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Non-Cut Points in Hausdorff Continua

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Dedicated to the memory of Tim Murphy

Open Problem. (Tim Murphy and friends) Alice and Bob play the following game. Starting with an empty $n \times n$ matrix for odd $n \in \mathbb{N}$ they take turns to input one real number entry each. When all entries are filled Alice wins if the determinant of the completed matrix is nonzero. Otherwise Bob wins. If Alice goes first who has the winning strategy?

Contents

Declaration

This thesis is my own work. I wrote it and no one else wrote any part of it for me. The thesis has not been submitted to any other institution or for any other academic award. Where the work of others is used proper references are provided in the Bibliography section.

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Standard Terminology and Notation

Throughout a *continuum* means a compact connected Hausdorff space. Note some authors only consider metric continua but we specify when a continuum admits a compatible metric by calling it a *metric continuum*. Often we use the redundant terminology *Hausdorff continuum* to emphasise when a given continuum need not be metric.

We call the topological space T *separable* to mean it has a countable dense subset, *hereditarily separable* to mean each subset is separable, and *dense-hereditarily separable* to mean each dense subset is separable. Metric continua are separable but the converse can fail.

For X a continuum we call the subset $K \subset X$ a *subcontinuum* to mean it is closed and connected. We write $C(X)$ for the family of subcontinua of X equipped with the *Vietoris topology* with open basis of sets

$$\langle U_1, \ldots, U_N \rangle = \big\{K \in C(X) : K \subset U_1 \cup \ldots \cup U_N \text{ and each } K \cap U_n \neq \varnothing\big\}$$

for all $N \in \mathbb{N} = \{1, 2, \ldots\}$ and open subsets $U_1, \ldots, U_N \subset X$.

For any point $p \in X$ we omit the curly brackets and write for example $X - p$ instead of $X - \{p\}$. The point $p \in X$ is called a *cut point* to mean $X - p$ is disconnected and a *non-cut point* otherwise. The set $S \subset X$ is called a *semicontinuum* to mean for each $a, b \in S$ there exists a subcontinuum $K \subset X$ with $\{a, b\} \subset K \subset S$. Each Hausdorff space – in particular each subspace of a continuum – is partitioned into maximal semicontinua called the *continuum components*. The point $p \in X$ is called a *weak cut point* to mean $X - p$ has more than one continuum component and a *strong non-cut point* otherwise.

For the closed subspace $M \subset X$ we equip the quotient space $X/M = \big\{M\big\} \cup \big\{\{x\} : x \in X - M\big\}$ with the topology where the set $U \subset X/M$ is open exactly if $\bigcup U \subset X$ is open. This makes X/M into a continuum.

We call X *decomposable* to mean it is the union of two proper subcontinua and *indecomposable* otherwise. The *composant* $\kappa(p)$ of the point $p \in X$ is the union of all proper subcontinua that contain p. Metric indecomposable continua have exactly $\mathfrak{c}$ many pairwise disjoint composants.

Throughout ω^* is the set of free ultrafilters on $\omega = \{0, 1, 2, \ldots\}$ and $\mathbb{H}^*$ is the set of free ultrafilters of closed subsets of $\mathbb{H} = \{x \in \mathbb{R} \colon x \geq 0\}$. We give $\mathbb{H}^*$ the topology generated by the sets $\widetilde{U} = \{\mathcal{A} \in \mathbb{H}^* \colon A \subset U$ for some $A \in \mathcal{A}\}$ for all open subsets $U \subset \mathbb{H}$. It is known that $\mathbb{H}^*$ is a non-metric indecomposable continuum whose composant number is independent of ZFC.

The ultrafilter $\mathcal{D} \in \omega^*$ is called a *Q-point* to mean each finite-to-one function $f : \omega \to \omega$ is injective when restricted to some element of $\mathcal{D}$. Two ultrafilters $\mathcal{D}$ and $\mathcal{E}$ are said to *nearly cohere* to mean there exists a finite-to-one function $f : \omega \to \omega$ with $f(\mathcal{D}) = f(\mathcal{E})$ where we define $f(\mathcal{D})$ as the ultrafilter generated by $\{f(D) : D \in \mathcal{D}\}$ and likewise for $f(\mathcal{E})$. Near coherence defines an equivalence relation on ω^*. The number of equivalence classes is independent of ZFC and equals the number of composants of $\mathbb{H}^*$. The axiom *Near Coherence of Filters* states there is exactly one equivalence class.

Introduction

This project is about looking for special types of non-cut points in Hausdorff continua. The line of research began in 1923 when Moore [42] proved the original non-cut point theorem. It states every metric continuum has two or more non-cut points.

The result was adapted in 1968 by Whyburn [61] to the non-metric case. Starting with the original metric proof, sequences of points become nets; chains of closed sets become directed families; inductive arguments become appeals to Zorn's lemma; and the non-cut point theorem immediately generalises to Hausdorff continua.

Recent research [16, 33] takes an approach perpendicular to Whyburn. Rather than dropping the metric assumption, authors Leonel, Bobok, Pyrih and Vejnar are interested in more refined types of non-cut points in metric continua. For this project the most important types are strong non-cut points, *shore points* and *non-block points*.

We call $p \in X$ a *shore point* to mean that $X \in C(X)$ is in the closure of the hyperspace $\{K \in C(X) : K \subset X - p\}$. We call $p \in X$ a *non-block point* to mean that $X - p$ has a dense continuum component. Every non-block point is a shore point but the converse can fail [16].

Every metric continuum has two or more non-block points. This improves Moore's original theorem and follows from a 1948 result of Bing [12]. Indeed Bing proves something stronger: for X metric each $x \in X$ is a *coastal point*. That means some $p \in X$ makes the continuum component of x in $X - p$ dense. Then by definition p is a non-block point. If we change the base-point from x to p and apply Bing's theorem a second time we get a second non-block point distinct from p.

This project is an attempt to complete the other half of the picture, by investigating special types of non-cut points in Hausdorff continua. The three definitions lead to three motivating questions, ordered strongest to weakest.

(1) Is every point of every nondegenerate continuum coastal?

(2) Does every nondegenerate continuum have two non-block points?

(3) Does every nondegenerate continuum have two shore points?

The questions are nontrivial because Bing's proof uses second countability in a fundamental way that cannot be directly adapted to the non-metric case as with Whyburn.

Indeed our investigation is quickly confronted with facts about metric continua that do not generalise at all to the Hausdorff case. Chief among these is the existence of indecomposable continua with exactly one composant, henceforth called *Bellamy continua*. These creatures are a peculiarity of the non-metric realm. They quickly become the second theme of the project.

