$$D_{s_2} = I_{12}, \quad P_{s_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and therefore

$$M_{s_2} = D_{s_2} P_{s_2} = P_{s_2}$$

(d) if $\sigma = s_3$ then

$$D_{s_3} = I_{12}, \quad P_{s_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and therefore

$$M_{s_3} = D_{s_3} P_{s_3} = P_{s_3}$$

4. For $\lambda = (\varnothing, \varnothing, (4))$, we find:

(a) if $\sigma = \delta_1$ then

$$D_{\delta_1} = \begin{bmatrix} \omega_3^2 \end{bmatrix}, P_{\delta_1} = \begin{bmatrix} 1 \end{bmatrix} \text{ and } M_{\delta_1} = \begin{bmatrix} \omega_3^2 \end{bmatrix}$$

(b) if $\sigma = s_i$ for $i \in \{1, 2, 3\}$ then

$$D_{s_i} = \begin{bmatrix} 1 \end{bmatrix}, P_{s_i} = \begin{bmatrix} 1 \end{bmatrix} \text{ and } M_{s_i} = D_{s_i} P_{s_i} = \begin{bmatrix} 1 \end{bmatrix}$$

4.8 Characters and Character Tables

Similar to the construction of the symmetric groups, it is important that for this construction we also have a way to validate the accuracy of the method, and therefore we follow the same route of matching the character table entries existing in the literature with the traces of the representing matrices obtained using our construction.

For each $\lambda \overset{r}{\vdash} n$, we can pick a conjugacy class representative. Again for each conjugacy class representative, we can find the character value corresponding it to be the trace of its representing matrix.

Due to the very large number of multipartitions of the monomial groups, their character tables become very large in dimensions. The smallest sized group is $G_{3,3}$ that is reasonably large and is of interest from the perspective of multipartitions. There are 22 multipartitions of $G_{3,3}$ and therefore it is time consuming to go through each of the multipartitions for all of the conjugacy classes separately. Due to the efficient GAP programmes, we obtain the character table for $G_{3,3}$.

	111..	11.1.	11..1	1.11.	1.1.1	1..11
111..	1	1	1	1	1	1
11.1.	3	$(3+ER(-3))/2$	$(3-ER(-3))/2$	$ER(-3)$	0	$-ER(-3)$
11..1	3	$(3-ER(-3))/2$	$(3+ER(-3))/2$	$-ER(-3)$	0	$ER(-3)$
1.11.	3	$ER(-3)$	$-ER(-3)$	$(-3+ER(-3))/2$	0	$(-3-ER(-3))/2$
1.1.1	6	0	0	0	-3	0
1..11	3	$-ER(-3)$	$ER(-3)$	$(-3-ER(-3))/2$	0	$(-3+ER(-3))/2$
.111.	1	$E3$	$E3^2$	$E3^2$	1	$E3$
.11.1	3	$(-3+ER(-3))/2$	$(-3-ER(-3))/2$	$(3-ER(-3))/2$	0	$(3+ER(-3))/2$
.1.11	3	$(-3-ER(-3))/2$	$(-3+ER(-3))/2$	$(3+ER(-3))/2$	0	$(3-ER(-3))/2$
..111	1	$E3^2$	$E3$	$E3$	1	$E3^2$
21..	2	2	2	2	2	2
1.2.	3	$ER(-3)$	$-ER(-3)$	$(-3+ER(-3))/2$	0	$(-3-ER(-3))/2$
1..2	3	$-ER(-3)$	$ER(-3)$	$(-3-ER(-3))/2$	0	$(-3+ER(-3))/2$
2.1.	3	$(3+ER(-3))/2$	$(3-ER(-3))/2$	$ER(-3)$	0	$-ER(-3)$
.21.	2	$2E3$	$2E3^2$	$2E3^2$	2	$2E3$
.1.2	3	$(-3-ER(-3))/2$	$(-3+ER(-3))/2$	$(3+ER(-3))/2$	0	$(3-ER(-3))/2$
2..1	3	$(3-ER(-3))/2$	$(3+ER(-3))/2$	$-ER(-3)$	0	$ER(-3)$
.2.1	3	$(-3+ER(-3))/2$	$(-3-ER(-3))/2$	$(3-ER(-3))/2$	0	$(3+ER(-3))/2$
..21	2	$2E3^2$	$2E3$	$2E3$	2	$2E3^2$
3..	1	1	1	1	1	1
.3.	1	$E3$	$E3^2$	$E3^2$	1	$E3$
..3	1	$E3^2$	$E3$	$E3$	1	$E3^2$

Table 4.4: Character table of the monomial group $G_{3,3}$

Taking Table 4.4, Table 4.5 and Table 4.6 altogether side-by-side, we obtain the character table of dimension 22×22 of the group $G_{3,3}$. The notations used in the figures are how square roots and roots of unity are represented in GAP. Here $E(r) = \omega_r$ and $ER(n) = \sqrt{n}$.

	.111.	.11.1	.1.11	..111	21..	1.2.	1..2	2.1.	.21.
111..	1	1	1	1	−1	−1	−1	−1	−1
11.1.	3E3	(−3 + ER(−3))/2	(−3 − ER(−3))/2	$3E3^2$	−1	−1	−1	−E3	−E3
11..1	$3E3^2$	(−3 − ER(−3))/2	(−3 + ER(−3))/2	3E3	−1	−1	−1	$-E3^2$	$-E3^2$
1.11.	$3E3^2$	(3 − ER(−3))/2	(3 + ER(−3))/2	3E3	−1	−E3	$-E\,3^2$	−1	−E3
1.1.1	6	0	0	6	0	0	0	0	0
1..11	3E3	(3 + ER(−3))/2	(3 − ER(−3))/2	$3E3^2$	−1	$-E3^2$	−E3	−1	$-E3^2$
.111.	1	E3	$E3^2$	1	−1	−E3	$-E3^2$	−E3	$-E3^2$
.11.1	3E3	− ER(−3)	ER(−3)	$3E3^2$	−1	−E3	$-E3^2$	$-E3^2$	−1
.1.11	$3E3^2$	ER(−3)	− ER(−3)	3E3	−1	$-E3^2$	−E3	−E3	−1
..111	1	$E3^2$	E3	1	−1	$-E3^2$	−E3	$-E\,3^2$	−E3
21..	2	2	2	2	0	0	0	0	0
1.2.	$3E3^2$	(3 − ER(−3))/2	(3 + ER(−3))/2	3E3	1	E3	$E3^2$	1	E3
1..2	3E3	(3 + ER(−3))/2	(3 − ER(−3))/2	$3E3^2$	1	$E3^2$	E3	1	$E3^2$
2.1.	3E3	(−3 + ER(−3))/2	(−3 − ER(−3))/2	$3E3^2$	1	1	1	E3	E3
.21.	2	2E3	$2E3^2$	2	0	0	0	0	0
.1.2	$3E3^2$	ER(−3)	− ER(−3)	3E3	1	$E3^2$	E3	E3	1
2..1	$3E3^2$	(−3 − ER(−3))/2	(−3 + ER(−3))/2	3E3	1	1	1	$E3^2$	$E3^2$
.2.1	3E3	− ER(−3)	ER(−3)	$3E3^2$	1	E3	$E3^2$	$E3^2$	1
..21	2	$2E3^2$	2E3	2	0	0	0	0	0
3..	1	1	1	1	1	1	1	1	1
.3.	1	E3	$E3^2$	1	1	E3	$E3^2$	E3	$E3^2$
..3	1	$E3^2$	E3	1	1	$E3^2$	E3	E^2	E3

Table 4.5: Character table of the monomial group $G_{3,3}$

	.1.2	2..1	.2.1	..21	3..	.3.	..3
111..	−1	−1	−1	−1	1	1	1
11.1.	−E3	$-E3^2$	$-E3^2$	$-E3^2$	0	0	0
11..1	$-E3^2$	−E3	−E3	−E3	0	0	0
1.11.	$-E3^2$	−1	−E3	$-E3^2$	0	0	0
1.1.1	0	0	0	0	0	0	0
1..11	−E3	−1	$-E3^2$	−E3	0	0	0
.111.	−1	$-E3^2$	−1	−E3	1	E3	$E3^2$
.11.1	−E3	−E3	$-E3^2$	−1	0	0	0
.1.11	$-E3^2$	$-E3^2$	−E3	−1	0	0	0
..111	−1	−E3	−1	$-E3^2$	1	$E3^2$	E3
21...	0	0	0	0	−1	−1	−1
1.2.	$E3^2$	1	E3	$E3^2$	0	0	0
1..2	E3	1	$E3^2$	E3	0	0	0
2.1.	E3	$E3^2$	$E3^2$	$E3^2$	0	0	0
.21.	0	0	0	0	−1	−E3	$-E3^2$
.1.2	$E3^2$	$E3^2$	E3	1	0	0	0
2..1	$E3^2$	E3	E3	E3	0	0	0
.2.1	E3	E3	$E3^2$	1	0	0	0
..21	0	0	0	0	−1	$-E3^2$	−E3
3..	1	1	1	1	1	1	1
.3.	1	$E3^2$	1	E3	1	E3	$E3^2$
..3	1	E3	1	$E3^2$	1	$E3^2$	E3

Table 4.6: Character table of the monomial group $G_{3,3}$

Chapter 5

Hyperoctahedral Groups

In this chapter we are going to look at a special construction of the Specht modules for the hyperoctahedral groups H_n, the group of symmetries of a n-dimensional cube. These groups also comes under the family of Coxeter groups, more specifically, group of signed permutations is same as the Coxeter group of type-B on n points.

The H_n or B_n groups can also be attained by restricting the parameter r to 2 in the monomial groups, which would then give us the elements in the matrices up to the second roots of unity, which are 1 and -1. However the construction discussed in this chapter is different from the $r = 2$ monomial case. We will explore more on it as we keep developing the concepts throughout the course of this chapter by introducing all the relevant ideas specific to the hyperoctahedrals.

5.1 Hyperoctahedral Groups, Generators and Class Representatives

Let us start by defining a signed permutation. A hyperoctahedral group can then be defined as a group of signed permutations.

Definition 5.1.1 (Signed Permutation). *A signed permutation on the points* $1, 2, \ldots, n$ *is a permutation* σ *on the set* $\{-n, -(n-1), \ldots, -1, 1, \ldots, (n-1), n\}$ *such that* $\sigma(-i) = -\sigma(i)$.

For every $n \in \mathbb{N}$ the *hyperoctahedral group* can be defined to be a permutation group, as a subgroup of a symmetric group on $2n$ points.

Definition 5.1.2 (Hyperoctahedral Group). *Let $n \geqslant 1$. Then the hyperoctahedral group H_n is the group of all signed permutations on n points.*

Alternatively, we can define the H_n group as a group generated by the sign-change permutations. Then, we have an equivalent definition of H_n in Definition 5.1.3.

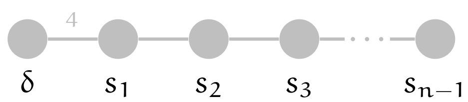
Definition 5.1.3. *Let $n \geqslant 1$ and take $X = \{-n, -(n-1), \dots, -1, 1, \dots, (n-1), n\}$. Then the hyperoctahedral group H_n is the subgroup of S_X generated by the elements $(-1\ 1)$ and $(i\ \ i{+}1)(-i\ \ -(i{+}1))$ for $1 \leqslant i < n$.*

A hyperoctahedral group can also be viewed as a group isomorphic to a wreath product of cyclic group $\mathbb{Z}_2$ and symmetric group S_n, with respect to the natural S_n-action on $\{1, 2, \dots, n\}$, such that for $i \in \{1, 2, \dots, n\}$ and $\pi \in S_n$ we have $(i, \pi) \mapsto \pi(i)$ where $\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order 2. Here the cyclic group $\mathbb{Z}_2$ operates by sign change.

Given the similarities in the generators of both S_n and H_n, the elements in H_n are signed permutations, which implies that for certain elements as permutations in the group contain negatives of the $\{1, 2, \dots, n\}$. Therefore we have that the order of the group, $|H_n| = 2^n \cdot n!$. To generate permutations with negative entries, a specific element $(-i\ i)$ changes any i to $-i$ by right multiplication, and we refer to it by a *sign change*. Likewise, we have the other generators that are product of a transposition $(i\ \ i{+}1)$ and its negative transposition $(-i\ \ -(i{+}1))$ for all $1 \leqslant i < n$, i.e., $(i\ \ i{+}1)(-i\ \ -(i{+}1))$. Therefore the generating set for the hyperoctahedral group H_n is $\{(-1\ 1), (1\ 2)(-2\ \ -1), \dots, (n{-}1\ \ n)(-n\ \ -n{+}1)\}$.

Now let us have a look at the Coxeter diagram that also describes the hyperoctahedral group but in terms of generators.

Definition 5.1.4. *Type-B_n group is a finite irreducible Coxeter group understood by the following Coxeter diagram*



B_n ($n \geqslant 1$)

where the nodes are the generators $\{\delta, s_i | i \in \{1, 2, \dots, n-1\}\}$ of the group such that $s_i = (i\ \ i{+}1)(-i\ \ -(i{+}1))$ and $i \in \{1, 2, \dots, n{-}1\}$ and $\delta = (-1\ 1)$.

It is to be noted that H_n and B_n are essentially the same group but with different presentations. We consider the following elements as defined in [8].

$$\delta_1 = \delta \quad \text{and} \quad \delta_i = s_i \delta_{i-1} s_i \quad \text{for} \quad i = 1, 2, \dots, n{-}1.$$

Then we have $\delta_i\delta_j = \delta_j\delta_i$ for all $i, j = 0, \dots, n-1$, and

$$\delta_i s_j = \begin{cases} s_i\delta_{i-1} & \text{if } j = 1, \\ s_{i+1}\delta_{i+1} & \text{if } j = i+1 \\ s_j\delta_i & \text{otherwise.} \end{cases}$$

Similar to the other families of groups discussed before, each element in the hyperoctahedral group has an associated *cycle-type*. The cycle structure of an element in the H_n group can be described in the same way as it has been done for the monomial groups as hyperoctahedral is nothing but a special type of monomial group.

Definition 5.1.5 ([10, Section 1.6: **Length Function**]). *The length of a signed permutation* σ *for all* $\sigma \in B_n$ *is defined as*

$$\ell(\sigma) = \min\{m \geqslant 0 \mid \sigma = \delta_{i_1} s_{i_1} s_{i_2} \dots s_{i_n} \textit{ for some } i_1, \dots, i_m \in \{0, \dots, n-1\}\}.$$

Definition 5.1.6 ([10, Section 1.6]). *The sign of a signed permutation* σ *for all* $\sigma \in B_n$ *is defined as*

$$\text{sgn}(\sigma) = (-1)^{\ell(\sigma)}.$$

By fixing $r = 2$ in $G_{r,n}$, a bipartition is defined below which will be used throughout this chapter.

Definition 5.1.7 (Bipartition). *Let* $n = \alpha + \beta$ *and* $\alpha, \beta, n \in \mathbb{N}$. *Let* $\lambda^0 \vdash \alpha$ *and* $\lambda^1 \vdash \beta$. *Then the pair* $\lambda = (\lambda^0, \lambda^1)$ *is called a bipartition of* n *denoted by* $\lambda \models n$.

For an example, let us consider $n = 5$. Now for $\alpha = 3$ and $\beta = 2$ we have that $\alpha + \beta = n$. If $\lambda^0 = (2, 1) \vdash \alpha$ and $\lambda^1 = (1, 1) \vdash \beta$ then we have $\lambda = (\lambda^0, \lambda^1) = ((2, 1), (1, 1)) \models 5$.

The conjugacy classes of the hyperoctahedral group H_n are parameterized by the bipartitions of the natural number n. We will discuss further on positive and negative cycles in the next section when talking about bipartitions.

Therefore, each distinct cycle-type corresponds to a conjugacy class of the group H_n. This implies that we have as many conjugacy classes in the group H_n as there are distinct cycle types.

Theorem 5.1.8 ([12, Theorem 4.2.8]). $\sigma, \rho \in H_n$ *have the same cycle type if and only if* σ *is a conjugate to* ρ.

If we consider the Coxeter description of the hyperoctahedral groups then we can simply use the generators δ_i and s_j to describe the class representatives of the signed permutations in the group B_n. In Section 3.4 of [8], this idea is well described using blocks and signed block forms. This is another way of looking into the cycle types of group elements as here the idea that has been used is of the positive and negative cycles.

5.2 Bipartitions and Words

Like seen for the other groups, we need to define a notion of words in order to build the Specht modules for the hyperoctahedral groups. Most of the definitions in this chapter are different to the monomial groups with r being restricted to 2.

In general, this construction is different than that of the $G_{2,n}$ groups as here we do not restrict the group action only to just symmetric group but we let the full H_n group act on words corresponding to each bipartition. Also, later we will see that the words being used to build the module are very different to that of the monomial construction. This results into the Specht matrices having a full H_n orbit of nonzero entries as opposed to just $n!$ in the $G_{2,n}$ case.

Bipartitions are nothing but a special case of a multipartitions of n by fixing one of the parameters. The reason for redefining the bipartition is to introduce the notions of positive and negative cycles. We say that the for a bipartition $\lambda = (\lambda^0, \lambda^1) \models n$ we have the parts of $\lambda^0 \vdash \alpha$ is corresponding to a positive cycle and the parts of $\lambda^1 \vdash \beta$ is corresponding to a negative cycle of an element of H_n. Due to the fact that the sign change functionality of the H_n groups affect the distinct letters in a word, we have that a bipartition consists of positive and negative cycles.

Now, for the construction of the Specht modules and more specifically the Specht matrices, the concept of words corresponding to a bipartition is crucial. We already have seen a construction of the monomial group words where we can use the indexed-letters by fixing $r = 2$. As this is a different construction, the new way to define the words is by embedding the H_n group as a subgroup inside the S_{2n} group and that gives us the words of length $2n$ in the following way.

So we make the convention of making words using two types of letters l and l' for $l \in \{1, \dots, n\}$. The letters l' are invariant under sign change and the letters l are sign variant and changes to $-l$ under the effect of sign change. We also make the convention of denoting $-l$ by $\bar{l}$ to be more efficient in using the space when we talk about Specht matrices.

Definition 5.2.1. *Let $\mathfrak{L} = \{\pm l, l' : l \in \{1, \ldots, n\}\}$ be the set of letters. Then we define $\bar{l} = -l$ if $l \in \{\pm 1, \ldots, \pm n\}$ and $\bar{l'} = l'$.*

Then we have for each bipartition $\lambda = (\lambda^0, \lambda^1) \models n$ we define ***canonical word*** $w_\lambda = (w_{\lambda^0})' w_{\lambda^1}$ is made of α number of sign invariant letters l' followed by β number of sign changing letters l so that $\alpha + \beta = n$.

As an example, let $\lambda = (\lambda^0, \lambda^1) = ((2,1,1)(2,2)) \models 8$. We have $(w_{\lambda^0})' = 1'1'2'3'$ and $w_{\lambda^1} = 1122$. Then the canonical word for $w_\lambda = (w_{\lambda^0})' w_{\lambda^1} = 1'1'2'3'1122$.

For any word w of length n, we define $\lambda(w) \models n$ to be the shape of w if $w = w_\lambda \cdot \sigma$ for any $\sigma \in H_n$. We now define a function that maps the length n words to length $2n$ as follows.

Definition 5.2.2 (Length-change Function). *Let w denote a word of length n. Let $w = (l_1 l_2 \ldots l_n)$ be a word of length n. We define the length-change function Ω that maps w to a word $\Omega(w)$ of length $2n$ by the map $(l_1 l_2 \ldots l_n) \mapsto (l_1 l_2 \ldots l_n \bar{l}_n \ldots \bar{l}_2 \bar{l}_1)$.*

Consider words of length n have letters arranged in such a way that the sign invariant letters come first and then followed by the sign changing letters. Whereas in the length $2n$ words, the first n letters are as they are in the original word and the next n letters are simply attained by a modified mirror-like effect so that the the first n letters are repeated in the reverse order with their respective signs changed. It also describes how the number of sign invariant, and sign changing letters are adding up to the positive and negative cycles α and β respectively.

The H_n groups get very complicated to work with due to the signed permutations and that is why the requirement to work flawlessly with computer programmes for experimental purposes, we embed it in S_{2n} group. As it is easier to work with the symmetric group without worrying too much about the extra generators as sign change and that is why this method of embedding inside S_{2n} has been adapted. In terms of computer language [9], each word is considered to be a list of letters where the sign changing letters are simple letters $l \in \mathbb{N}$ and the sign invariant letters l' are strings of letters.

As an example, let us consider the bipartition $\lambda = ((2,1),(1))$. Then the corresponding canonical word of length n is $w_\lambda = 1'1'2'1$. Therefore, the canonical word of length $2n$ is $\Omega(w_\lambda) = 1'1'2'1\bar{1}2'1'1'$. Similarly, if we take the word $w = 1'12'1'$ which is non-canonical, then by the map we get the word $\Omega(w) = 1'12'1'1'2'\bar{1}1'$.

We also define a map that takes any permutation σ in S_{2n} and sends it to a signed permutation in H_n. Let us define a map $\upsilon : \{\pm 1, \pm 2, \ldots, \pm n\} \longrightarrow \{1, 2, \ldots, 2n\}$ by

$$i \mapsto ((i-1) \mod (2n+1)) + 1.$$

Therefore the embedding of H_n inside S_{2n} is now concrete by implementation of the map υ in such a way that a point i in a signed permutation $\sigma \in H_n$ gets transformed into a point $j = \upsilon(i)$ in a permutation $\pi \in S_{2n}$. The reverse can be done by simply considering the inverse function.

We then have the inverse map $\upsilon^{-1} : \{1, 2, \ldots, 2n\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm n\}$ by

$$i \mapsto ((i+n) \mod (2n+1)) - n.$$

As an example, for $n = 5$ we have the list of integers $[1, 2, 3, 4, 5, -5, -4, -3, -2, -1]$ where simply the points belong to set $\{\pm 1, \ldots, \pm 5\}$. Now for every $i \in [1, 2, 3, 4, 5, -5, -4, -3, -2, -1]$, one can apply the map and list out all the $\upsilon(i)$ in that order to obtain the list $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$.

It is now time to understand the group action of H_n on the words on length n. Given a word w of length n, we have the symmetric group S_n acting transitively on it by rearranging the letters of the word in the usual way as we have seen before and that gives us $w \cdot S_n$. Also, we have that H_n group acts on it from the right sign change on the respective positions where the sign changing letters are present in the word as it has no visual changes on the sign invariant letters. Eventually as a result of these actions we retrieve the orbit set consisting of all possible rearrangements of w induced by signed permutations.

Definition 5.2.3. *Let* $w = w_1 \cdots w_{i-1} w_i w_{i+1} \cdots w_n$ *be a word of length* n*. For any* $\sigma \in S_n \leqslant H_n$ *we have the permuted action* $w \cdot \sigma = w_{1 \cdot \sigma^{-1}} \cdots w_{i \cdot \sigma^{-1}} \cdots w_{n \cdot \sigma^{-1}}$*, else for* $\delta_i \in H_n$ *we have* $w \cdot \delta_i = w_1 \cdots w_{i-1} \overline{w}_i w_{i+1} \cdots w_n$.

Let $\sigma \in H_n$ be any arbitrary signed permutation. Let w be any word of length n. Then by the right-action of H_n on w we obtain a rearrangement $w \cdot \sigma$. Similarly, if we take the length $2n$ version of the word $\Omega(w)$, then we obtain the rearrangement $\Omega(w) \cdot \sigma$. As there is a well defined correspondence between w and $\Omega(w)$, the H_n-actions on them are compatible. Therefore we have the following proposition.

Proposition 5.2.4. *Let* w *be any word of length* n *and* $\Omega(w)$ *be its corresponding word of* *length $2n$. Then for every signed permutation $\delta_i \in H_n$, we have that $\Omega(w \cdot \delta_i) = \Omega(w) \cdot \delta_i$.*

Proof. Let $w = w_1 \cdots w_i \cdots w_n$. Then we have

$$\begin{aligned}\Omega(w \cdot \delta_i) &= \Omega(w_1 \cdots \overline{w}_i w_n) \\ &= w_1 \cdots \overline{w}_i \cdots w_n \overline{w}_n \cdots w_i \cdots \overline{w}_1 \\ &= (w_1 \cdots w_i \cdots w_n \overline{w}_n \cdots \overline{w}_i \cdots \overline{w}_1) \cdot \delta_i \\ &= \Omega(w) \cdot \delta_i.\end{aligned}$$

□

Let $\alpha + \beta = n$ and w be any length n word. We see no visual change on the arrangement of letters of w by action of some permutations in H_n. For a word w this happens by the action of the stabilizer of the word.

Proposition 5.2.5. *Let w be any word of length $n = \alpha + \beta$ and $(\lambda^0, \lambda^1) \models n$ be its shape, where $\lambda^0 \vdash \alpha$ and $\lambda^1 \vdash \beta$. Then the length of the orbit $w \cdot H_n$ obtained by the right action of H_n on w is*

$$|w \cdot H_n| = \frac{|H_n|}{|H_{\lambda^0} \times S_{\lambda^1}|}.$$

Proof. Let w be any word of length n with shape $\lambda = (\lambda^0, \lambda^1)$. Let us take the action of H_n to obtain the the word w^+ that has all of its letters as positive letters. H_n has no effect on the sign invariant letters and only affects the sign variant letters. As the total number of sign changing letters in w is β, we have a total of 2^β possibilities of changing w to w^+.

We also have the S_n-action on this positive word w^+ giving us all the rearrangements and so we have that the length of the orbit by this action is

$$|w^+ \cdot S_n| = \frac{n!}{\prod\limits_{\substack{k=0,1 \\ i=1,\ldots,|\lambda^k|}} \lambda_i^k}$$

where $|\lambda^k|$ is same as α, β for k equals to $0, 1$ respectively.

Then by the above argument, the orbit length of the H_n-action w is at least $2^\beta \cdot |w^+ \cdot S_n|$, i.e.,

$$|w \cdot H_n| \leqslant 2^\beta \cdot \frac{n!}{\prod\limits_{\substack{k=0,1 \\ i=1,\ldots,|\lambda^k|}} \lambda_i^k}.$$

On the other hand, $H_{\lambda^0} \times S_{\lambda^1}$ stabilizes w. Therefore, from the orbit-stabilizer theorem it follows that all inequalities are in fact equalities:

1. the length of the orbit of w is exactly $2^\beta \cdot |w^+ \cdot S_n|$

2. $H_{\lambda^0} \times S_{\lambda^1}$ is the full stabilizer of w

Therefore, as a result of the overall H_n group action, we have

$$|w{\cdot}H_n| = 2^\beta \cdot \frac{n!}{\prod\limits_{\substack{k=0,1\\ i=1,\dots,|\lambda^k|}} \lambda_i^k} = 2^\beta \cdot \frac{n!}{|S_{\lambda^0}|! \cdot |S_{\lambda^1}|!} = \frac{2^{\alpha+\beta} \cdot n!}{2^\alpha \cdot |\lambda^0|! \cdot |\lambda^1|!} = \frac{2^n n!}{|H_{\lambda^0}| \cdot |S_{\lambda^1}|} = \frac{|H_n|}{|H_{\lambda^0} \times S_{\lambda^1}|}.$$

Hence, by the group action of H_n groups on the word w we obtain the set of all possible signed rearrangements induced by H_n group.

□

As an example of such an orbit, let us consider the bipartition $\lambda = ((1),(1,1))$. The associated canonical word of length $n = 3$ is $w_\lambda = 1'12$ which can be extended to a word of length $2n = 6$ by using the length-change function Ω as seen earlier. Then by the action of S_3 on w_λ we first obtain $3! = 6$ rearrangements of the all positive lettered word which are ordered as

$$\{1'12, 1'21, 11'2, 121', 21'1, 211'\}.$$

Now there are two letters $1, 2$ that change signs under the H_3 action, therefore for each of the six rearrangements we will have $2^2 = 4$ possibilities. Therefore by the overall action of H_3 on the word we obtain all possible $2^2 \cdot 3! = 4 \cdot 6 = 24$ rearrangements as shown below

$$\{1'12, 211', 1'\bar{1}2, 1'21, 1'1\bar{2}, \bar{2}11', 2\bar{1}1', 21'1, 1'2\bar{1}, 1'\bar{1}\bar{2}, 121', 1'\bar{2}1,$$
$$\bar{2}\bar{1}1', \bar{2}1'1, 21'\bar{1}, 11'2, \bar{1}21', 1'\bar{2}\bar{1}, 1\bar{2}1', \bar{2}1'\bar{1}, 11'\bar{2}, \bar{1}1'2, \bar{1}\bar{2}1', \bar{1}1'\bar{2}\}.$$

We also get from Proposition 5.2.5 that the stabilizer of a word is of the form $H_{\lambda^0} \times S_{\lambda^1}$, where H_α is a direct product of hyperoctahedral groups corresponding to the positive cycles α and S_β is a direct product of symmetric groups corresponding to the negative cycles β. The group H_α stabilizes the l'-letters and the group S_β stabilizes the l-letters.

Definition 5.2.6. *Let (u, v) be the pair of words of length n corresponding to the bitableau t. Then the row stabilizer subgroup of t is same as the stabilizing subgroup of word u, and* *so* $R(t) = \mathrm{Stab}_{H_n}(u)$. *Likewise, the column stabilizer subgroup of* t *is same as the stabilizing subgroup of word* v*, and so* $C(t) = \mathrm{Stab}_{H_n}(v)$.

For any given word w of length n made up of sign-changing and sign-invariant letters, one can obtain its shape which is a bipartition $\lambda(w) = \lambda$. This process is same as how it is done for the monomial groups. While counting the repetitions of letters in the word w of length $2n$, one thing to keep in mind is that we eventually have to look for the letters in the first half, i.e., up to length n in order to achieve a bipartition λ of n.

Proposition 5.2.7. *As permutation action of* H_n*, the rearrangements of a word* w_1 *of length* $2n$ *and the rearrangements of a word* w_2 *of length* $2n$ *are isomorphic if and only if* w_1 *and* w_2 *have the same shape, i.e., if* $\lambda(w_1) = \lambda(w_2)$.

Proof. The proof of this proposition follows in the same fashion as the proof of Proposition 3.3.3. The only difference now is that now we need to consider that H_n action on the words are slightly different than the S_n actions.

Let $\alpha, \beta \in \mathbb{N}$ with $\alpha + \beta = n$. By the action of $H_{\lambda^0} \times S_{\lambda^1}$ on the word w_1 for some λ^0, λ^1 such that $|\lambda^0| = \alpha$ and $|\lambda^1| = \beta$ and $\lambda(w_1) = (\lambda^0, \lambda^1)$, we find all the rearrangements of w_1. Similarly, By the action of $H_{\mu^0} \times S_{\mu^1}$ on the word w_2 for some μ^0, μ^1 such that $|\mu^0| = \alpha$ and $|\mu^1| = \beta$ and $\lambda(w_2) = (\mu^0, \mu^1)$, we find all the rearrangements of w_2.

The actions on the orbits are isomorphic because the stabilizers $H_{\lambda^0} \times S_{\lambda^1}$ and $H_{\mu^0} \times S_{\mu^1}$ are conjugates to each other, and this possible if and only if $(\lambda^0, \lambda^1) = (\mu^0, \mu^1)$, i.e., $\lambda(w_1) = \lambda(w_2)$. □

We can now restrict $r = 2$ in the Definition 4.4.2, Definition 4.4.3 and Proposition 4.4.4 to obtain an analogue of the same that fits the theme of this chapter. By fixing $r = 2$, we obtain a *bidiagram* from Defintion 4.4.2 as a subset of a multidiagram.

To construct the Specht modules, we are interested in bidiagrams where each component diagram of it is corresponding to a partition. Therefore, for each word in the orbit of w_λ, we can associate a bihistogram to it. The idea of a bihistogram corresponding to a word is different to the idea of constructing multihistograms for words in $G_{2,n}$ representations. Then, for any word w, we can find the bipartition $\lambda(w) = (\lambda^0, \lambda^1)$, and therefore we have two histograms each corresponding to a λ^i for $i = 0, 1$, together giving us a bihistogram. Therefore, a bihistogram for λ is a pair of histograms corresponding to each λ^i.

As there are now letters of two different types, sign variant and sign invariant, we consider that in the bipartition $\lambda = (\lambda^0, \lambda^1)$, the partition λ^0 correspond to the l'-letters and the partition λ^1 correspond to the l-letters. For this reason, we make the convention that in the pair of histograms, the rows of the first histogram is labelled by the sign invariant letters and the rows of the second histogram is labelled by the sign changing letters.