Only two classes of Bellamy continua are known [11, 14, 50, 51]. Continua of the first class come from carefully constructed inverse limits of ω_1-chains of metric continua. Continua of the second class are Stone-Čech remainders of spaces called *waves* $-$ of which the half-line $\mathbb{H}$ is the best known example. All wave remainders have the same composant number but that composant number is axiom-sensitive. Wave remainders having exactly one composant is equivalent to the set theoretic axiom Near Coherence of Filters.

Chapter 1 explores the proposed consequences of when a point x of a non-metric continuum X fails to be coastal. Our main result is the existence of a proper subcontinuum M that contains x and such that X/M is indecomposable and non-coastal at M. Properties of indecomposable continua immediately imply X/M has just one composant. Thus we reduce Question (1) to the special case of Bellamy continua. As a corollary we show all separable continua are coastal. This improves upon the results of Leonel and Bobok et al.

Chapter 2 searches for coastal and non-block points in a particular Bellamy continuum. Our main result is that $\mathbb{H}^*$ having only trivial coastal and non-block points (we make this precise in the sequel) is equivalent to the absence of Q-points in ω^*. Moreover $\mathbb{H}^*$ being without non-block points and coastal points is equivalent to Near Coherence of Filters. This gives a consistent negative answer to Questions (1) and (2). For the ω_1-chain class of Bellamy continua it is straightforward to show each point is coastal. It follows that each continuum examined to-date is either coastal at every point or coastal at no point. We proceed to examine the structure of the (proposed) exceptions. Our main result is that the coastal points of such a continuum comprise a maximal semicontinuum with dense interior.

Chapter 3 shows the shore point existence problem is equivalent to the non-block point existence problem. Combined with Chapter 2 this gives a consistent negative answer to Question (3). Our example is decomposable and we show this holds in general. We obtain some partial results about the coastal points of the decomposition elements. As a side result we prove shore points $p \in X$ in arbitrary continua are *proper* in that $X \in C(X)$ is in the closure of the hyperspace $\{K \in C(X) : K \subset \kappa(p) - p\}$.

Chapter 4 is about weak cut points of indecomposable continua. Indecomposable metric continua have $\mathfrak{c}$ many composants. This quickly proves each point is a weak cut point. Our main result is that this does not generalise to Hausdorff continua. We use a modification of Bellamy's original inverse system to construct a Bellamy continuum with exactly one strong non-cut point.

Chapter 5 is about the disparity between metric and non-metric indecomposable continua. Bellamy continua are *large* in the sense of being non-metric. It is natural to ask whether they can be *small* in other ways. Our main result is that there exists a Bellamy continuum that is *almost metric* in being separable. We use a modification of Smith's inverse system. Our example is separable but not hereditarily separable or even dense-hereditarily separable. The problem is open whether there exists a Bellamy continuum with either property.

Chapter 6 is an extension of Chapter 2. Our main result is that each wave remainder has the same pattern of (proper) non-block points as $\mathbb{H}^*$. We prove each point of a wave remainder weakly cuts its composant and show this fact is nontrivial by giving examples of remainders that diverge from $\mathbb{H}^*$. The upshot of the final chapter is that all known Bellamy continua are either coastal at every point or coastal at no point. Thus partially coastal Bellamy continua (if they exist) require entirely new methods to construct.

Chapter 1

Shore and Non-Block Points in Hausdorff Continua

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Abstract

We study the shore and non-block points of non-metric continua. We reduce the problem of showing a continuum to have non-block points to that of showing an indecomposable continuum to have non-block points. As a corollary we prove that separable continua have at least two non-block points – and moreover are irreducible about their set of non-block points.

1.1 Introduction

In 1923 Moore [7] proved that every metric continuum has two or more non-cut points. Whyburn [12] extended this result in 1968 to cover non-metric continua. Shore points were introduced by Puga-Espinosa et al [6, 9, 10] as a strengthening of the notion of a non-cut point and used in their study of dendrites.

Recently Leonel [5] has improved upon the result of Moore by showing every metric continuum has two or more shore points. Bobok et al [3] pointed out how the two points discovered by Leonel are not only shore points, but satisfy a stronger property which they called being a non-block point.

As the metric assumption is not necessary to guarantee the existence of non-cut points, this prompts the question of whether it is necessary to guarantee the existence of shore or non-block points.

We reduce the existence problem to the case of indecomposable continua. Moreover we show that if each indecomposable member of a class of continua $\mathcal{P}$ – which is closed under certain quotient maps – has two or more non-block points then every member of $\mathcal{P}$ has two or more non-block points, and is moreover irreducible about its set of non-block points. We apply this result to prove the existence of non-block points in separable continua.

1.2 Notation and Terminology

For sets A and B define $A - B = \{a \in A \colon a \notin B\}$. If B is the singleton $\{b\}$ we will write $A - b$ without confusion. Whenever we write $A \subset B$ we do not presume A is a proper subset of B.

For a subset $S \subset X$ denote by S° and $\overline{S}$ the interior and closure of S respectively. The *boundary* of S is the subset $\overline{S} \cap \overline{(X - S)}$.

X is a nondegenerate Hausdorff continuum. That is to say a compact connected Hausdorff space that contains more than one point. $C(X)$ denotes the hyperspace of subcontinua of X with the Vietoris topology.

The subset $P \subset X$ is called a *cut set* if $X - P$ fails to be connected. In this case P is said to cut X. If $\{p\}$ is a cut set, we say p is a *cut point* and that p cuts X. If p is not a cut point it is called a *non-cut point*.

The subset $P \subset X$ is called a *shore set* if X is the limit in $C(X)$ of a net of subcontinua of $X - P$. Equivalently for each finite family of nonempty open sets $U_1, U_2, \ldots, U_n$ some subcontinuum in $X - P$ meets each of $U_1, U_2, \ldots, U_n$. If $\{p\}$ is a shore set we say p is a *shore point*.

The subset $P \subset X$ is called a *non-block set* if there exists a family of subcontinua of $X - P$ whose intersection is nonempty and whose union is dense in X. If $\{p\}$ is a non-block set we say p is a *non-block point*.

We write $X = Y \oplus Z$ to mean that Y and Z are proper subcontinua of X for which $X = Y \cup Z$. Then Y and Z are said to form a *decomposition* of X. In case X admits no decomposition it is said to be *indecomposable*. The subcontinuum $K \subset X$ is said to be *thick* if it is both proper and has nonvoid interior. Being indecomposable is equivalent to having no thick subcontinua.

For each $x \in X$ denote by $\kappa(x)$ the *composant* of x, meaning the union of all proper subcontinua that contain x. X is called *aposyndetic at x with respect to y* if x is contained in the interior of a subcontinuum disjoint from y. X is called *aposyndetic at x* if it is aposyndetic at x with respect to each $y \neq x$. X is said to be *aposyndetic* if it is aposyndetic at each point. X is said to be *null-aposyndetic* at x if it is aposyndetic at x with respect to no other point of X. In other words no proper subcontinuum contains x in its interior. In this case we also refer to the point x as null-aposyndetic.