Definition 5.2.8 (Bihistogram). *Let w be a word as a function $w : \{-n, -(n-1), \ldots, -1, 1, \ldots, (n-1), n\} \to \{1', 2', 3', \ldots, \pm 1, \pm 2, \pm 3, \ldots\}$ such that $w(i) = \Omega(w)[\upsilon^{-1}(i)]$ and $w(-i) = \overline{w(i)}$ where $\Omega(w)[\upsilon^{-1}(i)]$ denotes the letter at position $\upsilon^{-1}(i)$ in $\Omega(w)$. Then the bihistogram, denoted by $\mathcal{H}_w$, of the word w is the set $\{w^{-1}(1'), w^{-1}(2'), w^{-1}(3'), \ldots, w^{-1}(1), w^{-1}(2), w^{-1}(3), \ldots\}$ of pre-images of its distinct letters.*

It is to be noted that due to the correspondence of each of the component histograms to a positive and negative cycles, we only record the absolute values of the $i \in \{\pm 1, \cdots, \pm n\}$ as entries in rows of the histogram corresponding to the positive cycle. Also, as in the histogram corresponding to the positive cycle the row labels are sign invariant letters, the change of sign has no effect on them. Therefore, negative entries only occur in the histogram corresponding to λ^1 in the bihistogram $\mathcal{H}$ of shape $\lambda = (\lambda^0, \lambda^1)$.

As an example, consider $\lambda = ((3,1),(2,2))$ and $w_\lambda = 1'1'1'2'1122$. The rows of each histogram corresponding to each partition is labelled by distinct sign invariant and variant letters. Let $\sigma = (1\ -3\ 7\ 6)(4\ -5)(-1\ 3\ -7\ -6)(-4\ 5) \in H_8$ be a signed permutation. Then we have another rearrangement $w_\lambda \cdot \sigma = 1'1'\bar{2}\bar{1}2'1'12$ by the right action of σ on w_λ. The respective bihistograms are shown in Figure 5.1.

Again, for each component histogram we get a sequence of subsets of $\{\pm 1, \pm 2, \ldots, \pm n\}$ decreasing in size. Union of the collection of all these subsets returns the set $\{\pm 1, \pm 2, \ldots, \pm n\}$. One can recover the word corresponding to a given bihistogram.

This recovery process is similar to how it was for the monomial groups, the difference being that a bihistogram can take negative integers as well. However, it can only take value as either the positive or the negative of a same integer, i.e., the integers with same absolute value can appear only once in a bihistogram.

If the integer corresponding to a letter in the bihistogram is negative, we need a way to retrieve the words with negative letters. If in the bihistogram, we have a negative integer $i \in \{-1, -2, \ldots, -n\}$ then we first check whether i belongs to component histogram of shape λ^0 or λ^1. If i belongs to the histogram of shape λ^0 then i is corresponding an l' letter and therefore

$$\begin{array}{c|ccc} 1' & 1 & 2 & 3 \\ 2' & 4 & & \end{array} \; , \; \begin{array}{c|cc} 1 & 5 & 6 \\ 2 & 7 & 8 \end{array}$$

(a) w_λ bihistogram

$$\begin{array}{c|ccc} 1' & 1 & 2 & 6 \\ 2' & 5 & & \end{array} \; , \; \begin{array}{c|cc} 1 & 4 & 7 \\ 2 & \overline{3} & 8 \end{array}$$

(b) $w_\lambda \cdot \sigma$ bihistogram

Figure 5.1: Bihistograms for $w_\lambda = 1'1'1'2'1122$ and $w_\lambda \cdot \sigma = 11'1'\bar{1}2'21'2$

we consider the l' letter as it is. If i belongs to the histogram of shape λ_1 then we replace l by $-l$ at position i in the word w.

Let w be a word with distinct letters $\{1', 2', 3', \dots, \pm 1, \pm 2, \pm 3, \dots\}$. Let $\mathcal{H} = \{h_{w_{\lambda^0}}, h_{w_{\lambda^1}}\}$ be a bihistogram. For $i = 0, 1$, each row of $h_{w_{\lambda^i}}$ has some elements producing a subset of $\{\pm 1, \pm 2, \dots, \pm n\}$ corresponding to a row tabloid. Each row of $h_{w_{\lambda^i}}$ is labelled by a sign invariant or sign changing letter. Let us denote the row subsets by $h_{w_{\lambda^0}}[l']$ or $h_{w_{\lambda^1}}[l]$ where l' and l are the letter labelling the corresponding rows of $h_{w_{\lambda^0}}$ and $h_{w_{\lambda^1}}$ respectively. Then we have that $|h_{w_{\lambda^0}}[l']| \cup_l |h_{w_{\lambda^1}}[l]| = \{1, 2, \dots, n\}$ where $|h_{w_{\lambda^0}}[l']|$ and $|h_{w_{\lambda^1}}[l]|$ denote the sets of absolute values of the integers corresponding to the letters l' and l respectively. These subsets describe the position of the letter l' or l in the word w. Therefore, running through all letters l' and l, we obtain the n-lettered word w corresponding to $\mathcal{H}$.

For example, if we consider the bihistogram in Figure 5.1(b), then we see that we have the corresponding set of distinct letters $l', l \in \{1', 2', 1, 2\}$ with respective set of subsets $D(\lambda^0)[l'] \in \{\{2, 3, 7\}, \{5\}\}$ and $D(\lambda^1)[l] \in \{\{1, -4\}, \{6, 8\}\}$ describing the positions of each letter in the word. Therefore, we obtain the word corresponding to this bihistogram is $w = 11'1'\bar{1}2'21'2$.

The action of H_n groups on the words w of length n can be understood with the help of their respective bihistograms. Now, as we have sign invariant and sign changing letters, we introduce a new lexicographic ordering on these letters.

Definition 5.2.9. *Let us set the convention of the ordering of the letters for the hyperoctahedral groups*

$$1' < 2' < 3' < \cdots < \bar{1} < \bar{2} < \bar{3} < \cdots < 1 < 2 < 3 < \cdots .$$

Let $w = w_1w_2\dots w_n$ and $v = v_1v_2\dots v_n$ be two words of same length n. Then for all i if $w_i = v_i$, we say that $w = v$. Also, $w < v$ or w appears earlier than v in the lexicographic order if for some $w_i < v_i$ and $w > v$ or w appears later than v in the lexicographic order if for some $w_i > v_i$ and $w_j = v_j$ for all $j < i$.

5.3 Bitableaux and Pair of Words

A bidiagram is nothing but a multidiagram corresponding to a bipartition of a natural number n, each row of the corresponding component diagrams has a length. We convert it into a Young bidiagram by assigning an empty box to each of these positions in the rows of the diagrams. Clearly, there are n boxes in a Young bidiagram.

Now, filling the n boxes with numbers $\{\pm 1, \pm 2, \dots, \pm n\}$ will give us a bitableau, which is different to that of the Young multitableau that are used to represent the $G_{r,n}$.

Definition 5.3.1 (Young Bitableau). *If $\lambda \models n$, then a λ-bitableau (or Young bitableau of shape λ) is a 3-dimensional array t of integers obtained by placing $\{\pm 1, \pm 2, \dots, \pm n\}$ in the boxes of two Young diagrams corresponding to each λ^i for $i = 0, 1$ respectively satisfying the criterion that only one of the numbers with same absolute value can appear in the boxes at a time, i.e., $t = (t^0, t^1)$ is a Young bitableau where t^i is a Young tableau of shape λ^i for $i = 0, 1$.*

As an example, we have $\lambda = (\lambda^0, \lambda^1) = ((3,1),(2,2)) \models 8$ and therefore one of the corresponding bitableaux is as follows

$$\begin{array}{|c|c|c|}\hline 1 & \bar{5} & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 6 & \bar{2} \\ \hline 8 & \bar{7} \\ \hline\end{array}.$$

Clearly, by the right-action of H_8, we will have a corresponding tableau for each element in B_8. This implies there will be a total of $2^8 \cdot 8!$ Young bitableaux of shape $((3,1),(2,2))$.

Definition 5.3.2 (Standard Bitableau). *A Young bitableau is said to be a standard bitableau when the integers in the boxes of each t^i for $i = 0, 1$ in t are all positive and increasing to the right and to the bottom for each row and each column respectively.*

As it is allowed for a bitableau to contain both positive and negative entries, for any bipartition of n we have a total of $2^n \cdot n!$ Young bitableaux. From now on we will use the abbreviation *SYBT* for standard Young bitableau.

Definition 5.3.3 (Canonical Bitableau). *A canonical bitableau t_λ of shape λ is a special case of $r = 2$ for a canonical multitableau corresponding to the group $G_{2,n}$.*

As an example, for $\lambda = ((3,1),(2,2)) \models 8$ we have the standard canonical λ-bitableau as

$$t_\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & 8 \\ \hline\end{array}.$$

By natural action of hyperoctahedral group on the labelling of the boxes of the bitableau we get the numbers permuted with their signs changed in the boxes.

As an example, $(1\ -3\ 7\ 6)(4\ -5)(-1\ 3\ -7\ -6)(-4\ 5) \in H_8$ acts from the right on the above bitableau as follows

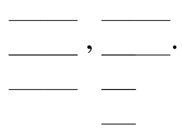
$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & 8 \\ \hline\end{array} \cdot (1\ -3\ 7\ 6)(4\ -5)(-1\ 3\ -7\ -6)(-4\ 5) = \begin{array}{|c|c|c|}\hline \bar{3} & 2 & \bar{7} \\ \hline \bar{5} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \bar{4} & 1 \\ \hline 6 & 8 \\ \hline\end{array}$$

by preserving the shape of the bitableau but with a different numbering of the diagram. Then each bitableau corresponds to an element in H_n. If the canonical SYBT corresponds to the id $\in H_n$ then by applying the group action on the canonical SYBT we can generate all the possible numbering of the bitableau.

The definition of conjugate bipartition is going to be different for this construction from the one already seen for the monomial groups.

Definition 5.3.4 (Conjugate Bipartition). *Let $\lambda = (\lambda^0, \lambda^1) \models n$. Then the conjugate bipartition of λ is $\mu = (\mu^0, \mu^1) \models n$ such that $\mu^0 = {\lambda^1}^T$ and $\mu^1 = {\lambda^0}^T$. Therefore we have $\mu = \lambda^T = ({\lambda^1}^T, {\lambda^0}^T)$ to denote the conjugate bipartition of λ.*

For every conjugate bipartition λ^T we also have a corresponding bidiagram of which the first component diagram is the transpose diagram of the second one in the bidiagram corresponding to λ and vice versa. As an example, for $\lambda = ((3,1),(2,2))$ we have its conjugate bipartition $\lambda^T = ((2,2),(2,1,1))$ which is obtained from the bidiagram



Then we have a bitableau

$$t = \begin{array}{|c|c|c|}\hline 4 & \bar{1} & 8 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \bar{2} & 7 \\ \hline 6 & 5 \\ \hline\end{array}$$

corresponding to the bipartition λ, then the transpose of t, say t^T, corresponds to the conjugate bipartition λ^T, so that the shape of t^T is λ^T. It is possible by following the same idea of a conjugate bipartition of a bipartition on to the transposed bitableau of a bitableau. Then we have that

$$t^{\mathsf{T}} = \begin{array}{|c|c|}\hline \bar{2} & 6 \\ \hline 7 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 3 \\ \hline \bar{1} \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}.$$

Clearly, it can be seen from the above example is that there is no effect of the conjugation on the signs attached to any of the integers present in the boxes of the bitableau t.

The canonical word for $\lambda^{\mathsf{T}} = ((2,2),(2,1,1))$ is $w_{\lambda^{\mathsf{T}}} = 1'1'2'2'1123$. So we have the bihistogram associated to $w_{\lambda^{\mathsf{T}}}$ as:

$$\begin{array}{c|cc} & \multicolumn{2}{c}{} \\ \hline 1' & 1 & 2 \\ \hline 2' & 3 & 4 \\ \hline \end{array}\ , \quad \begin{array}{c|cc} & \multicolumn{2}{c}{} \\ \hline 1 & 5 & 6 \\ \hline 2 & 7 & \\ \cline{1-2} 3 & 8 & \\ \cline{1-2} \end{array}$$

Definition 5.3.5 (Bitabloid). *A bitabloid of shape λ gives us the row equivalence class for the underlying bitableau corresponding to it. Similarly, one can define the column-equivalence class for a λ-bitableau. We denote the row and column equivalence classes of t by $\{t\}$ and $[t]$ respectively.*

For any two bihistograms corresponding to two words of same length n, we get an intersection bihistogram. For the intersection to work, we need to have that the component histograms intersect in a crosswise fashion.

Definition 5.3.6 (Intersection Bihistogram). *Let u and v be two words of same length n consisting sign invariant and sign changing letters, and $\mathcal{H}_u = \{h_{u_0}, h_{u_1}\}$ and $\mathcal{H}_v = \{h_{v_0}, h_{v_1}\}$ be their corresponding bihistograms respectively. We define the intersection bihistogram $\mathcal{H} = \mathcal{H}_u \cap \mathcal{H}_v$ such that $\mathcal{H} = ((\pm h_{u_0}) \cap h_{v_1}, h_{u_1} \cap (\pm h_{v_0}))$ where $\pm h_{u_0}$ and $\pm h_{v_0}$ denote the sets h_{u_0} and h_{v_0} have their entries with both signs $\pm$.*

This crosswise intersection of bitableaux does indeed always work for two bitableaux corresponding to any pair of words of same length. Like we have seen for the other cases, here as well, we have the notion of good and bad intersection bihistograms. Due to the effect of this cross intersection between the bihistograms, the intersection bihistogram that we attain has its rows and columns labelled by two different types of letters and later on we will make it a concrete criterion to be satisfied in order to retain such intersection bihistograms. Same as before, we only are interested in intersection bihistograms that preserve the shapes of the original bipartitions that we start with.

Definition 5.3.7 (Simultaneous Bihistogram). *Let (u, v) be a pair of words of same length n consisting of sign invariant and sign changing letters. Then there exists a simultaneous bihistogram $\mathcal{H} = (h^0, h^1)$ corresponding to (u, v) obtained by the intersection of two bihistograms corresponding to u and v respectively if the following criteria hold*

1. *each component histogram h^i for $i = 0, 1$ is a simultaneous histogram,*

2. *the shape of $\mathcal{H}$ is a bidiagram that represents a bipartition of n.*

As usual, we consider a pair of n-lettered words (u, v) where $\lambda(u), \lambda(v) \models n$. As the H_n words are now made up of two types of letters, we further need to impose a criterion to be satisfied for a pair of signed-lettered words to be fit in a good intersection bihistogram. In general, a *simultaneous bihistogram* is a Young bitableau such that each of the component tableaux has its rows and columns labelled by the distinct letters in the row and column word respectively and the pair (u, v) has complementary words. Therefore, the shapes of bihistograms of the corresponding words that intersect each other diagonally need to be conjugates to each other.

Proposition 5.3.8. *Let u and v be two words of same length, and h be the intersection bihistogram of their corresponding bihistograms, then shape of u must be the transpose of the shape of v.*

Example 5.3.9. For $n = 8$, let us have $w_1 = 1'1'1'2'1122$ and $w_2 = 1'1'2'2'1123$. Clearly, if we use the pair

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1' & 1' & 1' & 2' & 1 & 1 & 2 & 2 \\ 1' & 1' & 2' & 2' & 1 & 1 & 2 & 3 \end{pmatrix}$$

then they not only fail to satisfy criterion 1 in Definition 5.3.7 but also the criterion for the cross intersection of the bihistograms, and therefore cannot be fitted in a simultaneous bihistogram. Instead if we consider the word $w_1 = 1'1'1'2'1122$ as it is and take a rearrangement of w_2 as $w_2' = 12311'2'1'2'$. Now we have the pair

$$\begin{pmatrix} w_1 \\ w_2' \end{pmatrix} = \begin{pmatrix} 1' & 1' & 1' & 2' & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 1 & 1' & 2' & 1' & 2' \end{pmatrix}.$$

that satisfies all the criteria to become a simultaneous bihistogram as shown in Figure 5.2.

Clearly, $\lambda(w_1) = ((3, 1), (2, 2))$ and $\lambda(w_2) = ((2, 2), (2, 1, 1)) = \lambda(w_1)^T$. As usual, in the pair (w_1, w_2'), each column is corresponding to a numbering in the boxes of tableaux in the

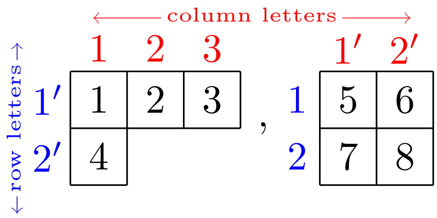


Figure 5.2: Simultaneous bihistogram of the pair $(1'1'1'2'1122, 12311'2'1'2')$

bihistogram and the respective numbers in the boxes denote the position of the column in the sequence of columns in the word pair.

	1	2	3	4	5	6	7	8
$w_1 \rightarrow$	$1'$	$1'$	$1'$	$2'$	1	1	2	2
$w_2' \rightarrow$	1	2	3	1	$1'$	$2'$	$1'$	$2'$

Hyperoctahedral group H_n acts on the pair of words from the right and we get different orderings of the columns and in return we get all the possible numbering of the boxes in the simultaneous bihistogram.

We find that the bitableau found for the pair of words (w_1, w_2) in Example 5.3.9 is a simultaneous bihistogram. An interesting thing to notice is that the letters in any column of the pair are of different types. This indicates the cross intersection of component histograms of the bihistograms for the words.

Here we get the H_n group action on the words as described earlier. Therefore we get possible arrangements of the words with their signed letters. For two words u, v of same length n, consider the sets of rearrangements U, V by the H_n action on u, v respectively. Therefore, H_n acts on the pairs (u, v), which gives us a subset of $U \times V$.

As seen in [18] and for the monomials, we have an analog for the case of the H_n groups. Two words w_1, w_2 of same length n have complementary rearrangements if the action of H_n on the product of rearrangements of the words has a unique free orbit. The idea of complementary words is same as that of the symmetric groups, with the added criterion that letters in each column are of different types.

Definition 5.3.10. *A pair (u, v) of words consisting of sign invariant and sign changing letters is said to be complementary if and only if the following hold:*

- *there is no repeated columns in the pair* (u, v),
- *the letters in each column are of different types, i.e., if any one of the letters is a sign-invariant letter* l' *then the other one must be a sign-changing letter* l.

Clearly, the H_n-orbit of a complementary pair of words is a free orbit.

Theorem 5.3.11. *Let* v *and* w *be two words of length* n *made up of sign-invariant and sign-changing letters. Then the action of* H_n *on the pair* (v', w'), *where* v' *and* w' *are some signed-rearrangements of* v *and* w *respectively, has a unique free orbit if and only if* $\lambda(w)$ *is the transposed bipartition of* $\lambda(v)$.

Proof. The proof of this theorem uses the same theme of the proofs of Theorem 3.4.11 and 4.5.11.

Let v, w be two words with sign-invariant and sign-changing letters of same length n. Let $\lambda = (\lambda^0, \lambda^1) \models n$ and $\mu = (\mu^0, \mu^1) \models n$ such that $\lambda(v) = \lambda$ and $\lambda(w) = \mu$. Now, let v', w' be some rearrangement of the words v, w respectively for which we have a simultaneous bihistogram $\mathcal{H} = (h^0, h^1)$. Therefore, $\mathcal{H}$ can be considered as a matrix that has its columns made of letters of the complementary words such that no column has letters of same type.

Now, as explained in Section 5.2, for each of the words v' and w' we have a corresponding bihistogram. We also know that bihistograms are set of subsets of $\{\pm 1, \pm 2, \ldots, \pm n\}$. Then for a pair of words (v', w'), we have two bihistograms as two sets of subsets of $\{\pm 1, \pm 2, \ldots, \pm n\}$ corresponding to each word v' and w' respectively.

The simultaneous bihistogram $\mathcal{H}$ therefore is simply the intersection of the two bihistograms corresponding to v' and w' such that $\mathcal{H} = (h^0, h^1) = (h_{\lambda^0} \cap h_{\mu^1}, h_{\lambda^1} \cap h_{\mu^0})$ where the bihistograms individually corresponding to the words are $\mathcal{H}_{v'} = (h_{\lambda^0}, h_{\lambda^1})$ and $\mathcal{H}_{w'} = (h_{\mu^0}, h_{\mu^1})$ respectively. Now, for each of these component tableau h_i in $\mathcal{H}$, we can apply the recovery process to avoid the failures (F_1) and (F_2) as explained in Theorem 3.4.11 and as follows.

Let $\pm R = \{+r, -r : r \in R\}$ denote any row of a simultaneous bihistogram where $+r$ and $-r$ are the positive and negative entries in the corresponding rows R respectively. Similarly, let C denote the columns of the simultaneous bihistograms consisted of just the absolute values of the entries in the same column. If $\mathcal{H}$ is the simultaneous bihistogram corresponding to the some rearrangements of w_λ and w_μ then $\{\mathcal{H}\} = \bigcup \pm R$ and $[\mathcal{H}] = \bigcup C$. Then for $k_1, k_2 = 0, 1$ and $1 \leqslant i \leqslant |\lambda^k|$ and $1 \leqslant j \leqslant |\mu^k|$ we have that $\mathcal{H}_{(i,j)} = \pm R_i^{\lambda^{k_1}} \cap C_j^{\mu^{k_2}}$ for $k_1 \neq k_2$. Now if we have that the entry in the box having coordinate (i, j) has a negative sign, then we take the letters corresponding to the row and column words with their signs negative, although it has no effect for one of the letters as it is sign-invariant.

We also do the vice versa of this process whenever we are going in the reverse direction, i.e., while finding out the simultaneous bihistogram for a pair of word if we have that in any column of the pair we have a signed letter then we take assign a negative sign to the integer that goes into the corresponding box in the bihistogram.

Therefore, we obtain the pair of word in the same way as described in earlier chapters as well as given by the aforementioned mechanism for deciding the signs of the corresponding letters. Iterating through all the letters for each word, we find that the shape of k_1-th histogram of $H_{v'}$ is the transpose of the k_2-th histogram of $H_{w'}$ such that $k_1 \neq k_2$. Hence, $\mu = \lambda^T \implies \lambda(w) = \lambda(v)^T$.

□

Just like in the other cases, we have a canonical bitableau which corresponds to a canonical pair of word. A canonical pair of word for the H_n groups is chosen by following the guidelines mentioned below, which is not going to be exactly the same as that of the monomial groups. Here we need to have a careful consideration of the fact that due to the properties of a conjugate bipartition we have the effect of the cross intersection, which eventually results in having different types of letters in each column of the word pair.

Definition 5.3.12 (Canonical Pair). *For the canonical bitableau of shape λ, let (u, v) be the corresponding pair of complementary words of same length made of sign invariant and sign changing letters.*

Then a pair (u, v) is the canonical pair if we keep u as it is and take v in such a way that all the distinct sign-changing letters appear first once each in the increasing order and then they repeat the pattern until their multiplicities are exhausted, then repeat the same for sign-invariant letters until all the letters of v are exhausted.

Quite clearly, there can be only one canonical pair of such words, but overall there are $2^n \cdot n!$ ways to pair up words of same length corresponding to a bipartition and its conjugate bipartition so that each of them correspond to a Young bitableau. In Example 5.3.9 we find all the criteria for satisfying a pair to be canonical hold, we can say that the chosen pair (w_1, w_2') is canonical. Canonical bitableau always corresponds to canonical pair of words. Therefore, we consider the canonical pair/bitableau to be a representative of $\mathrm{id} \in H_n$. Hyperoctahedral group acts on the canonical bitableau from the right by giving us all possible $2^n \cdot n!$ possible bitableaux of the same shape.

Similar to that of the other groups, we have the following as a direct implication of Definition 5.3.10 and Theorem 5.3.11.

Corollary 5.3.13. *Let U and V be the set of all rearrangements of complementary words with sign-invariant and sign-changing letters u and v respectively. Then for $u' \in U$ and $v' \in V$, the pairs (u', v') is in the free orbit if and only if the intersection bihistogram is a bitableau.*

Corollary 5.3.14. *Let $\lambda \models n$. The function $\mathcal{H} : (v, w) \mapsto$ simultaneous bihistogram of (v, w) establishes a bijection between the elements of the unique free H_n-orbit on the set $\{$rearrangements of $w_\lambda\} \times \{$ rearrangements of $w_{\lambda^T}\}$ and the set of bitableaux of shape λ.*

We can also find nonstandard ***inter-bitableaux*** for the case of hyperoctahedral groups. An inter-bitableau is a special case of inter-multitableau obtained by fixing $r = 2$. The intersection of two bitableaux follows the same description as that of the monomial groups. Therefore, one can refurbish Example 4.5.14 by ignoring its third component to get an example of an inter-bitableau. Let us have a look at another example.

Example 5.3.15. Let us consider two SYBT of shape $\lambda = ((3, 2), (1, 1))$,

$$t_1 = \begin{array}{|c|c|c|}\hline 1 & \overline{3} & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline \end{array} \quad \text{and} \quad t_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|}\hline \overline{6} \\ \hline 7 \\ \hline \end{array}$$

respectively. Then for t_1 we have the row stabilizer subset as $\{t_1\} = \{\{\{1, \overline{3}, 5\}, \{2, 4\}\}, \{\{6\}, \{7\}\}\}$ and similarly for the transposed t_2 we have $\{t_2^T\} = \{\{\{\overline{6}, 7\}\}, \{\{1, 4\}, \{2, 5\}, \{3\}\}\}$. Therefore, we obtain a inter-bitableau $t = (t^0, t^1)$ such that $t = (\pm\{t_1^0\} \cap \{(t_2^T)^1\}, \pm\{t_1^1\} \cap \{(t_2^T)^0\})$ where $\pm\{t_1^0\} \cap \{(t_2^T)^1\}$ and $\pm\{t_1^1\} \cap \{(t_2^T)^0\}$ denote the sets $\{t_1^0\} \cap \{(t_2^T)^1\}$ and $\{t_1^1\} \cap \{(t_2^T)^0\}$ can take their entries with sign positive or negative if there is an element in the intersection which is the absolute value of elements in the tabloids. In such cases where there are elements in both the tabloids that are intersecting each other and in the intersection have at least one entry which are the absolute values of corresponding entries in the tabloids then we put this entry with its sign in the box of the intersection bitableau. Therefore, we have

$$
\begin{aligned}
t^0_{(1,1)} &= \pm\{1,\bar{3},5\}\cap\{1,4\} = \{1\}\\
t^0_{(1,2)} &= \pm\{1,\bar{3},5\}\cap\{2,5\} = \{5\}\\
t^0_{(1,3)} &= \pm\{1,\bar{3},5\}\cap\{3\} = \{\bar{3}\}\\
t^0_{(2,1)} &= \pm\{2,4\}\cap\{1,4\} = \{4\}\\
t^0_{(2,2)} &= \pm\{2,4\}\cap\{2,5\} = \{2\}\\
t^0_{(2,3)} &= \pm\{2,4\}\cap\{3\} = \varnothing\\
t^1_{(1,1)} &= \pm\{6\}\cap\{\bar{6},7\} = \{\bar{6}\}\\
t^1_{(2,1)} &= \pm\{7\}\cap\{\bar{6},7\} = \{7\}
\end{aligned}
$$

implying that we have

$$
t = \begin{array}{|c|c|c|}\hline 1 & 5 & \bar{3} \\ \hline 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}, \begin{array}{|c|}\hline \bar{6} \\ \hline 7 \\ \hline \end{array}.
$$

By changing the order of the tableaux t_1 and t_2 and considering the row stabilizers $\{t_2\}$ and $\{t_1^T\}$, then we find that in the intersection tableau s (say), we have

$$
|s^0_{(1,1)}| = |\{1,2,3\}\cap\{1,2\}| = |\{1,2\}| \neq 1.
$$

Then intersection bitableau, in this case, only exists for (t_1, t_2) and not for (t_2, t_1). Therefore, we can sum up this property as a proposition for this group by an extension of Proposition 3.4.15 and Proposition 4.5.15.

Proposition 5.3.16. *Let* $t_1 = (t_1^0, t_1^1)$ *and* $t_2 = (t_2^0, t_2^1)$ *be two bitableaux of shape* $\lambda = (\lambda^0, \lambda^1)$. *Then there exists an inter-bitableau* $t = (t^0, t^1)$ *such that* $t = (\{\pm t_1^0\}\cap\{(t_2^T)^1\}, \{t_1^1\}\cap\{\pm(t_2^T)^0\})$ *where* $\{\pm t_1^0\}$ *and* $\{\pm(t_2^T)^0\}$ *denote the sets* $\{t_1^0\}$ *and* $\{(t_2^T)^0\}$ *have their entries with both signs* $\pm$, *if and only if for any* (i,j) *we have that* $|t^k_{(i,j)}| = 1$ *for* $k = 0, 1$ *where* (i,j) *is a position of a box in the Young tableau* t^k *of shape* λ^k.

5.4 Specht Objects

It is time now to construct the Specht modules for the hyperoctahedral groups. The idea of a *flat-bitableau* can be drawn in a similar way to that of a flat-multitableau, except for that now it can contain negative entries. As usual, for any bitableau t, we denote the flat-bitableau corresponding to it by $t^\flat$ and the canonical flat bitableau is denoted by $t_\lambda^\flat$.

Then we have a permutation action on the $2n$-lettered words similar to that of the symmetric group action, whereas we still preserve the sign action whenever we are looking at it from the perspective of the n lettered words, which is simply the first half of each $2n$-lettered word.

The Young character of a bitableau t is decided by the sign of the permutation σ that converts the canonical multitableau t_λ to t.

Definition 5.4.1 (Young Character for Bitableau). *Let $t_\lambda^\flat$ be the canonical flat-bitableau corresponding to the bitableau t of shape $\lambda \models n$. For $\sigma \in H_n$ if we obtain $t^\flat = t_\lambda^\flat \cdot \sigma$ of same shape λ then we define the Young character denoted by $Y(t)$ as $Y(t) = \mathrm{sgn}(t^\flat)$ where $\mathrm{sgn}(t^\flat)$ is as given in Definition 5.1.6.*

As the Young character of a bitableau t is decided by a signed permutation σ such that $t_\lambda \cdot \sigma = t$, the Definition 3.5.6 holds true for the case of H_n.

Then we have the Specht matrix for shape λ. As seen before, if we consider the alternate notation of the flat-bitableau $t^\flat = (w_1^\lambda, w_2^\lambda)$ where the pair of words corresponds to the bitableau t, then we can simply write $Y(t) = Y(w_1', w_2')$.

Definition 5.4.2 (Specht Matrix for H_n Group). *Let $\lambda \models n$. Let $w_1 = w_\lambda$ and $w_2 = w_{\lambda^T}$ be the complementary canonical words of length n and of shapes λ and λ^T respectively. Let A and B be the sets of all rearrangements of w_1 and w_2 respectively. Then we define the Specht matrix $\varphi(\lambda) : A \times B \to \{0, \pm 1\}$ defined by $(w_1', w_2') \mapsto Y(w_1', w_2')$ where $Y(w_1', w_2') = Y(t)$ only if (w_1', w_2') corresponds to the bitableau t.*

The elements in the Specht matrix are decided by the sign of the simultaneous bihistograms corresponding to pairs (w_1', w_2'). Clearly, there will be $2^n \cdot n!$ nonzero entries in the Specht matrix $\varphi(\lambda)$.

For a concrete understanding , let us have a look at an example on how we build the Specht matrix for monomial groups.

Example 5.4.3. Let $n = 3$ and $\lambda = ((1,1),(1))$. Then the conjugate multipartition is $\lambda^{T} = ((1),(2))$. Therefore, we have the corresponding canonical words $u_{\lambda} = 1'2'1\bar{1}2'1'$ and $v_{\lambda^T} = 1'11\bar{1}\bar{1}1'$ respectively. We choose the canonical pair of these words as our row and column words pair. Then the row word is $u_{\lambda} = 1'2'1\bar{1}2'1'$ as it was but we take a rearrangement of letters of v_{λ^T} and so our column word is $v'_{\lambda^T} = 111'1'\bar{1}\bar{1}$. If we consider the words as a pair then we have

$$\begin{pmatrix} u_{\lambda} \\ v'_{\lambda^T} \end{pmatrix} = \begin{pmatrix} 1' & 2' & 1 & \bar{1} & 2' & 1' \\ 1 & 1 & 1' & 1' & \bar{1} & \bar{1} \end{pmatrix}$$

and clearly it is now complementary. As we already have obtained the free orbit for this pair, it implies that this pair satisfies the criteria for a simultaneous bihistogram. As usual, this pair is corresponding to the canonical bitableau of shape λ and therefore we can obtain the Specht matrix.