The subset $A \subset B$ of a partially ordered set B is said to be *cofinal* if each element of B is bounded from above by an element of A. Define the *cofinality* of a totally-ordered set B as the unique least ordinal β which is order-isomorphic to a cofinal subset of B.

1.3 Coastal Continua

It was proved by Leonel [5] that every metric continuum has two or more shore points. This improves the classical result of Moore [7] that every metric continuum has two or more non-cut points. Bobok et al [3] observed how the two shore points discovered by Leonel are in fact non-block points. (They also show that a shore point of a metric continuum need not be a non-block point.) It is unknown whether these results extend to Hausdorff continua.

Definition 1.3.1. For a subcontinuum $K \subset X$ and subset $P \subset X$ define the *composant* of K *relative to* P:

$$\kappa(K;P) = \bigcup \big\{ M \in C(X) \colon K \subset M, M \neq X, M \cap P = \varnothing \big\}$$

When for example $K = \{x\}$ and $P = \{p\}$ we will write $\kappa(x;p)$. Note that if $P = \varnothing$ then $\kappa(x;P)$ is the composant of x. That the composant of a point is dense follows from the boundary bumping theorem for Hausdorff continua. The proof is found in §47, III Theorem 2 of [4].

The following theorem has been proved by Bing [2] when X is metric. The proof immediately generalizes to spaces whose degree of Baireness is no less than the weight.

Theorem 1.3.2. For each $x \in X$ some $p \in X$ makes $\kappa(x;p)$ a dense subspace.

This motivates the following definition.

Definition 1.3.3. The continuum X is called *coastal* at $x \in X$ to mean $\kappa(x;p)$ is dense for some $p \neq x$. We call X *coastal* to mean it is coastal at each point.

By Theorem 1.3.2 all metric continua are coastal. Leonel [5] essentially proved the following.

Theorem 1.3.4. A coastal continuum has at least two non-block points.

Bobok, Pyrih and Vejnar [3] have proved each metric continuum is irreducible about its set of non-block points. This strengthens the classic result that metric continua are irreducible about their sets of non-cut points. Theorem 1.3.8 shows how this result can be generalized. Since maps of the following type are ubiquitous later they warrant a name.

Definition 1.3.5. Let $K \subset X$ be a subcontinuum. The canonical quotient map $\theta \colon X \to X/K$ obtained by treating K as a single point is called the K-*bloom*. A map from X is called a *bloom* if it is the K-bloom for some subcontinuum $K \subset X$. A class $\mathcal{P}$ of continua is called a *bloom class* if it is closed under blooms.

Note that blooms are continuous and monotone. Therefore the image of X under the K-bloom is compact and connected. Moreover it follows from normality of X that X/K is Hausdorff, hence a continuum.

Examples of bloom classes are the class of metric continua, the class of separable continua, and the class of finitely-irreducible continua. We will make frequent use of the following lemma.

Lemma 1.3.6. Let $K \subset X$ be a subcontinuum. Suppose X/K is coastal at the point K. Then $\kappa(K; y)$ is dense for some $y \notin K$.

Proof. Suppose X/K is coastal at the point K. Then there exists a point $m \neq K$ of X/K and a family $\mathcal{C}$ of subcontinua of X/K whose every element contains the point K and is disjoint from m, such that $\bigcup \mathcal{C}$ is dense in X/K.

The point $m \in X/K$ is the image of a unique point $y \in X$ under the K-bloom θ. Since θ is monotone we have for each $C \in \mathcal{C}$ that $\theta^{-1}(C)$ is a subcontinuum of X containing K. Moreover $y \notin \theta^{-1}(C)$ as C is disjoint from m. Therefore since θ is surjective $\theta^{-1}\big(\bigcup \mathcal{C}\big) = \bigcup \big\{\theta^{-1}(C) : C \in \mathcal{C}\big\}$ is a family of subcontinua whose union is dense in X and disjoint from y. $\square$

Corollary 1.3.7. Let $K \subset X$ be a subcontinuum with $x \in K$. Suppose X/K is coastal at the point K. Then X is coastal at x.

Theorem 1.3.8. Suppose the elements of the bloom class $\mathcal{P}$ are coastal. Each element of $\mathcal{P}$ is irreducible about its set of non-block points.

Proof. Suppose X is of class $\mathcal{P}$. Denote by B the nonempty set of non-block points of X and assume a proper subcontinuum K contains B. Choose a point $x \in K$. Since $\mathcal{P}$ is a bloom class X/K is coastal at the point K. By Lemma 1.3.6 there is some $y \notin K$ for whick $\kappa(K; y)$ is dense in X. But since $K \subset \kappa(K; y)$ this implies $y \notin K$ is a non-block point of X, contradicting the assumption that $B \subset K$. □

Once we have shown in Section 1.5 that separable continua are coastal, Theorem 1.3.8 can be employed to show separable continua are irreducible about their set of non-block points.

1.4 Blooms

The purpose of this section is to show that in order to prove all continua are coastal we may restrict our attention to indecomposable continua. As an intermediate step we shall first reduce the problem of showing all continua to be coastal to that of showing them to be coastal at their null-aposyndetic points. We first prove some results about null-aposyndetic points for later use.

Lemma 1.4.1. Each thick subcontinuum $T \subset X$ contains all null-aposyndetic points of X.

Proof. Suppose $x \in X$ is null-aposyndetic. Since T is thick $\overline{(X - T)}$ is a proper subset of X. Suppose first $\overline{(X - T)}$ is connected. Then we have a decomposition $X = T \oplus \overline{(X - T)}$. But $X - T$ is contained in the interior of the proper subcontinuum $\overline{(X - T)}$. Therefore $x \notin X - T$ and as a result $x \in T$.

Now suppose $\overline{(X - T)}$ is disconnected. Then $X - T$ is disconnected and there exist nonempty disjoint clopen subsets A and B of $X - T$ such that $A \cup B = X - T$. If $x \notin T$ then without loss of generality $x \in A$. By boundary bumping $T \cup A$ and $T \cup B$ are proper subcontinua and $X = (T \cup A) \oplus (T \cup B)$ is a decomposition. But this implies x is an element of the open subset $X - (T \cup B)$ of the continuum $T \cup A$, contradicting how x is null-aposyndetic. $\square$

Only indecomposable continua fail to contain a thick subcontinuum. In this case all points are null-aposyndetic.

Lemma 1.4.2. Each thick subcontinuum $T \subset X$ contains all null-aposyndetic points of X in its boundary.