Let $w_1 = u_{\lambda}$ and $w_2 = v'_{\lambda^T}$. Then, the Specht matrix $\varphi(\lambda)$ is shown in Table 5.4.3. It can be seen that each row in the matrix is labelled by some rearrangement of the row word $1'2'1\bar{1}2'1'$ and each column is labelled by some rearrangement of the column word $111'1'\bar{1}\bar{1}$.

	$111'1'\bar{1}\bar{1}$	$11'1\bar{1}1'\bar{1}$	$1'11\bar{1}\bar{1}1'$	$1\bar{1}1'1'1\bar{1}$	$11'\bar{1}11'\bar{1}$	$1'1\bar{1}1\bar{1}1'$	$1'\bar{1}1\bar{1}11'$	$\bar{1}11'1'\bar{1}1$	$\bar{1}1'1\bar{1}1'1$	$\bar{1}\bar{1}1'1'11$	$\bar{1}1'\bar{1}11'1$	$1'\bar{1}\bar{1}111'$
$1'2'1\bar{1}2'1'$	1	·	·	−1	·	·	·	−1	·	1	·	·
$12'1'1'2'\bar{1}$	·	·	−1	·	·	1	1	·	·	·	·	−1
$1'12'2'\bar{1}1'$	·	−1	·	·	1	·	·	·	1	·	−1	·
$1'2'\bar{1}12'1'$	−1	·	·	1	·	·	·	1	·	−1	·	·
$\bar{1}2'1'1'2'1$	·	·	1	·	·	−1	−1	·	·	·	·	1
$11'2'2'1'\bar{1}$	·	·	1	·	·	−1	−1	·	·	·	·	1
$2'11'1'\bar{1}2'$	·	1	·	·	−1	·	·	·	−1	·	1	·
$1'\bar{1}2'2'11'$	·	1	·	·	−1	·	·	·	−1	·	1	·
$\bar{1}1'2'2'1'1$	·	·	−1	·	·	1	1	·	·	·	·	−1
$2'1'1\bar{1}1'2'$	−1	·	·	1	·	·	·	1	·	−1	·	·
$2'\bar{1}1'1'12'$	·	−1	·	·	1	·	·	·	1	·	−1	·
$2'1'\bar{1}11'2'$	1	·	·	−1	·	·	·	−1	·	1	·	·

Table 5.1: Specht matrix $\varphi(((1,1),(1)))$

As seen in the other cases, we claim that the column space of the Specht matrix corresponding to the bipartition λ is an irreducible representation of H_n and for any two distinct bipartitions they are nonisomorphic to each other.

Let us take A to be the set of all rearrangements of w_{λ}. Then we have a signed permutation module M^{λ} described by A.

Definition 5.4.4 (Signed Permutation Module). *Let $\lambda \models n$ be a bipartition and w_{λ} be the* *canonical word of shape λ. We define the signed permutation module*

$$M^\lambda = \langle q_w : w \in A \rangle_{\mathbb{C}}$$

as a vector space over $\mathbb{C}$, where A is the set of all rearrangements of w_λ.

Let $v \in M^\lambda$. Then by Definition 5.4.4 we have $v = \sum_{w \in A} \alpha_w q_w$ for some $\alpha_w \in \mathbb{C}$. Then for any $\sigma \in H_n$ we have that

$$v \cdot \sigma = (\sum_{w \in A} \alpha_w q_w) \cdot \sigma = \sum_{w \in A} (\alpha_w q_w) \cdot \sigma = \sum_{w \cdot \sigma \in A} \alpha_{w \cdot \sigma} q_{w \cdot \sigma} = v_\sigma.$$

Therefore we clearly have that M^λ for $\lambda \models n$ is a H_n-module.

Proposition 5.4.5. *Let $\lambda \models n$ and M^λ be the corresponding signed permutation module. Then M^λ is a H_n-module.*

Consider t, a bitableau of shape $\lambda \models n$ and $\mathbb{C}[H_n]$ be a group ring. Let the row and column stabilizers of t be $R(t)$ and $C(t)$ respectively, both as subgroups of H_n. Then we define the formal sums

$$p_t = \sum_{p \in R(t)} p, \quad c_t = \sum_{c \in C(t)} \bar{c}$$

where $\bar{c} = \mathrm{sgn}(c) \cdot c$. Then $p_t c_t$ is an element of $\mathbb{C}[H_n]$. Let $\{t\}$ be the row bitabloid corresponding to t expressed as $\{t\} = t \cdot p_t$.

Like in the symmetric group case, we then have p_t to be the sum of all bitableaux in the same row and c_t to be signed sum of all bitableaux in the same column corresponding to t in $\varphi(\lambda)$ where t is a λ-bitableau. Similarly, $\{t\}$, the row bitabloid of t corresponds to all the bitableaux in the same row of t in $\varphi(\lambda)$ and so can be indicated by the word $w_1' \in A$ corresponding to it.

Definition 5.4.6. *We define a polybitabloid v_t for a bitableau t given by $v_t = \{t\} \cdot c_t$.*

Like seen for the other cases, we have $v_t = t \cdot p_t c_t$. Then for $t = t_\lambda$ we find $v_\lambda = p_\lambda c_\lambda$ as t_λ corresponds to the identity in H_n. We obtain the elements in the same column of the bitableau t ordered differently by a signed permutation action of $C(t)$ on $\{t\}$ and so for all $\pi \in C(t)$ we get row bitabloids $\{t \cdot \pi\}$.

Definition 5.4.7. *Let w_1 and w_2 be complementary words of shape $\lambda \models n$ and corresponding to a Specht matrix $\varphi(\lambda)$. We define the elements*

$$v_{w_2} = \sum_{w_1'} Y(w_1', w_2) q_{w_1'}$$

where $Y(w_1', w_2)$ is the Young character corresponding to the pair (w_1', w_2) and $q_{w_1'}$ is the vector corresponding to w_1' in M^λ.

By the action of $C(t)$ on bitabloid $\{t\}$ with its corresponding element as its coefficient, we obtain v_t. Recording all the coefficients of these bitabloids in a sequence, we obtain the column corresponding to t in $\varphi(\lambda)$ labelled by w_2.

Irrespective of an ordinary or signed permutation σ, and for a bitableau t we always have the property that $\{t\} \cdot \sigma = \{t \cdot \sigma\}$. Therefore Proposition 3.5.14 also holds in this case as follows.

Proposition 5.4.8. *For a bipartition λ of n we have that $v_{w_2} = \mathrm{sgn}(t^\flat) \cdot v_t$.*

Proof. Proof follows from the proof of Proposition 3.5.14 by considering t to be a bitableau of shape $\lambda \models n$. □

Now we define the Specht module for the hyperoctahedral groups in a similar fashion to that of the monomial groups.

Definition 5.4.9 (Specht Module for H_n Group). *For a bipartition $\lambda \models n$, the Specht module S^λ is the submodule of M^λ spanned by all v_t.*

Proposition 5.4.10. *The Specht module S^λ is the column space of the Specht matrix $\varphi(\lambda)$, spanned by v_{w_2} where w_2 is the column word in $\varphi(\lambda)$.*

Now we have that Lemma 3.5.15 also applies in this case for a bipartition λ.

Lemma 5.4.11. *Let t be a λ-bitableau of shape $\lambda \models n$. Then the following hold.*

1. *For $\sigma \in R(t)$ and $\pi \in C(t)$, we have that $p_t \cdot \sigma = \sigma \cdot p_t = p_t$ and $c_t \cdot \pi = \pi \cdot c_t = \mathrm{sgn}(\pi) c_t$.*

2. *We have that $p_t \cdot p_t = |R(t)| p_t$ and $c_t \cdot c_t = |C(t)| c_t$.*

Proof. The proof follows like Lemma 3.5.15. □

We need to now show that the Specht module S^λ is closed under the action of H_n group.

Lemma 5.4.12. *Let* t *be any bitableau of shape* λ *and* $\sigma \in H_n$ *be any arbitrary group element. Then we have that* $v_t \cdot \sigma = v_{t \cdot \sigma} \in S^\lambda$.

Proof. Take a signed permutation $\sigma \in H_n$. Then the proof follows using the same steps of the proof of Lemma 3.5.16. □

Then we have from Lemma 5.4.12 that the Specht module S^λ is closed under H_n-action and so we have it as follows.

Proposition 5.4.13. S^λ *is a* H_n *module for* $\lambda \models n$.

Now let us understand if we can pair up row word of one bitableau with the column word of another bitableau and call the pair to be complementary, given that the two words are of same length. Let us consider $\alpha, \beta, \alpha', \beta' \in \mathbb{N}$ such that $\alpha + \beta = \alpha' + \beta' = n$ for some $n \in \mathbb{N}$. Let $\lambda = (\lambda^0, \lambda^1)$ and $\lambda' = (\lambda'^0, \lambda'^1)$ be two bipartitions of n such that $\lambda^0 \vdash \alpha, \lambda^1 \vdash \beta$ and $\lambda'^0 \vdash \alpha', \lambda'^1 \vdash \beta'$. Let t, t' be two bitableaux of shapes λ, λ' corresponding to the word pairs (u, v) and (u', v') respectively. Therefore, all four words u, v, u', v' are of same length n.

We now take u' and v, and pair them up. It is to be noticed that the shape of u' is λ' and the shape of v is $\lambda^T = (\lambda^{1^T}, \lambda^{0^T})$. If in the pair (u', v), we find that u' has more sign-invariant letters than in v then we have that $\alpha' + \beta > n$. It implies that in the pair of words, we have at least one column with both letters being sign-invariant. Therefore $\sigma = (i\ -i)$ is present in both $C(t)$ and $R(t')$.

Now, $\alpha' + \beta' = n \implies \alpha' = n - \beta'$. Then $\alpha' + \beta > n \implies n - \beta' + \beta > n \implies \beta > \beta'$ and so $\alpha' > \alpha$.

If $\sigma \in C(t)$ such that $\text{sgn}(\sigma) = -1$, e.g. $\sigma = (i\, j)(-i\ -j)$ or $\sigma = (i\ -i)$, then we have that

$$(-\sigma) \cdot c_t = (-\sigma) \cdot (\sum_{\pi \in C(t)} \text{sgn}(\pi) \cdot \pi) = \sum_{\sigma\pi \in C(t)} \text{sgn}(\sigma\pi) \cdot \sigma\pi = \sum_{c \in C(t)} \text{sgn}(c) \cdot c = c_t.$$

So we find that $\sigma \cdot c_t = -c_t$.

An element $p \in R(t)$ also belongs to the stabilizer subgroup $\text{Stab}(u)$ where u is the row word corresponding to t. Then clearly p acts on any pair of words with row word u in such a way that u remains unchanged while the column words get rearranged. Therefore, all pairs of words that have their row words fixed to u, can move from one pair to another by the action of $\mathrm{Stab}(u)$. Likewise, $\mathrm{Stab}(v)$ acts the same way on the pair of words with their column words being fixed to v.

If $\lambda = \lambda'$, then the two bitableaux t, t' are of same shape. Then for some $p \in R(t')$ and $q \in C(t)$ we have that $t' \cdot p = t \cdot q$.

Lemma 5.4.14. *Let* $\lambda, \lambda' \models n$ *and* $\lambda \geqslant \lambda'$. *Let* t, t' *be the bitableaux corresponding to* λ, λ' *respectively. If the row word of* t' *and column word of* t *make a complementary pair of words, or if there exists a sign change permutation* $(i\ \ -i)$ *in both* $R(t')$ *and* $C(t)$ *then we have* $\{t'\}\cdot c_t = 0$ *and otherwise* $\{t'\} \cdot c_t = \pm v_t$.

Proof. If such a complementary pair of words exists then let $\sigma \in R(t')$ be the transposition that permutes them. Now since $\sigma \in C(t)$, we have that $\sigma \cdot c_t = -c_t$ but $\{t'\} \cdot \sigma = \{t'\}$. Then by considering $\{t'\} = u'$ where u' is the row word corresponding to t', we obtain that

$$u' \cdot c_t = u' \cdot \sigma \cdot c_t = u' \cdot (-c_t) = -u' \cdot c_t.$$

So we have $u' \cdot c_t = 0 \implies \{t'\} \cdot c_t = 0$.

If no such pair exists, then for $\sigma \in R(t')$ and $\pi \in C(t)$ we have $t' \cdot \sigma = t \cdot \pi$. Then

$$u' \cdot c_t = \{t'\} \cdot c_t = \{t' \cdot \sigma\} \cdot c_t = \{t \cdot \pi\} \cdot c_t = \{t\} \cdot \pi \cdot c_t = \mathrm{sgn}(\pi)\{t\} \cdot c_t = \mathrm{sgn}(\pi) \cdot v_t.$$

□

It is to be noted that if t is a SYBT of shape λ and $p \in R(t)$, $c \in C(t)$, then we have that $t \cdot p \geqslant t$ and $t \cdot c \leqslant t$ as can be decided by an extension of the linear ordering on tableaux given in Section 7.1 in [6].

Lemma 5.4.15. *For two standard bitableaux* t, t' *with the property that* $t' > t$, *we find that there is a pair of integers in the same row of* t' *and the same column of* t *or there exists a sign change permutation* $(i\ \ -i)$ *in both* $R(t')$ *and* $C(t)$.

Proof. The shape of t cannot dominate the shape of t' since $t' > t$. If no such pair exists then for some $\sigma \in R(t')$ and $\pi \in C(t)$ we have that $t' \cdot \sigma = t \cdot \pi$. Since t, t' are standard, we have that $t' \cdot \sigma \geqslant t'$ and $t \cdot \pi \leqslant t$. It is contradictory to the assumption $t' > t$. Hence proven. □

Then as a consequence of Lemma 5.4.14 we have the following and the proof of it can be drawn in a similar fashion from the proof of Corollary 3.5.19.

Corollary 5.4.16. *For two standard Young bitableaux* t *and* t' *with the property that* $t' > t$ *we find that* $\{t'\} \cdot c_t = 0$*.*

Therefore, from Lemma 5.4.11, Lemma 5.4.14 and Corollary 5.4.16 we have the following.

Theorem 5.4.17. *For a Young bitableaux* t *of shape* $\lambda \models n$ *we have that*

$$M^{\lambda} \cdot c_t = S^{\lambda} \cdot c_t = \mathbb{C} \cdot v_t \neq 0; \tag{5.1}$$

$$M^{\lambda'} \cdot c_t = S^{\lambda'} \cdot c_t = 0 \text{ if } \lambda' > \lambda. \tag{5.2}$$

Proof. It follows from the proof of Theorem 3.2 by considering t to be a bitableau of shape $\lambda \vdash n$. □

Now the equations in Theorem 5.4.17 implies that each Specht module S^{λ} is irreducible for λ ranging over all bipartitions of n. Over a field with characteristic zero, irreducibility is same as indecomposability. So we can bring a similar argument in here to that of the symmetric groups.

If $S^{\lambda} = V \oplus W$, then $\mathbb{C} \cdot v_t = S^{\lambda} \cdot c_t = V \cdot c_t \oplus W \cdot c_t$ implies that v_t must be contained in either V or W. If say v_t is contained in $V \cdot c_t$ without loss of generality, then it must be true that V contains v_t. As $\mathbb{C} \cdot v_t$ is a one-dimensional space, then for $c_t \in \mathbb{C}[H_n]$ and V being a H_n-module, we have that $V \cdot c_t \in V$. Therefore, if v_t is contained in V then $S^{\lambda} = \mathbb{C}[H_n] \cdot v_t = V$.

For $\lambda' > \lambda$, equation (5.2) gives us that c_t acts as 0 on $S^{\lambda'}$, which implies that for any two distinct bipartitions λ, λ' we have that their corresponding Specht modules $S^{\lambda}, S^{\lambda'}$ are non-isomorphic. It is also true that there are as many irreducible representations of H_n as the number of bipartitions of n and hence we have the following.