Proof. Suppose $x \in X$ is null-aposyndetic. The previous lemma says $x \in T$. But since x is null-aposyndetic it is not in the interior of any proper subcontinuum. Therefore x is an element of the boundary of T. $\square$

Corollary 1.4.3. If x is null-aposyndetic each subcontinuum that does not contain x has void interior.

We are now ready to prove our first main theorem.

Theorem 1.4.4. Suppose there exists a non-coastal continuum. There also exists a continuum that is not coastal at a null-aposyndetic point.

Proof. Suppose X fails to be coastal at x. We will construct a proper subcontinuum $M \subset X$. The image of X under the M-bloom will fail to be coastal at a null-aposyndetic point.

Let $(D^\alpha)_{\alpha<\xi}$ be an open base for the topology of X where ξ is a cardinal. Let $N_0 = \{x\}$ and define a nest of proper subcontinua N_α by transfinite recursion as follows:

Where $\alpha = \beta + 1$ is a successor ordinal: If some proper subcontinuum R contains N_β in its interior and meets D^α then let $N_\alpha = R$. If no such subcontinuum exists let $N_\alpha = N_\beta$.

Where α is a limit ordinal: Consider $L_\alpha = \bigcup_{\beta<\alpha} N_\beta$. We cannot have $L_\alpha = X$ since then the interiors of the sets N_β would form an increasing chain of proper open subsets with union X, and compactness of X forbids this. L_α cannot be dense as then $\kappa(x;p)$ would be dense for every $p \notin L_\alpha$ which would imply X to be coastal at x. If some proper subcontinuum R contains $\overline{L_\alpha}$ in its interior and meets D^α then let $N_\alpha = R$. If no such subcontinuum exists let $N_\alpha = \overline{L_\alpha}$.

The union $N = \bigcup_{\beta<\xi} N_\beta$ must be proper and non-dense for the same reason each L_α is proper and non-dense. Define the proper subcontinuum $M = \overline{N}$. Let $\pi\colon X \to X/M$ be the M-bloom and $X' = X/M$. Note that $\pi(M) = \pi(x)$ as $x \in M$. It follows from Corollary 1.3.7 that X' fails to be coastal at $\pi(x)$.

We claim that X' is null-aposyndetic at $\pi(x)$. To prove this suppose there is an open set $U \subset X'$ and proper subcontinuum $K \subset X'$ such that $\pi(x) \in U \subset K$. Since π is continuous and monotone we get an open set $\pi^{-1}(U) \subset X$ and proper subcontinuum $\pi^{-1}(K) \subset X$ such that $M \subset \pi^{-1}(U) \subset \pi^{-1}(K)$. Then $\pi^{-1}(K) - M$ contains a basic open set D^α disjoint from M. Therefore N_α is disjoint from D^α. Consider what happened at stage α of our construction.

Where $\alpha = \beta + 1$ is a successor ordinal: It follows that $N_\alpha = N_\beta$. But this implies no proper subcontinuum contains N_β in its interior while intersecting D^α. But $\pi^{-1}(K)$ is a proper subcontinuum containing N_β in its interior, leading to a contradiction.

Where α is a limit ordinal: It follows that no proper subcontinuum R contains $\overline{L_\alpha}$ in its interior while intersecting D^α. But $\pi^{-1}(K)$ is a proper subcontinuum containing $\overline{L_\alpha}$ in its interior, leading to a contradiction. □

The previous theorem demonstrates how in proving all continua are coastal it is sufficient to look at the null-aposyndetic points. We now proceed to further reduce the problem to examining indecomposable continua.

For the remainder of the section X is a fixed continuum and $x \in X$ a fixed null-aposyndetic point. For X indecomposable we are already done. Hence for convenience we assume X is decomposable and so contains at least one thick subcontinuum.

Lemma 1.4.5. The set of null-aposyndetic points of X is nowhere dense.

Proof. Let T be a thick subcontinuum. By Lemma 1.4.2 we know each null-aposyndetic point is contained in the boundary of T. But the boundary of T is closed with void interior. Therefore the set of null-aposyndetic points is nowhere dense. □

The next lemma is used in the proof of Theorem 1.4.7 which is itself a precursor to Theorem 1.4.15.

Lemma 1.4.6. For each thick subcontinuum $T \subset X$ the point x is in the closure of each component of $X - T$.

Proof. If $X - T$ is connected $\overline{(X - T)}$ is a thick subcontinuum. Then it contains x by Lemma 7. Otherwise we can assume the family $\mathcal{L}$ of components of $X - T$ has more than one element. Also assume for a contradiction some $L \in \mathcal{L}$ does not have x in its closure. Corollary 1.4.3 says $\overline{L}$ and hence L have void interior. Fix some $p \in L$.

By boundary bumping $T \cup R$ is a subcontinuum for each $R \in \mathcal{L}$. Define the family $\mathcal{F} = \{T \cup R \colon R \in \mathcal{L}, R \neq L\}$ of subcontinua. Lemma 7 says $x \in T$. Hence we have $\kappa(x,p) \subset \bigcup \mathcal{F} = X - L$. Since L is nowhere dense this shows $\kappa(x,p)$ is dense. Thus x is coastal contrary to assumption. We conclude that $x \in \overline{L}$ for each component L of $X - T$. $\square$

Theorem 1.4.7. For each thick subcontinuum $T \subset X$ the set $\overline{(X - T)}$ is a thick subcontinuum.

Proof. By Lemma 1.4.6 every component of $\overline{(X - T)}$ meeting $X - T$ must contain x. Therefore only one component L of $\overline{(X - T)}$ meets $X - T$ and all other components are contained in the boundary of T. But this implies that $(X - T) \subset L$. Since L is connected it follows that $\overline{L} = \overline{(X - T)}$ is also connected. $\square$

The next lemma readily follows from – and will be employed in the proof of – the stronger statement of Corollary 1.4.12: that every thick subcontinuum has connected interior.

Lemma 1.4.8. Let $T \subset X$ be a subcontinuum. The components of T° have nonvoid interior.

Proof. This is clear if $T^\circ = X$ or $T^\circ = \varnothing$. Hence we can assume T is thick. Note the relation $X - \overline{(X - T)} = T^\circ$. Therefore the components of $X - \overline{(X - T)}$ are the components of T°.