Theorem 5.4.18. *The set of Specht modules* $\{S^{\lambda}$ *for all* $\lambda \models n\}$ *gives the complete set (up to isomorphism) of pairwise non-isomorphic irreducible representations of* H_n*.*

Proof. Following this construction we obtain that there are as many irreducible representations of H_n as the number of bipartitions of n. Each S^{λ} corresponding to $\lambda \models n$ is an irreducible representation, and any two of them are pairwise non-isomorphic. Therefore, $\{S^{\lambda}$ for all $\lambda \models n\}$ produces the complete set of irreducible representations of H_n up to isomorphism. □

We now prove that the columns corresponding to the standard bitableaux create a basis for the Specht module S^{λ}.

Lemma 5.4.19. *The standard Young bitableaux entries in the Specht matrix belong to distinct rows and columns.*

Proof. This proof follows the same pattern as that of the ones seen before for the other groups.

In the Specht matrix all the entries sitting in any row belong to the bitableaux of same row-equivalence class or row bitabloid. If t is a standard λ-bitableau then in the row bitabloid $\{t\}$ there can be only one SYBT, i.e., t itself. This implies that every SYBT belong to different row-equivalence classes and therefore they all sit in different rows.

Likewise, every SYBT belong to different column-equivalence classes and therefore they all sit in different columns.

Hence the standard Young bitableaux entries in the Specht matrix belong to distinct rows and columns. □

We will also show that SYBT entries are the first nonzero entries in the respective rows and columns to prove the linear independence of these rows and columns.

Theorem 5.4.20. *In the Specht matrix corresponding to the bipartition $\lambda \models n$, we have the standard bitableaux elements as the first nonzero elements in their respective rows and columns.*

Proof. In the Specht matrix the rows and columns are labelled by the rearrangements of the row word u and the rearrangements of the column word v respectively, where $\lambda(u) = \lambda$ and $\lambda(v) = \mu$ such that $\mu = (\mu^0, \mu^1) = (\lambda^{1^T}, \lambda^{0^T}) = \lambda^T$.

In each column we have a column-equivalence class, i.e. a column bitabloid. Then each column consisting a SYBT entry has exactly one such corresponding simultaneous bihistogram that is column-standard.

By the construction of simultaneous bihistogram for a pair of words, it is clear that the distinct letters of the word u appear in positions corresponding to the entries in the first column of each of the component tableau t^k and they have repetitions in respective positions corresponding to the further columns in each t^k for $k = 0, 1$. As each t^k is column-standard, if the smaller entry in a column has row word letter e and larger entry in the same column has row word letter f then it must be that $e < f$ where e and f both are letters of same type.

Now if we label the boxes of each t^k by (i, j) to denote its position then we have that $t^k_{(i,j)} < t^k_{(i+1,j)}$. For any $id \neq \pi \in C = H_{\mu^1} \times S_{\mu_0}$, column stabilizer, we have $t^k_{(i,j)} \cdot \pi >$

$t^k_{(i+1,j)} \cdot \pi$ implying that at least one of the columns in $t^k_{(i+1,j)} \cdot \pi$ is non-standard. Therefore, for $u \cdot \pi$ we have $u_{(i,j)} > (u \cdot \pi)_{(i+1,j)}$ implying that $u < u \cdot \pi$, i.e., $u \cdot \pi$ comes later than u in lexicographical order. In the Specht matrix $\varphi(\lambda)$, if we order the row words lexicographically then the column standard multitableaux entries appearing on top of the respective columns making it the nonzero elements. Likewise we can establish a similar argument that the row standard multitableaux entries are the first nonzero elements in their respective rows.

Hence, for t being both row and column standard, i.e., t being a SYBT we have that the corresponding element in the respective rows and columns are the first nonzero entries.

□

Therefore, by combining Lemma 5.4.19 and Theorem 5.4.20 together we prove that the rows (and columns) containing SYBT entries in the Specht matrix are linearly independent.

Now, by extracting from Table 5.1 only the rows and columns containing the SYBT entries and replace them by their corresponding simultaneous bihistograms then we obtain Table 5.2.

	$111'1'\bar{1}\bar{1}$	$11'1\bar{1}1'\bar{1}$	$1'\bar{1}1\bar{1}11'$
$1'2'1\bar{1}2'1'$	$\boxed{1}\boxed{2},\boxed{3}$	·	·
$1'12'2'\bar{1}1'$	·	$\boxed{1}\boxed{3},\boxed{2}$	·
$11'2'2'1'\bar{1}$	·	·	$\boxed{2}\boxed{3},\boxed{1}$

Table 5.2: SYBT-extracted submatrix of Specht matrix $\varphi(((1,1),(1)))$

By converting the SYBT in the submatrix back to their Young character values, we obtain the submatrix shown in Table 5.3.

	$111'1'\bar{1}\bar{1}$	$11'1\bar{1}1'\bar{1}$	$1'\bar{1}1\bar{1}11'$
$1'2'1\bar{1}2'1'$	1	·	·
$1'12'2'\bar{1}1'$	·	−1	·
$11'2'2'1'\bar{1}$	·	·	1

Table 5.3: SYBT-extracted submatrix of Specht matrix $\varphi(((1,1),(1)))$

It can be seen that each row and column has exactly one nonzero entry the submatrix takes a nice form with its nonzero entries on its main diagonal. Therefore, evidently, the rows (respectively columns) containing the SYBT elements in the Specht matrix are linearly independent.

In this case as well, we have noticed that for $n \geqslant 5$ we find the inter-bitableau for some of the bipartitions of n. The idea here is quite similar to that of the monomial groups. Let us demonstrate it using Example 5.4.21.

Example 5.4.21. Let $n = 5$. Let us consider $\lambda = ((3,2), \varnothing)$ and the corresponding conjugate bipartition $\lambda^T = (\varnothing, (2,2,1))$. The respective row and column words are $1'1'1'2'2'2'2'1'1'1'$ and $11223\bar{3}\bar{2}\bar{2}\bar{1}\bar{1}$. In the truncated version of the Specht matrix $\varphi(\lambda)$, we find a lower-triangular matrix with all the diagonal entries being the SYBT entries.

Now considering the SYBT entries with the corresponding row and column simultaneous bihistograms being

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \varnothing \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \varnothing$$

then we have an entry at the intersection of these two, giving us a inter-bitableau

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 3 \\ \hline 4 & 2 \\ \cline{1-2}\end{array}, \varnothing.$$

Again, the reason behind the existence of this inter-bitableau is very similar to that of the other groups and as explained earlier.

Now we need to prove that the set of SYBT columns is a spanning set for the module. We are going achieve this in a similar way how it was done for the symmetric groups.

The hooklength formula for the bitableaux is going to be a special case to that of the monomial groups and therefore it needs no special treatment other than fixing the parameter $r = 2$.

Definition 5.4.22. *Let $a + b = n$ and $\lambda = (\lambda^0, \lambda^1) \models n$ for $\lambda^0 \vdash a$ and $\lambda^1 \vdash b$. Then by using the generalised hooklength formula (Definition 4.6.19) we find that*

$$f^{\lambda} = \binom{n}{a, b} f^{\lambda^0} \cdot f^{\lambda^1}.$$

Theorem 5.4.23. *The number of standard Young bitableau for the hyperoctahedral groups corresponding to a bipartition λ of n is decided by the hooklength formula f^{λ}.*

Proof. The proof of this theorem follows from that of the Theorem 4.6.21. □

Theorem 5.4.24. *The RSK correspondence for hyperoctahedral groups is*

$$\sum_{\lambda \models n} (f^{\lambda})^2 = 2^n n!.$$

Proof. The proof of this theorem follows from the proof of Theorem 4.6.22 by simply fixing $r = 2$. □

It is now time that we establish a similar global argument to that of the other cases in order to prove for the spanning set. As shown already, the SYBT columns in the Specht matrix are linearly independent and there are precisely f^{λ} of them. It is not possible to have anymore linearly independent columns as otherwise it would fail the general RSK correspondence. Therefore, there are exactly as many linearly independent columns as the number of t_{λ}, where t_{λ} is a SYBT of shape λ.

In a spanning set all the vectors have to be linearly independent. Hence, the span of the set of SYBT-columns in the Specht matrix for each λ is the module S^{λ}.

Again, from [12] and [8] we know that each S^{λ} is irreducible and for any two bipartitions $\lambda, \mu \models n$ and $\lambda \neq \mu$ we have that S^{λ} and S^{μ} are non-isomorphic. It implies that there are as many Specht modules as the number of bipartitions λ of n. This forms the complete set of irreducible representations.

General RSK-correspondence implies that the sum of the squares of the degrees of irreducible representations is same as the order of the group H_n.

Let for our construction $d^{\lambda} = \dim(S^{\lambda})$. Then we from the general RSK-correspondence, for each λ we have that $d^{\lambda} \geqslant f^{\lambda}$. Now

$$\sum_{\lambda \models n} (f^{\lambda})^2 = 2^n n!$$

implies that $d^{\lambda} = f^{\lambda}$ must be true. Hence, dimension of each S^{λ} is equal to the number of the SYBT of shape λ.

Then, the set of SYBT-columns of the Specht matrix are linearly independent and they span the module S^{λ}. Therefore, we have proved that SYBT columns in $\varphi(\lambda)$ is a basis of S^{λ}.

Similar to the other cases, the coefficients of the row bitabloids $\{t \cdot \sigma\}$ for all permutations $\sigma \in C(t)$ in a sequence is same as the columns corresponding to the bitableaux t in the Specht matrix. Then the polybitabloids v_t are same as these columns.

For each bipartition of n we have a conjugacy class of H_n and for each pair of conjugacy classes we have a unique Specht matrix. Hence, for any two bipartitions $\lambda, \mu \models n$, we have that S^λ is non-isomorphic to S^μ except for when $\lambda = \mu$.

Throughout the chapter we have established that the SYBT columns of the Specht matrix are linearly independent and span the module S^λ. Therefore, we have the following.

Theorem 5.4.25. *Let $\lambda \models n$. Then the set of SYBT-columns in $\varphi(\lambda)$ form a basis of S^λ.*

5.5 Representing Matrices

As usual, now let us go through the process of making representing matrices.

Let us pick an element σ, a signed permutation from the hyperoctahedral group on n points H_n. Let u be a row word of length $2n$ made of sign-invariant and sign-changing letters. Let U be the set of all rearrangements of u and $|U| = k$.

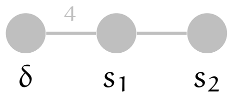
Now we define a map $\psi : H_n \longrightarrow S_k$, which is a homomorphism such that it gives a permutation action of H_n on U. Now let $\pi \in S_k$ be a permutation such that $\pi = \psi(\sigma) = \sigma^\psi$, i.e., π is a permutation obtained by the image of σ under ψ.

Let $\varphi(\lambda)$ be the Specht matrix corresponding to the row and column words u and v of shape λ respectively. Let C be the set of all columns of $\varphi(\lambda)$. Let $C' = \{c^\pi | c \in C\}$ be another rearrangement of C, of which elements are obtained by the permutation action of π on each of the elements of C individually.

We now consider the SYBT column vectors from C' and express them as a linear combination of the SYBT columns from C. Like seen in other groups, if we now extract the coefficients of these linear combinations and put them in the rows of a matrix, we obtain the representing matrix corresponding to the element σ. Let us denote the representing matrix corresponding to the permutation σ by M_σ.

If there are f^λ SYBTs corresponding to the module S^λ, then the representing matrices for each of these irreducible representations are of dimensions $f^\lambda \times f^\lambda$. As usual, this method of finding the matrix M_σ fits similar linear algebraic description of representing matrices found in the chapters concerning the other groups. The example that follows shows explicitly the compatibility of this construction with the hyperoctahedral group elements.

Example 5.5.1. Let us consider that $n = 3$. The Coxeter diagram of the group H_3 is



H_3

Let $S = \{\delta, s_1, s_2\}$ be the generator set of H_3 where

$$\delta = (1\ \ -1),\ \ s_1 = (1\,2)(-2\ \ -1),\ \ s_2 = (2\,3)(-3\ \ -2).$$

We have a total of 10 bipartitions of 3, namely

$$\{((1,1,1),\varnothing),\ \ ((1,1),(1)),\ \ ((1),(1,1)),\ \ (\varnothing,(1,1,1)),\ \ ((2,1),(\varnothing),$$
$$((1),(2)),\ \ ((2),(1)),\ \ (\varnothing,(2,1)),\ \ ((3),\varnothing),\ \ (\varnothing,(3))\}.$$

Now we pick each of the generators, $\sigma \in S$, and for each of them we find the corresponding representing matrix M_σ.

1. For $\lambda = ((1,1,1),\varnothing)$, we find:

   (a) if $\sigma = \delta$ then
   $$M_\delta = \begin{bmatrix} 1 \end{bmatrix}$$

   (b) if $\sigma = s_i$ for $i = 1,2$ then
   $$M_{s_i} = \begin{bmatrix} -1 \end{bmatrix}$$

2. For $\lambda = ((1,1),(1))$, we find:

   (a) if $\sigma = \delta$ then
   $$M_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   (b) if $\sigma = s_1$ then
   $$M_{s_1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

   (c) if $\sigma = s_2$ then
   $$M_{s_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

3. For $\lambda = ((1),(1,1))$, we find:

   (a) if $\sigma = \delta$ then
$$M_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   (b) if $\sigma = s_1$ then
$$M_{s_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   (c) if $\sigma = s_2$ then
$$M_{s_2} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

4. For $\lambda = (\varnothing,(1,1,1))$, we find:

   (a) if $\sigma = \delta$ then
$$M_\delta = \begin{bmatrix} -1 \end{bmatrix}$$

   (b) if $\sigma = s_i$ for $i = 1,2$ then
$$M_{s_1} = \begin{bmatrix} -1 \end{bmatrix}$$

5. For $\lambda = ((2,1),\varnothing)$, we find:

   (a) if $\sigma = \delta$ then
$$M_\delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   (b) if $\sigma = s_1$ then
$$M_{s_1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

   (c) if $\sigma = s_2$ then
$$M_{s_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

6. For $\lambda = ((1),(2))$, we find:

(a) if $\sigma = \delta$ then

$$M_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) if $\sigma = s_1$ then

$$M_{s_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) if $\sigma = s_2$ then

$$M_{s_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

7. For $\lambda = ((2),(1))$, we find:

(a) if $\sigma = \delta$ then

$$M_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) if $\sigma = s_1$ then

$$M_{s_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) if $\sigma = s_2$ then

$$M_{s_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. For $\lambda = (\varnothing,(2,1))$, we find:

(a) if $\sigma = \delta$ then

$$M_\delta = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) if $\sigma = s_1$ then

$$M_{s_1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

(c) if $\sigma = s_2$ then

$$M_{s_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

9. For $\lambda = ((3), \varnothing)$, we find:

(a) if $\sigma = \delta$ then

$$M_\delta = \begin{bmatrix} 1 \end{bmatrix}$$

(b) if $\sigma = s_i$ for $i = 1, 2$ then

$$M_{s_i} = \begin{bmatrix} 1 \end{bmatrix}$$

10. For $\lambda = (\varnothing, (3))$, we find:

(a) if $\sigma = \delta$ then

$$M_\delta = \begin{bmatrix} -1 \end{bmatrix}$$

(b) if $\sigma = s_i$ for $i = 1, 2$ then

$$M_{s_i} = \begin{bmatrix} 1 \end{bmatrix}$$

5.6 Characters and Character Tables

As usual, the way to validate the accuracy of this new construction, we again follow the same way of matching the traces of the representing matrices with the existing character table for this group.

For every bipartition of a number n we have a corresponding cycle structure, which is associated to a conjugacy class of the hyperoctahedral groups H_n. Again, we have our way of choosing representatives for each of these conjugacy classes and each of them is a signed permutation as a group element. The traces of the representing matrices corresponding to each of these class representatives, therefore, give us the character values. If we find the character values corresponding to all the conjugacy classes for all the bipartitions and put them in a table then we find the character table of the group H_n.

We have been using the group H_3 or B_3 as an example to demonstrate multiple ideas in the previous sections and it has a reasonable size of the set of bipartitions and therefore we will use the same to demonstrate the character table.

The set of generators for H_3 is $\{\delta, s_1, s_2\} = \{(1\ -1), (1\ 2), (2\ 3)\}$. Now, in order to find the class representatives of each conjugacy class corresponding to each bipartition, let us denote the sign changes on the first position by $\delta = \delta_1 = (1\ -1)$, in the second position by $\delta_2 = (2\ -2)$ and in the third position by $\delta_3 = (3\ -3)$. We now have that $\delta_2 = s_1\delta_1 s_1$ and $\delta_3 = s_2\delta_2 s_2$. Then we have the list of class representatives as

$$[\text{id},\ \delta_3,\ \delta_2\delta_3,\ \delta_1\delta_2\delta_3,\ s_1,\ \delta_2 s_2,\ s_1\delta_3,\ \delta_1 s_1\delta_3,\ s_1 s_2,\ \delta_1 s_1 s_2]$$

corresponding to the list of bipartitions

$$[((1,1,1),\varnothing),\ ((1,1),(1)),\ ((1),(1,1)),\ (\varnothing,(1,1,1)),\ ((2,1),\varnothing),$$
$$((1),(2)),\ ((2),(1)),\ (\varnothing,(2,1)),\ ((3),\varnothing),\ (\varnothing,(3))].$$

Therefore we obtain the character table for H_3 as shown in Table 5.4.

	id	δ_3	$\delta_2\delta_3$	$\delta_1\delta_2\delta_3$	s_1	$\delta_2 s_2$	$s_1\delta_3$	$\delta_1 s_1\delta_3$	$s_1 s_2$	$\delta_1 s_1 s_2$
$((1,1,1),\varnothing)$	1	1	1	1	−1	−1	−1	−1	1	1
$((1,1),(1))$	3	1	−1	−3	−1	−1	1	1	0	0
$((1),(1,1))$	3	−1	−1	3	−1	1	−1	1	0	0
$(\varnothing,(1,1,1))$	1	−1	1	−1	−1	1	1	−1	1	−1
$((2,1),\varnothing)$	2	2	2	2	0	0	0	0	−1	−1
$((1),(2))$	3	−1	−1	3	1	−1	1	−1	0	0
$((2),(1))$	3	1	−1	−3	1	1	−1	−1	0	0
$(\varnothing,(2,1))$	2	−2	2	−2	0	0	0	0	−1	1
$((3),\varnothing)$	1	1	1	1	1	1	1	1	1	1
$(\varnothing,(3))$	1	−1	1	−1	1	−1	−1	1	1	−1

Table 5.4: Character table of Coxeter group B_3

Chapter 6

Conclusion and Outlook

In this concluding chapter, I summarize the work presented throughout the thesis. While traditional methods in the representation theory of finite groups have described the irreducible representations of finite Coxeter and monomial groups, the challenge of finding a base change matrix that converts the large permutation matrices from these representations into block forms has remained unresolved. Through this novel approach, leveraging the concept of Specht matrices, I introduce a new construction of Specht modules.

I have identified these base change matrices for several large families of finite groups, specifically for symmetric groups (type-A_{n-1} Coxeter groups), monomial groups, and hyperoctahedral groups (type-B_nCoxeter groups). This method has proven to be robust, relying on straightforward concepts from linear algebra (such as solving systems of linear equations using Gaussian elimination) and combinatorics. The effectiveness of this approach is now established both theoretically and computationally, and it is easily comprehensible. Consequently, this new construction has the potential to complement existing literature in addressing challenging questions in the representation theory of finite groups. Looking ahead, I plan to explore several future directions stemming from this research, which are outlined below.

- Given that this method has consistently proven effective and compatible with major families of groups such as symmetric groups, monomial groups, and hyperoctahedral groups, the next question arises: Can this construction be generalized to other finite group representations, particularly for other finite Coxeter groups?

- For Coxeter groups, there are associated Hecke algebras. This raises another open question: first, to test the compatibility of this construction with the Hecke algebras of the group families discussed in this work, and subsequently with those of other finite Coxeter groups. I have already begun experimenting with the Hecke algebras for type-A_{n-1} groups, drawing on ideas from Chapter 9 in [8]. Initial findings suggest that Specht matrices can be constructed using similar concepts, but with the introduction of an indeterminate. Further research is ongoing.

- Although the irreducible representations of symmetric groups are fully classified over fields with characteristic zero, the question of whether all irreducible representations of S_n can be classified over a field with prime characteristic has remained a widely researched open problem for the past few decades.

  Given that the elements in the Specht matrices, as determined by the Young character, are all rational numbers, specifically integers $\{-1, 0, 1\}$, this construction can be extended to any field $\mathbb{F}_p$, where p is a prime, by taking $i \mod p$ for $i \in \{-1, 0, 1\}$. The next step is to verify whether the remainder of the method is applicable over $\mathbb{F}_p$. Since modular representation theory is generally more complex than representation theory over fields of characteristic zero, it requires more careful and sophisticated analysis, making it an excellent topic for future research.