Let $\mathcal{L}$ denote the family of components of T°. By Theorem 1.4.7 $\overline{(X - T)}$ is a subcontinuum. It follows from boundary bumping that $L \cup \overline{(X - T)}$ is a subcontinuum for each $L \in \mathcal{L}$. If some $L \in \mathcal{L}$ has void interior then the family $\mathcal{F} = \{R \cup \overline{(X - T)} \colon R \in \mathcal{L}, R \neq L\}$ can be used to repeat the argument of Lemma 1.4.6 to show X is coastal at x, contrary to our assumptions. $\square$

Since X is decomposable it has at least one thick subcontinuum T. Theorem 1.4.7 says $\overline{(X - T)}$ is a second thick subcontinuum. It is certainly not the case that X only has two thick subcontinua. For Theorem 5.5 of [8] shows how to expand a subcontinuum within a prescribed open set. Obviously expansion preserves the property of being thick.

However we will show T and $\overline{(X - T)}$ comprise *almost all* thick subcontinua of X. That is to say we cannot expand either to a proper subcontinuum with a strictly larger interior.

Lemma 1.4.9. Suppose U is an open subset of the subcontinuum $T \subset X$ and $x \notin \overline{U}$. Then $X - U$ is disconnected. One component of $X - U$ contains $\overline{(X - T)}$ and all other components are contained in T°.

Proof. $X - U$ cannot be connected as then $\overline{(X - U)}$ would be a thick subcontinuum with boundary contained in $\overline{U}$. Since $x \notin \overline{U}$ this contradicts Lemma 1.4.2.

Since $\overline{(X - T)}$ is connected by Theorem 1.4.7 and disjoint from U it is contained in a component of $X - U$. Because $X - \overline{(X - T)} = T^\circ$ each remaining component of $X - U$ is contained in T°. $\square$

Lemma 1.4.10. Suppose $S, T \subset X$ are thick subcontinua with $T \subset S$. Then $S^\circ = T^\circ$.

Proof. Suppose otherwise that $T^\circ \neq S^\circ$. It follows that $S^\circ - T$ is nonempty and open. Then there exists $p \in S^\circ - T$. Choose an open $U \subset T^\circ$ such that $x \notin \overline{U}$. Note also that $U \subset S^\circ$. Let $\mathcal{L}$ denote the family of components of $X - U$. By the previous lemma $\mathcal{L}$ has more than one element; some $N \in \mathcal{L}$ contains $\overline{(X - S)}$; and all $R \neq N$ are contained in S°.

Each $R \in \mathcal{L}$ meets the boundary of U. Since $U \subset T$ each R meets T and hence $T \cup R$ is a subcontinuum. Let $L \in \mathcal{L}$ be such that $p \in L$. Note that $L \not\subset T$. The closure $\overline{L}$ is formed by adding to L a subset of $\overline{U}$. Since $x \notin \overline{U}$ it follows from lemma 1.4.3 that $\overline{L}$ has void interior. Hence L has void interior.

The family of subcontinua $\{N \cup T \cup R \colon R \neq L\}$ contains $X - L$ in the union and shows $\kappa(x; p)$ is dense, contrary to our assumptions. □

Theorem 1.4.11. Suppose S and T are thick subcontinua. Either $S \cup T = X$ or $S^\circ = T^\circ$.

Proof. If $S^\circ \neq T^\circ$ then without loss of generality $S^\circ - T$ is nonempty and open. $S \cup T$ is a subcontinuum since T and S meet at x by Lemma 1.4.2. Then $T \subset S \cup T$ but the interior of $S \cup T$ is strictly larger than the interior of T. By the Lemma 1.4.10 we know $S \cup T$ cannot be thick. Therefore $S \cup T = X$. □

As mentioned earlier we can combine Theorem 1.4.11 with Lemma 1.4.8 to get the following.

Corollary 1.4.12. Every thick subcontinuum has connected interior.

We are now ready to prove the result mentioned in the section's preamble.

Theorem 1.4.13. $X = S \oplus T$ where S and T are indecomposable and $S \cap T$ is nowhere dense.

Proof. Let T be a thick subcontinuum and $S = \overline{(X - T)}$. By Corollary 1.4.12 and Theorem 1.4.11 we can replace each of S and T with the closure of the interior. Hence we can assume $T = \overline{(T^\circ)}$ and $S = \overline{(S^\circ)}$ and still have $X = S \cup T$. Since S° and T° are disjoint the intersection $S \cap T$ is nowhere dense. We claim each of S and T is indecomposable under the subspace topology.

To prove this suppose some subcontinuum $K \subset T$ has interior in T. Then there exists an open set U of X such that $T \cap U \subset K$. Since $T = \overline{(T^\circ)}$ the open set U must intersect T°. Then $U \cap T^\circ \subset K$ and therefore K is thick in X.

It follows from Lemma 1.4.10 that $K^\circ = T^\circ$. But since we assumed $K \subset T$ and $T = \overline{(T^\circ)}$ this guarantees $K = T$. We conclude that no proper subcontinuum of T has interior in T. Equivalently T is indecomposable under the subspace topology. The proof is identical for S. $\square$

The next lemma will allow us to obtain an indecomposable quotient by treating either of the two indecomposable continua S or T as a single point.

Lemma 1.4.14. Suppose $X = S \oplus T$ where S and T are indecomposable. The image of X under the S-bloom is indecomposable.

Proof. Since T contains the open set $X - S$ it is thick. Likewise S is thick. Therefore $x \in S \cap T$. Let $\varphi\colon X \to X/S$ be the S-bloom. We claim X/S is indecomposable. To prove this suppose $K \subset X/S$ is a thick subcontinuum. Since blooms are monotone $\varphi^{-1}(K)$ is a thick subcontinuum of X. By Lemma 7 the null-aposyndetic point $x \in \varphi^{-1}(K)$. It follows that $\varphi(x) \in K$.

Since K has interior in X/S there exists an open set U of X/S such that $\varphi(x) \notin U \subset K$. Then $x \notin \varphi^{-1}(U) \subset \varphi^{-1}(K)$. In particular $\varphi^{-1}(U)$ is disjoint from S. It follows that the interior of $\varphi^{-1}(K)$ is strictly larger than the interior of S. But then $\varphi^{-1}(K)$ is a subcontinuum of X that contains S and whose interior strictly contains the interior of S. Then theorem 1.4.11 implies $\varphi^{-1}(K) = X$. This in turn implies $K = X/S$ contradicting the assumption that K is thick. $\square$

Theorem 1.4.15. If there exists a non-coastal continuum, there exists a non-coastal indecomposable continuum.

Proof. Suppose X fails to be coastal at x. Let $\pi\colon X \to X/M$ be the bloom constructed in Theorem 1.4.4. If X/M is indecomposable let φ be the identity mapping on X/M.

Otherwise X/M is decomposable and fails to be coastal at the null-aposyndetic point $\pi(x)$. In this case write $X/M = S \oplus T$ as the union of two indecomposable subcontinua and let φ be the S-bloom constructed in Lemma 1.4.14.