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Appendix A

GAP Code

```
####Section-3.1. The group, generators and class reps

#Symmetric group generators
SymmGens:= function(n)
    local gens, i;
    gens:= [];
    for i in [1..n-1] do
        Add(gens, (i, i+1));
    od;
    return gens;
end;

#Symmetric Group
SymmGroup:= function(n)
    local G;
    G:= GroupWithGenerators(SymmGens(n));
    return G;
end;

#Class representatives
SymmRep:= function(lambda)
    local cycle, p, classes, cycle_list, cycle_structure;
    cycle_list:= [];
    cycle:= function(n)
        return ([1..n] mod n) + 1;
    end;
    for p in lambda do
        Append(cycle_list, cycle(p) + Length(cycle_list));
    od;
    cycle_structure:= PermList(cycle_list);
    return cycle_structure;
end;

####Section-3.3. Partitions and words

#Conjugate partition
ConjugatePartition:= function(lambda)
    return AssociatedPartition(lambda);
end;

#Word corresponding to a partition
WordPartition:= function(lambda)
    local i, l, W;
```

```
    W:= [];
    for l in [1..Length(lambda)] do
        for i in [1..lambda[l]] do
            Add(W, l);
        od;
    od;
    return W;
end;

#Partition corresponding to a word
PartitionWord:= function(word)
    local lambda, i;
    Sort(word);
    lambda:= [];
    for i in Set(word) do
        Add(lambda, Size(Positions(word, i)));
    od;
    return lambda;
end;

#####Section-3.4. Tableaux and pairs of words

#Standard Young tableaux for a partition lambda
SYTs:= function(lambda)
    local   isOneHook, removeOneHook, addOneHook, n, list, i, new, t;
    isOneHook:= function(lambda, i)
        return i = Length(lambda) or lambda[i] > lambda[i+1];
    end;
    removeOneHook:= function(lambda, i)
        local new;
        if i = Length(lambda) and lambda[i] = 1 then
            return lambda{[1..i-1]};
        fi;
        new:= ShallowCopy(lambda);
        new[i]:= new[i]-1;
        return new;
    end;
    addOneHook:= function(t, i, n)
        if i > Length(t) then
            Add(t, [n]);
        else
            Add(t[i], n);
        fi;
        return t;
    end;
    # trivial case first
    if lambda = [1] then
        return [[[1]]];
    fi;
    # initialize
    n:= Sum(lambda);
    list:= [];
    # loop
    for i in Reversed([1..Length(lambda)]) do
        if isOneHook(lambda, i) then
            new:= removeOneHook(lambda, i);
            for t in SYTs(new) do
                addOneHook(t, i, n);
```

```
                Add(list, t);
            od;
        fi;
    od;
    return list;
end;

#Canonnical tableau for a partition lambda
CanonicalTableau:= function(lambda)
    local t, o, l;
    t:= [];
    o:= 0;
    for l in lambda do
        Add(t, o + [1..l]);
        o:= o + l;
    od;
    return t;
end;

#Word corresponding to a tableau
WordsTableau:= function(tableau)
    local rows, cols, i, k, a;
    rows:= [];
    cols:= [];
    for i in [1..Length(tableau)] do
        for k in [1..Length(tableau[i])] do
            a:= tableau[i][k];
            rows[a]:= i;
            cols[a]:= k;
        od;
    od;
    return rec(rows:= rows, cols:= cols);
end;

#Tableau corresponding to a word
TableauWords:= function(row_word, col_word)
    local pair, pairTransposed, lambda, tableau, i;
    pair:= [row_word, col_word];
    pairTransposed:= TransposedMat(pair);
    lambda:= PartitionWord(row_word);
    tableau:= CanonicalTableau(lambda);
    if Size(Set(pairTransposed)) < Length(pairTransposed) then
        return false;
    else
        for i in pairTransposed do
            tableau[i[1]][i[2]]:= Position(pairTransposed, i);
        od;
    fi;
    return tableau;
end;

#Transpose of a tableau corresponding to the conjugate partition
ConjugateTableau:= function(t)
    local i, j, result;
    result:= List(t[1], x -> []);
    for i in [1..Length(t)] do
        for j in [1..Length(t[i])] do
            result[j][i]:= t[i][j];
```

```
        od;
    od;
    return result;
end;

#Intersection tableau at the intersection of two tableaux t and s
IntersectionTableau:= function(t,s)
    local cols, tt, a, uu, row, i, k;
    cols:= WordsTableau(t).cols;
    uu:= List(t[1], x -> []);
    for row in s do
        for i in [1..Length(row)] do
            a:= row[i];
            k:= cols[a];
            Add(uu[k], a);
        od;
    od;
    return TransposedMat(uu);
end;

####Section-3.5. Specht objects

#Young character of two words u and v
YoungCharSymm:= function(u, v)
    local c, d, pi;
    c:= TransposedMat([u,v]);
    if Size(Set(c)) < Length(c) then
        return 0;
    fi;
    d:= ShallowCopy(c);
    pi:= Sortex(d);
    return SignPerm(pi);
end;

#Specht matrix corresponding to the arrangements of the row words
and column word A and B respectively
SpechtMatSymm:= function(A, B)
    local u, v, matrix, row;
    matrix:= [];
    for u in A do
        row:= [];
        for v in B do
            Add(row, YoungCharSymm(u,v));
        od;
        Add(matrix, row);
    od;
    return matrix;
end;

#Specht object to record all the previous information needed for
further concepts
SpechtSymm:= function(lambda)
    local syt,  a,  A,  k,  b,  B,  sm;
    syt:= SYTs(lambda);
    a:= WordPartition(lambda);
    A:= Arrangements(a, Size(a));
    b:= WordPartition(ConjugatePartition(lambda));
    B:= Arrangements(b, Size(b));
```

```
    k:= List(syt, x -> [Position(A, WordsTableau(x).rows),
        Position(B, WordsTableau(x).cols)]);
    sm:= SpechtMatSymm(A, B);
    return rec(sm:= sm, A:= A, B:= B, syt:= syt, k:= k);
end;

####Section-3.6. Representing matrices

#Representing matrix corresponding to the permutation sigma of
symmetric group
SpechtRepPerm:= function(specht, perm)
    local pi, rows, cols, i, stdrows, stdcols, stdrowspermuted,
    stdcolspermuted, m;
    pi:= Permutation(perm, specht.A, Permuted);
    cols:= List(specht.k, i -> i[2]);
    stdcols:= TransposedMat(specht.sm){cols};
    stdcolspermuted:= List(stdcols, l -> Permuted(l, pi));
    m:= List(stdcolspermuted, l -> SolutionMat(stdcols, l));
    return m;
end;

####Section-3.7. Characters and character table

#Characters correspnding to a partition lambda for all the
class reps
SymmCharacter:= function(lambda, reps)
    local specht;
    specht:= SpechtSymm(lambda);
    return List(reps, sigma -> Trace(SpechtRepPerm(specht, sigma)));
end;

#Character table for all the partitions lambda of n
SymmCharTable:= function(n)
    local P, reps, CharTable;
    P:= Partitions(n);
    reps:= List(P, SymmRep);
    CharTable:= List(P, lambda -> SymmCharacter(lambda, reps));
    return CharTable;
end;

#################################################################

####Section-4.1. The group, generators and class reps

#Monomial group generators as matrices in GL_n(C) with r-th
roots of unity
MonoGens:= function(n, r)
    local gen, l, i;
    gen:= [];
    l:= ListWithIdenticalEntries(n, 1);
    l[1]:= E(r);
    Add(gen, DiagonalMat(l));
    for i in [1..(n-1)] do
        Add(gen, PermutationMat((i, i+1), n, 1));
    od;
    return gen;
end;
```

```
#Monomial Group
MonoGroup:= function(n, r)
    local gens;
    gens:= MonoGens(n, r);
    return GroupWithGenerators(gens);
end;

#Class representatives as a block matrix
MonoRep:= function(lambda)
    local n, M, o, i, a, mat, r;
    n:= Sum(lambda, Sum);
    r:= Length(lambda);
    M:= NullMat(n,n);
    o:= 0;
    for i in [1..r] do
        for a in lambda[i] do
            mat:= PermutationMat(PermList(([1..a] mod a) + 1), a);
            mat[a][1]:= E(r)^(i-1);
            M{o+[1..a]}{o+[1..a]}:= mat;
            o:= o + a;
        od;
    od;
    return M;
end;

####Section-4.4. Multipartitions and words

#Multipartitions of a number n having r partitions
MultiPartitions:= function(n, r)
    return PartitionTuples(n, r);
end;

#Conjugate multipartition
ConjugateMultiPartition:= function(lambda)
    return List(lambda, AssociatedPartition);
end;

#Word corresponding to a multipartition
WordMultiPartition:= function(multipartition)
    local WordPartition, W, u, l, i, j;
    W:= [];
    l:= List(multipartition, WordPartition);
    for i in [1..Length(multipartition)] do
        for j in l[i] do
            Add(W, [j, i-1]);
        od;
    od;
    return W;
end;

#Multipartition corresponding to a word
MultiPartitionWord:= function(word)
    local lambda, index, l, i;
    index:= Set(List(word, l -> l[2]));
    lambda:= List(index, i -> []);
    for i in Set(word) do
        Add(lambda[i[2]], Size(Positions(word, i)));
    od;
```

```
    return lambda;
end;

#####Section-4.5. Multitableaux and pair of words

#Standard Young multitableaux (SYMTs) for a multipartition lambda
SYMTs:= function(multipartition)
    local isOneHook, removeOneHook, addOneHook, n, list, i, k,
    lambda, new, t;
    #here lambda is an ordinary partition
    isOneHook:= function(lambda, i)
        return i = Length(lambda) or lambda[i] > lambda[i+1];
    end;
    removeOneHook:= function(multipartition, k, i)
        local lambda, new;
        lambda:= ShallowCopy(multipartition[k]);
        new:= ShallowCopy(multipartition);
        if i = Length(lambda) and lambda[i] = 1 then
            new[k]:= lambda{[1..i-1]};
        else
            lambda[i]:= lambda[i]-1;
            new[k]:= lambda;
        fi;
        return new;
    end;
    addOneHook:= function(t, k, i, n)
        if i > Length(t[k]) then
            Add(t[k], [n]);
        else
            Add(t[k][i], n);
        fi;
        return t;
    end;
    # trivial case first
    if ForAll(multipartition, x -> x = []) then
        return [List(multipartition, x -> [[]])];
    fi;
    # initialize
    n:= Sum(multipartition, Sum);
    list:= [];
    # loop
    for k in Reversed([1..Length(multipartition)]) do
        lambda:= multipartition[k];
        for i in Reversed([1..Length(lambda)]) do
            if isOneHook(lambda, i) then
                new:= removeOneHook(multipartition, k, i);
                for t in SYMTs(new) do
                    addOneHook(t, k, i, n);
                    Add(list, t);
                od;
            fi;
        od;
    od;
    return list;
end;

#Canonnical multitableau for a multipartition lambda
CanonicalMultiTableau:= function(lambda)
```

```
    local CanonicalTableau, tab, i;
    tab:= List(lambda, CanonicalTableau);
    for i in [2..Length(lambda)] do
        if lambda[i] <> [] then
            tab[i]:= tab[i] + Sum(List(lambda{[1..i-1]}, Sum));
        fi;
    od;
    return tab;
end;

#Word corresponding to a multitableau
WordsMultiTableau:= function(tab)
    local rows, cols, i, j, k, a;
    rows:= [];
    cols:= [];
    for i in tab do
        for j in [1..Length(i)] do
            for k in [1..Length(i[j])] do
                a:= i[j][k];
                rows[a]:= [j, Position(tab, i)-1];
                cols[a]:= [k, Position(tab, i)-1];
            od;
        od;
    od;
    return [rows, cols];
end;

####Section-4.6. Specht objects

#Young character of two words u and v
YoungCharMono:= function(u, v)
    local c, d, pi, i;
    c:= TransposedMat([u,v]);
    if Size(Set(c)) < Length(c) then
        return 0;
    fi;
    for i in c do
        if i[1][2] <> i[2][2] then
            return 0;
        fi;
    od;
    d:= ShallowCopy(c);
    pi:= Sortex(d);
    return SignPerm(pi);
end;

#Specht matrix corresponding to the arrangements of the row word
and column words A and B respectively
SpechtMatMono:= function(A1, A2)
    local u, v, matrix, row;
    matrix:= [];
    for u in A1 do
        row:= [];
            for v in A2 do
                Add(row, YoungCharMono(u,v));
            od;
        Add(matrix, row);
    od;
```

```
    return matrix;
end;

#Specht object to record all the previous information needed for
further concepts
SpechtMono:= function(lambda)
    local n, w1, w1reversed, w2reversed, w2, A, sigma, B, sm,
    symt, words, k;
    n:= Sum(lambda, Sum);
    w1:= WordMultiPartition(lambda);
    #sorting the words lexicographically in the set of all
    arrangements of w1
    w1reversed:= List(Arrangements(List(w1, Reversed), Length(w1)));
    A:= List(w1reversed, w -> List(w, Reversed));
    sigma:= ConjugateMultiPartition(lambda);
    w2:= WordMultiPartition(sigma);
    #sorting the words lexicographically in the set of all
    arrangements of w2
    w2reversed:= List(Arrangements(List(w2, Reversed), Length(w2)));
    B:= List(w2reversed, w -> List(w, Reversed));
    sm:= SpechtMatMono(A, B)
    symt:= SYMTs(lambda);
    words:= List(symt, WordsMultiTableau);
    k:= List(words, i -> [Position(A, i[1]), Position(B, i[2])]);
    return rec(sm:= sm, A:= A, B:= B, k:= k, symt:= symt);
end;

####Section-4.7. Representing matrices

###Representing matrix corresponding to a matrix element in the
monomial group is a product of two matrices, a diagonal matrix
and a permutation matrix

#The diagonal matrix D corresponding to the decomposition of the
rep mat of an elemet in the group, we say that the diagonal is
consisted of the signs
SpechtRepSign:= function(specht, mat)
    local n, so, l, sign, index, D;
    n:= Length(mat);
    l:= List(specht.k, i -> i[1]);
    sign:= List(mat, row -> First(row, i -> i <> 0));
    index:= List(specht.A{l}, i -> Product([1..n], j ->
            sign[j]^i[j][2]));
    D:= DiagonalMat(index);
    return D;
end;

#The permutation matrix P corresponding to the decomposition of
the rep mat of an elemet in the group, we say that the diagonal
is consisted of the signs
SpechtRepPermMat:= function(specht, mat)
local n, sigma, pi, smT, l, submat, k, M, P;
    n:= Length(mat);
    sigma:= PermList(List(mat, x -> PositionProperty(x, i ->
            i <> 0)));
    pi:= Permutation(sigma, specht.A, Permuted);
    smT:= TransposedMat(specht.sm);
    l:= List(specht.k, i -> i[2]);
```

```
    submat:= smT{l};
    k:= List(submat, i -> Permuted(i, pi));
    P:= List(k, i -> SolutionMat(submat, i));
    return P;
end;

#The representing matrix corresponding to a matrix element Mat
in G(r,1,n) as a product of the diagonal mat D and Permutation
mat P, i.e., Mat:= D*P
SpechtRepMono:= function(specht, mat)
    local D, P;
    D:= SpechtRepSign(specht, mat);
    P:= SpechtRepPermMat(specht, mat);
    return D*P;
end;

####Section-4.8. Characters and character table

#Characters correspnding to a multipartition lambda for all the
class reps
MonoCharacter:= function(lambda, reps)
    local n, r, ClassReps, specht, RepMats;
    n:= Sum(lambda, Sum);
    r:= Length(lambda);
    specht:= SpechtMono(lambda);
    RepMats:= List(reps, x -> SpechtRepMono(specht, x));
    return List(RepMats, x -> Trace(x));
end;

#Character table for all the multipartitions lambda of n
MonoCharTable:= function(n, r)
    local P, reps, CharTable;
    P:= MultiPartitions(n, r);
    reps:= List(P, MonoRep);
    CharTable:= List(P, lambda -> MonoCharacter(lambda, reps));
    return CharTable;
end;

#################################################################

####Section-5.1. The group, generators and class reps

#Hyperoctahedral group (as a subgroup of S_2n) generators as perms
on 2n points [1,2,...,n,2n,2n-1,...,n+1] = [1,2,...n,-n,-n-1,...,]
HypoGens:= function(n)
local i, gens;
    gens:= [];
        for i in [0..n-2] do
            Add(gens, (i+1,i+2)(2*n-i,2*n-i-1));
        od;
    Add(gens, (1, 2*n), 1);
    return gens;
end;

#Hyperoctahedral Group
HypoGroup:= function(n)
    return GroupWithGenerators(HypoGens(n));
end;
```

```
#Class representatives as a generator word
HypoRepGenWord:= function(lambda)
    local w, o, l, pcycle, ncycle;
    pcycle:= function(o,l)
    return o+[2..l];
    end;
    ncycle:= function(o,l)
    return Concatenation([o+1,o..1],[2..o+l]);
    end;
    w:=[];
    o:= 0;
    for l in Reversed(lambda[2]) do
        Append(w, ncycle(o,l));
        o:= l+o;
    od;
    for l in lambda[1] do
        Append(w, pcycle(o,l));
        o:= l+o;
    od;
    return w;
end;

#Class representatives for a set of bipartitions
HypoReps:= function(all_bipartitions)
    local n, gens, classes, conj_classes, c;
    n:= Sum(Sum(all_bipartitions[1]));
    gens:= HypoGens(n);
    classes:= List(all_bipartitions, HypoRepGenWord);
    classes[Position(classes, [])]:= [1,1];
    conj_classes:= List(classes, c -> Product(gens{c}));
    return conj_classes;
end;

####Section-5.2. Bipartitions and words

#Bipartitions of a number n
BiPartitions:= function(n)
    return PartitionTuples(n, 2);
end;

#Conjugate bipartition
ConjugateBiPartition:= function(lambda)
    return Reversed(List(lambda, AssociatedPartition));
end;

#Word corresponding to a bipartition
WordBiPartition:= function(lambda)
local WordPartition, w_1, w_2, word;
    WordPartition:= function(lambda)
    local i, l, u;
        u:= [];
        for l in [1..Length(lambda)] do
            for i in [1..lambda[l]] do
                Add(u, l);
            od;
        od;
        return u;
```

```
    end;
    w_1:= List(WordPartition(lambda[1]), String);
    w_2:= WordPartition(lambda[2]);
    word:= Concatenation(w_1, w_2, -Reversed(w_2), Reversed(w_1));
    return word;
end;

#Arrangements of words by the action of hyperoctahdral group
HypoArrangements:= function(w)
local B_nGroup, a1, a2, sigma, pi;
    B_nGroup:= function(n)
        return Centralizer(SymmetricGroup(2*n),
        PermList(Reversed([1..2*n])));
    end;
    a1:= Orbit(B_nGroup(Length(w)/2), w, Permuted);
    a2:= Orbit(HypoGroup(Length(w)/2), w, Permuted);
    sigma:= Sortex(ShallowCopy(a1));
    pi:= Sortex(ShallowCopy(a2));
    return Permuted(a2, pi*sigma^-1);
end;

#Bipartition corresponding to a word
BiPartitionWord:= function(w)
local lambda, a;
    lambda:= [[],[]];
    for a in w{[1..Length(w)/2]} do
        if IsInt(a) then
            a:= AbsInt(a);
            if IsBound(lambda[2][a]) then
                lambda[2][a]:= lambda[2][a]+1;
            else
                lambda[2][a]:= 1;
            fi;
        else
            a:= Int(a);
            if IsBound(lambda[1][a]) then
                lambda[1][a]:= lambda[1][a]+1;
            else
                lambda[1][a]:= 1;
            fi;
        fi;
    od;
    return lambda;
end;

#Mirrored word corresponding to a word of length n and returns
a word of length 2n. Same as the Length Change function
MirroredWord:= function(w)
local w1, w2, i;
    w1:= [];
    w2:= [];
    for i in w do
        Add(w1, i);
        if IsInt(i) then
            Add(w2, -i);
        else
            Add(w2, i);
        fi;
```

```
    od;
    return Concatenation(w1, Reversed(w2));
end;

#####Section-5.3. Bitableaux and pair of words

#Standard Young biltitableaux (SYBTs) for a multipartition lambda
#tuple = [lambda1, lambda2]
SYBTs:= function(tuple)
local   isOneHook,  removeOneHook,  addOneHook,  n,  list,  i, k,
lambda, new,  t;
#here lambda is an ordinary partition
    isOneHook:= function(lambda, i)
        return i = Length(lambda) or lambda[i] > lambda[i+1];
    end;
    removeOneHook:= function(tuple, k, i)
        local lambda, new;
        lambda:= ShallowCopy(tuple[k]);
        new:= ShallowCopy(tuple);
        if i = Length(lambda) and lambda[i] = 1 then
            new[k]:= lambda{[1..i-1]};
        else
            lambda[i]:= lambda[i]-1;
            new[k]:= lambda;
        fi;
        return new;
    end;
    addOneHook:= function(t, k, i, n)
        if i > Length(t[k]) then
            Add(t[k], [n]);
        else
            Add(t[k][i], n);
        fi;
        return t;
    end;
    # trivial case first
    if tuple = [[],[]] then
        return [[[[]],[[]]]];
    fi;
    # initialize
    n:= Sum(tuple, Sum);
    list:= [];
    # loop
    for k in Reversed([1..Length(tuple)]) do
        lambda:= tuple[k];
        for i in Reversed([1..Length(lambda)]) do
            if isOneHook(lambda, i) then
                new:= removeOneHook(tuple, k, i);
                for t in SYBTs(new) do
                    addOneHook(t, k, i, n);
                    Add(list, t);
                od;
            fi;
        od;
    od;
    return list;
end;
```

```
#Canonnical biltitableau for a multipartition lambda
CanonicalBiTableau:= function(lambda)
local CanonicalTableau, t, t1, o, l;
    CanonicalTableau:= function(lambda)
    local t, o, l;
        t:= [];
        o:= 0;
            for l in lambda do
                Add(t, o + [1..l]);
                o:= o + l;
            od;
        return t;
    end;
    t:= [];
    t1:= [];
    Add(t, CanonicalTableau(lambda[1]));
    o:= Sum(lambda[1]);
        for l in lambda[2] do
            Add(t1, o + [1..l]);
            o:= o + l;
        od;
        Add(t, t1);
    return t;
end;

#Word corresponding to a biltitableau
WordsBiTableau:= function(tab)
local rows, cols, i, k, a, b, tab1, tab2;
    rows:= [];
    cols:= [];
    #tableau 1
    tab1:= tab[1];
        for i in [1..Length(tab1)] do
            for k in [1..Length(tab1[i])] do
                a:= tab1[i][k];
                if a > 0 then
                    rows[a]:= String(i);
                    cols[a]:= k;
                else
                    rows[-a]:= String(i);
                    cols[-a]:= -k;
                fi;
            od;
        od;
    #tableau 2
    tab2:= tab[2];
        for i in [1..Length(tab2)] do
            for k in [1..Length(tab2[i])] do
                b:= tab2[i][k];
                if b > 0 then
                    rows[b]:= i;
                    cols[b]:= String(k);
                else
                    rows[-b]:= -i;
                    cols[-b]:= String(k);
                fi;
            od;
        od;
```

```
    return List([rows, cols], MirroredWord);
end;

#Tableau corresponding to a word
BiTableauWords:= function(u,v)
local lambda, n, pair, pairTransposed, tableau, coords, t1, t2, i;
    lambda:= BiPartitionWord(u);
    n:= Sum(List(lambda, Sum));
    pair:= [u,v];
    pairTransposed:= TransposedMat(pair);
    tableau:= CanonicalBiTableau(lambda);
    coords:= pairTransposed{[1..n]};
    t1:= tableau[1];
    t2:= tableau[2];
    if Size(Set(coords)) < n then
        return false;
    else
        for i in coords do
            if Number(i, IsInt) <> 1 then
                return false;
            elif IsString(i[1]) = true then
                t1[Int(i[1])][AbsInt(i[2])]:= SignInt(i[2])
                * Position(coords, i);
            else
                t2[AbsInt(i[1])][Int(i[2])]:= SignInt(i[1])
                * Position(coords, i);
            fi;
        od;
    fi;
    return [t1, t2];
end;

####Section-5.4. Specht objects

#Young character of two words u and v
YoungCharHypo:= function(w1, w2)
local c, p, n, list, l, pi, lambda, t, pi0, u1, w;
    c:= TransposedMat([w1, w2]);
    if Size(Set(c)) < Length(c) then
        return 0;
    fi;
    p:= Set(c, x -> Number(x, IsInt));
    if p <> [1] then
        return 0;
    fi;
    lambda:= BiPartitionWord(w1);
    t:= CanonicalBiTableau(lambda);
    w:= WordsBiTableau(t);
    pi0:= Sortex(ShallowCopy(TransposedMat(w)));
    pi:= Sortex(ShallowCopy(c));
    u1:= pi/pi0;
    if u1 <> () then
        n:= (LargestMovedPoint(u1) + SmallestMovedPoint(u1)-1)/2;
        list:= (ListPerm(u1){[1..n]} + n) mod (2*n + 1) - n;
        l:= List(list, AbsInt);
        pi:= PermList(l);
        return SignPerm(u1 * pi);
    else
```

```
        return 1;
    fi;
end;

#Specht matrix corresponding to the set of arrangements of
the row word and column words, A and B respectively
SpechtMatHypo:= function(A, B)
local u, v, matrix, row;
    matrix:= [];
        for u in A do
            row:= [];
            for v in B do
                Add(row, YoungCharHypo(u,v));
            od;
            Add(matrix, row);
        od;
    return matrix;
end;

#Specht object to record all the previous information
needed for further concepts
SpechtHypo:= function(lambda)
local w1, A, w2, B, mu, sm, sybt, words, k;
    w1:= WordBiPartition(lambda);
    A:= HypoArrangements(w1);
    mu:= ConjugateBiPartition(lambda);
    w2:= WordBiPartition(mu);
    B:= HypoArrangements(w2);
    sm:= SpechtMatHypo(A, B);
    sybt:= SYBTs(lambda);
    words:= List(sybt, i -> WordsBiTableau(i));
    k:= List(words, i -> [Position(A, i[1]), Position(B, i[2])]);
    return rec(sm:= sm, A:= A, B:= B, k:= k, sybt:= sybt);
end;

####Section-5.5. Representing matrices

###Representing matrix corresponding to a signed permutation
element in the hyperoctahedral group
SpechtRepHypo:= function(specht, sigma)
local act, pi, rows_label, new_rows_label, sm_rows;
    act:= ActionHomomorphism(HypoGroup(Length(specht.A[1])/2),
    specht.A, Permuted);
    pi:= sigma^act;
    rows_label:= List(specht.k, i -> i[1]);
    new_rows_label:= OnTuples(rows_label, pi);
    sm_rows:= List(specht.k, i -> specht.sm[i[1]]);
    return List(new_rows_label, i -> SolutionMat(sm_rows, specht.sm[i]));
end;

####Section-5.6. Characters and character tables

#Characters correspnding to a bipartition lambda for all of
the generators of hypo group
HypoCharacterWordGens:= function(lambda, gens)
    local n, Bipartitions, specht, classes, list;
    n:= Sum(lambda[1])+Sum(lambda[2]);
    Bipartitions:= BiPartitions(n);
```

```
    specht:= SpechtHypo(lambda);
    classes:= List(Bipartitions, HypoRepGenWord);
    classes[1]:= [1,1];
    list:= List(gens, sigma -> SpechtRepHypo(specht, sigma));
    return List(classes, c -> TraceMat(Product(list{c})));
end;

#Characters correspnding to a bipartition lambda for all of
the class reps (permutations)
HypoCharacter:= function(lambda, reps)
    local specht;
    specht:= SpechtHypo(lambda);
    return List(reps, sigma -> TraceMat(SpechtRepHypo(specht, sigma)));
end;

#Character table for all the bipartitions lambda of n
HypoCharTable:= function(n)
    local P, reps, CharTable;
    P:= BiPartitions(n);
    reps:= HypoReps(P);
    CharTable:= List(P, lambda -> HypoCharacter(lambda, reps));
    return CharTable;
end;
```