In both cases the continuum $\varphi\big(\pi(X)\big) = (X/M)/S$ is indecomposable. Let $\theta = \varphi \circ \pi$. It follows that θ is the K-bloom for $K = (\pi^{-1}(S) \cup M)$. Corollary 1.3.7 then implies that $\theta(X)$ fails to be coastal at $\theta(x)$. □

Corollary 1.4.16. If all indecomposable continua are coastal, all continua are coastal.

Corollary 1.4.16 can be strengthened by letting X be an element of some bloom class.

Theorem 1.4.17. Suppose $\mathcal{P}$ is a bloom class. If all indecomposable continua of type $\mathcal{P}$ are coastal, all continua of type $\mathcal{P}$ are coastal.

Proof. Let X be a member of the bloom class $\mathcal{P}$ and $x \in X$. Let the blooms $\pi\colon X \to X/M$ and $\varphi\colon (X/M) \to (X/M)/S$ be as defined in the proofs of Theorems 1.4.4 and 1.4.14 respectively. It follows that $\theta = \varphi \circ \pi$ is the K-bloom for $K = (\pi^{-1}(S) \cup M)$.

From the proof of Theorem 1.4.15 we know $\theta(X)$ is nondegenerate and indecomposable. Since $\mathcal{P}$ is a bloom class $\theta(X) \in \mathcal{P}$. By assumption $\theta(X)$ is coastal at $\theta(x)$. Then by Corollary 1.3.7 we have that X is coastal at x. □

Theorem 1.4.17 provides an alternate proof of Bing's Theorem 1.3.2 in the metric case: Let $\mathcal{P}$ denote the class of metric continua. Observe that if a metric space is mapped continuously onto a Hausdorff space, the image is metrizable. From this we conclude that $\mathcal{P}$ is closed under blooms. Therefore by Theorem 1.4.17 we may restrict our attention to indecomposable metric continua. Recall an indecomposable metric continuum has uncountably many pairwise disjoint composants. In that case we may, for each $x \in X$, choose $p \in X - \kappa(x)$. Then $\kappa(x;p) = \kappa(x)$ is dense.

We observe that if X is not coastal at x then $\theta(X)$ is the composant of $\theta(x)$. We can say slightly more.

Lemma 1.4.18. Suppose X fails to be coastal at x. There exists a subcontinuum $K \subset X$ including x such that X/K is indecomposable and fails to be coastal at the point K. Therefore X/K has exactly one composant.

Proof. In the notation of Theorem 1.4.15 take $K = (\pi^{-1}(S) \cup M)$ and let θ be the K-bloom. Then $\theta(X)$ is indecomposable. Moreover Corollary 1.3.7 implies that $\theta(X)$ is not coastal at $\theta(x)$. Therefore $\theta(X)$ is the composant of $\theta(x)$; otherwise $\kappa\big(\theta(x);p\big)$ would be dense for each $p \notin \kappa\big(\theta(x)\big)$. It is well known that distinct composants of an indecomposable continuum are disjoint. Therefore $\theta(X)$ must be the composant of each of its points. Therefore there is exactly one composant. $\square$

Thus the study of which continua are coastal reduces to the study of indecomposable continua with exactly one composant. Continua of this sort are a peculiarity of the non-metric realm and were shown to exist by Bellamy [1].

1.5 Baireness

The non-block point existence theorem of Leonel relies on Bing's Theorem 1.3.2. A slightly modified version of his proof applies to continua that satisfy a condition on its cardinal invariants. We first define the relevant cardinal invariants, then give the condition itself.

Definition 1.5.1. A space is called α-Baire if every family of α many open dense subsets has dense intersection.

Definition 1.5.2. The *weight* $w(X)$ of the space X is the least cardinality of an open base for the topology.

Definition 1.5.3. The *density* $d(X)$ of the space X is the least cardinality of a dense subset.

Bing's proof of Theorem 1.3.2 can be adapted to show that if the continuum X is $w(X)$-Baire, it is coastal. If X is metric it is second-countable and thus $w(X) = \aleph_0$. In addition metric continua satisfy the Baire category theorem and hence are $\aleph_0$-Baire. Combining these two facts yields that each metric continuum X is $w(X)$-Baire.

In this section we show it is enough to demand X be $d(X)$-Baire. Denote by $\mathcal{D}$ the family of continua meeting this condition and recall that $\mathcal{D}$ contains all separable continua. Since it follows immediately from their definitions that $d(X) \leq w(X)$ our result is a direct strengthening of Bing's.

We will require the following well-known facts about cardinal invariants.

Proposition 1.5.4. Suppose X is α-Baire and $U \subset X$ open. Then U is α-Baire.

Proposition 1.5.5. Suppose $U \subset X$ is open. Then $d(U) \leq d(X)$.

Let the blooms π and φ be as defined in the proofs of Theorems 1.4.4 and 1.4.14 respectively. In the proof of Theorem 1.4.17 we showed that $\theta = \varphi \circ \pi$ is the K-bloom for some subcontinuum $K \subset X$. It follows from the surjectivity of θ that $d\big(\theta(X)\big) \leq d(X)$.

We can invoke Proposition 1.5.4 for $U = X - K$ to get the following fact: If X is α-Baire then $\theta(X)$ is α-Baire. In other words θ cannot reduce the degree of Baireness. Combining these observations gives the following.

Corollary 1.5.6. The family of continua $\mathcal{D}$ is a bloom class.

The main result of this section relies on the following lemma.

Lemma 1.5.7. Suppose $X \in \mathcal{D}$. Each nest of open dense subsets of X has dense intersection.

Proof. Suppose $\mathscr{U}$ is a nest of open dense subsets of X. Since the $\mathscr{U}$ is totally-ordered by reverse-inclusion there is a well-ordered cofinal subset $\mathscr{V} \subset \mathscr{U}$. By cofinality we have $\bigcap \mathscr{V} = \bigcap \mathscr{U}$. By replacing $\mathscr{U}$ with $\mathscr{V}$ we can assume $\mathscr{U}$ has the form $\{U(\gamma) : \gamma < \Gamma\}$ for some ordinal Γ. Without loss of generality Γ equals its own cofinality.

Now suppose $V \subset X$ is an arbitrary open set. Choose a dense $D \subset V$ with $|D| = d(V)$. For a contradiction assume V is disjoint from $\bigcap \mathscr{U}$. In particular $x \notin \bigcap \mathscr{U}$ for each $x \in D$. So we can define the function $\gamma : D \to \Gamma$ by $\gamma(x) = \min\big\{\alpha \in \Gamma : x \notin U(\alpha)\big\}$.

Since each $U(\alpha)$ is open and dense there is some $x \in U(\alpha) \cap D$. It follows from definition $\alpha < \gamma(x)$. Letting $\alpha \in \Gamma$ be arbitrary we see $\gamma(D) \subset \Gamma$ is cofinal. Since Γ equals its own cofinality $\Gamma \cong \gamma(D)$ as ordinals. In particular $|\Gamma| = |\gamma(D)| \leq |D| = d(V)$ and so $|\Gamma| \leq d(V)$. Proposition 1.5.4 then implies $|\Gamma| \leq d(X)$. But since $X \in \mathcal{D}$ we have $\bigcap \mathscr{U}$ dense by definition. In particular $\bigcap \mathscr{U}$ meets V contrary to assumption. □

Corollary 1.5.8. Suppose $X \in \mathcal{D}$. The union of a nest of closed nowhere dense subsets of X is proper.

The hypothesis $X \in \mathcal{D}$ is essentially a restriction on the density of X. One might hope this hypothesis can be dropped. The next example shows otherwise.

Example 1.5.9. We give a continuum that is the union of a nst of nowhere dense subcontinua. The functional analysis terminology used in this example can be found in [11].

Let ω_1 be the first uncountable ordinal and $E = \ell^2(\omega_1)$ the Hilbert space of square-summable functions $x\colon \omega_1 \to \mathbb{R}$ with the inner-product $xy = \sum_{\alpha<\omega_1} x(\alpha)y(\alpha)$. Define the α-coordinate vector e_α by $e_\alpha(\beta) = \delta_\alpha^\beta$ and define the β-coordinate functional ϵ_β by $\epsilon_\beta(x) = x(\beta)$.

Our example is the closed unit ball $B \subset E$ under the weak topology. Theorem 3.15 of [11] says B is weak* compact. But Theorem 12.5 of [11] implies a real Hilbert space is linear-isomorphic to its dual. Therefore $E = E^{**}$, the weak and weak* topologies coincide on E, and the closed unit ball $B \subset E$ is weak-compact. We represent B as the union of a nest of nowhere-dense subsets.

For each $x \in B$ since $\|x\| < \infty$ only countably many values of $x(\alpha)$ are nonzero. Since ω_1 has uncountable cofinality some $\alpha < \omega_1$ has $x(\beta) = 0$ for all $\beta > \alpha$. Let $B(\alpha) = \{y \in B\colon y(\beta) = 0 \ \forall \beta > \alpha\}$. Then clearly $x \in B(\alpha)$ and $\bigcup_{\alpha<\omega_1} B(\alpha) = B$. Recall the functional ϵ_β is by definition weak-continuous. Therefore $B(\alpha) = B \cap \left(\bigcap_{\beta>\alpha} \epsilon_\beta^{-1}(0)\right)$ is closed in B.

To show $B(\alpha)$ is nowhere dense in B assume otherwise $x \in B(\alpha)$ is an interior point. Write $\beta = \alpha + 1$ and recall $\epsilon_\beta y = 0$ for all $y \in B(\alpha)$. Since addition and rescaling are weak-continuous so is the function $f(t) = x + te_\beta$. Observe $f(0) = x$ and since $x \in B(\alpha)$ is an interior point continuity gives $f(\varepsilon) \in B(\alpha)$ for some $\varepsilon > 0$. Thus $x + \varepsilon e_\beta \in B(\alpha)$. Now apply ϵ_β to both sides to get $\epsilon_\beta(x + \varepsilon e_\beta) = \epsilon_\beta x + \varepsilon \epsilon_\beta e_\beta = 0 + \varepsilon = 0$ which is a contradiction. We conclude

$B(\alpha)$ has void interior relative to B.

Theorem 1.5.10. The indecomposable elements of $\mathcal{D}$ are coastal.

Proof. Suppose X is an indecomposable continuum of class $\mathcal{D}$. Let $x \in X$. Since X is indecomposable distinct composants are disjoint. If $\kappa(x) \neq X$ choose $p \notin \kappa(x)$. Then $\kappa(x; p) = \kappa(x)$ is dense.

Now assume that $\kappa(x) = X$. Then x can be joined to any point by a proper subcontinuum which must be nowhere dense since X is indecomposable.

Let $Y = \{y_\alpha \colon \alpha < d(X)\}$ be a dense subset with $x = y_0$. For each $1 \leq \alpha < d(X)$ let L^α be a subcontinuum with void interior joining y_0 to y_α. We will define an increasing chain of proper subcontinua N_α by transfinite recursion on β beginning with $N_0 = \{y_0\}$.

When $\gamma = \alpha + 1$ is a successor ordinal: Let $N_\gamma = N_\alpha \cup L^\gamma$. By construction each of N_α and L^γ is closed and nowhere dense, hence proper. It follows that N_γ is closed, nowhere dense and proper as well. N_γ is connected since each of N_α and L^γ contains x.

When γ is a limit ordinal: Consider $M^\gamma = \bigcup_{\alpha<\gamma} N_\alpha$. By construction each N_α is closed and nowhere dense so Corollary 1.5.8 implies $M^\gamma \neq X$. M^γ is connected since each N_α contains x. If M^γ is dense choose $p \notin M^\gamma$. Then $\kappa(x; p)$ contains each subcontinuum N_α so is dense. This proves X is coastal at $x = y_0$. Otherwise $\overline{(M^\gamma)}$ is a proper subcontinuum, so must be nowhere dense. In this case it follows that $N_\gamma = \overline{(M^\gamma)} \cup L^\gamma$ is a proper and nowhere dense subcontinuum.

Assuming no M^γ is dense consider the union $M = \bigcup_{\alpha<d(X)} N_\alpha$. This is proper by Corollary 1.5.8. Moreover it contains the dense subset Y. Therefore it is dense. It follows that $\kappa(x; p)$ is dense for each $p \notin M$. Thus X is coastal at $x = y_0$. □

To prove the next theorem we apply Theorem 1.4.17 and Corollary 1.5.6 to the class $\mathcal{D}$.

Corollary 1.5.11. Continua of class $\mathcal{D}$ are coastal.

Corollary 1.5.12. For each separable continuum X and point $x \in X$ some $p \in X$ makes the continuum component of x in $X - p$ dense.

Finally we can apply Theorem 1.3.8 to Corollary 1.5.11 to get the following.

Theorem 1.5.13. Every continuum whose density does not exceed its Baire characteristic $-$ in particular every separable continuum $-$ is irreducible about its set of non-block points.

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Chapter 2

Continuum Without Non-Block Points

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Abstract

For any composant $E \subset \mathbb{H}^*$ and corresponding near-coherence class $\mathscr{E} \subset \omega^*$ we prove the following are equivalent : (1) E properly contains a dense semicontinuum. (2) Each countable subset of E is contained in a dense proper semicontinuum of E. (3) Each countable subset of E is disjoint from some dense proper semicontinuum of E. (4) $\mathcal{E}$ has a minimal element in the finite-to-one monotone order of ultrafilters. (5) $\mathcal{E}$ has a Q-point. A consequence is that NCF is equivalent to $\mathbb{H}^*$ containing no proper dense semicontinuum and no non-block points. This gives an axiom-contingent answer to a question of the author. Thus every known continuum has either a proper dense semicontinuum at every point or at no points. We examine the structure of indecomposable continua for which this fails, and deduce they contain a maximum semicontinuum with dense interior.

2.1 Introduction

Non-block points are known to always exist in metric continua [8, 12]. Moreover it follows from Theorem 5 of [4] that every point of a metric continuum is included in a dense proper semicontinuum. We call a point with this property a coastal point. A coastal continuum is one whose every point is coastal.

The author's investigation of whether non-metric continua are coastal began in [1]. The problem was reduced to looking at indecomposable continua. Specifically it was shown that every non-coastal continuum X admits a proper subcontinuum K such that the quotient space X/K obtained by treating K as a single point is indecomposable and fails to be coastal (as a corollary this proves separable continua are coastal).

Since every indecomposable continuum with more than one composant is automatically coastal, the heart of the problem rests in those indecomposable (necessarily non-metric) continua with exactly one composant. We henceforth call these *Bellamy continua*, after David Bellamy who constructed the first example in ZFC [3]. There are very few examples known. The best-studied candidate is the Stone-Čech remainder $\mathbb{H}^*$ of the half-line. The composant number of $\mathbb{H}^*$ is axiom sensitive, but under the axiom Near Coherence of Filters (NCF) the composant number is exactly one [6]. In the first section of this paper, we show under NCF that $\mathbb{H}^*$ has neither coastal nor non-block points. Thus there consistently exists a non-coastal continuum.

It remains unresolved whether such a continuum can be exhibited without auxiliary axioms. The only other Bellamy continua of which the author is aware arise from an inverse limit process [3] [16] [17] . The process in fact yields a continuum with exactly two composants – which are then combined by identifying a point of each. The nature of this construction ensures that what used to be a composant is still a dense proper semicontinuum, and so these examples are easily shown to be coastal.

Thus every known Bellamy continuum is either coastal at every point or at none. One might wonder whether these are the only options. This question is addressed in the paper's final section, where we show what pathology a partially coastal Bellamy continuum must display.

2.2 Notation and Terminology

By a *continuum* we mean a compact connected Hausdorff space. We do not presume metrisability. The interior and closure of a subspace B are denoted B° and $\overline{B}$ respectively. The continuum X is said to be *irreducible* between two points $a, b \in X$ if no proper subcontinuum of X contains the subset $\{a, b\}$.

The topological space T is called *continuumwise connected* if for every two points $a, b \in T$ there exists a continuum $K \subset T$ with $\{a, b\} \subset K$. We also call a continuumwise connected space a *semicontinuum*. Every Hausdorff space is partitioned into maximal continuumwise connected subspaces. These are called the *continuum components*. When X is a continuum and $S \subset X$ a subset, we call S *thick* to mean it is proper and has nonvoid interior. The point $p \in X$ of a continuum is called a *weak cut point* to mean the subspace $X - p$ is not continuumwise connected. If $a, b \in X$ are in different continuum components of $X - p$ we say that p is *between* a and b and write $[a, p, b]$.

When X is a continuum the *composant* $\kappa(p)$ of the point $p \in X$ is the union of all proper subcontinua that include p. Another formulation is that $\kappa(p)$ is the set of points $q \in X$ for which X is not irreducible between p and q. For any points $x, p \in X$ we write $\kappa(x; p)$ for the continuum component of x in $X - p$. The point $x \in X$ is called *coastal* to mean that $\kappa(x; p)$ is dense for some $p \in X$. We call $p \in X$ a *non-block point* if $\kappa(x; p)$ is dense for some $x \in X$. From the definition, a continuum has a coastal point if and only if it has a non-block point, f and only if it contains a dense proper semicontinuum.

Throughout ω^* is the space of nonprincipal ultrafilters on the set $\omega = \{0, 1, 2, \ldots\}$ with topology generated by the sets $\widetilde{D} = \{\mathcal{D} \in \omega^* \colon D \in \mathcal{D}\}$ for all subsets $D \subset \omega$. Likewise $\mathbb{H}^*$ is the space of nonprincipal closed ultrafilters on $\mathbb{H} = \{x \in \mathbb{R} \colon x \geq 0\}$ with topology generated by the sets $\widetilde{U} = \{\mathcal{A} \in \mathbb{H}^* \colon A \subset U \text{ for some } A \in \mathcal{A}\}$ for all open subsets $U \subset \mathbb{H}$. For background on such spaces the reader is directed to [10] and [22].

$\mathbb{H}^*$ is known to be an *hereditarily unicoherent* continuum. That is to say any pair of its subcontinua have connected intersection. Moreover $\mathbb{H}^*$ is *indecomposable*, meaning we cannot write it as the union of two proper subcontinua. This is equivalent to every proper subcontinuum having void interior. The composants of an indecomposable continuum are pairwise disjoint.

For any two subsets $A, B \subset \mathbb{H}$ we write $A < B$ to mean $a < b$ for each $a \in A$ and $b \in B$. By a *simple sequence* we mean a sequence $I_n = [a_n, b_n]$ of closed intervals of $\mathbb{H}$ such that $I_1 < I_2 < I_3 < \ldots$ and the sequence a_n tends to infinity. Suppose $\mathbb{I} = \{I_1, I_2, \ldots\}$ is a simple sequence. For each subset $N \subset \omega$ define $I_N = \bigcup\{I_n \colon n \in N\}$. For each $\mathcal{D} \in \omega^*$ the set $\mathbb{I}_{\mathcal{D}} = \bigcap \left\{ \overline{I_D} \colon D \in \mathcal{D} \right\}$ is a subcontinuum of $\mathbb{H}^*$. These are called *standard subcontinua*. In case each sequence element is the singleton $\{a_n\}$ the corresponding standard subcontinuum is also a singleton, called a *standard point*, and we denote it by $a_{\mathcal{D}}$.

Throughout $\mathbb{I} = \{I_1, I_2, \ldots\}$ and $\mathbb{J} = \{J_1, J_2, \ldots\}$ are simple sequences. Each $I_n = [a_n, b_n]$ and $J_n = [c_n, d_n]$. For any choice of $x_n \in I_n$ the point $x_{\mathcal{D}}$ is called a regular point of $\mathbb{I}_{\mathcal{D}}$. Observe that while every regular point is standard, being regular is a relative notion. It makes no sense to say 'x is a regular point', only 'x is a regular point of $\mathbb{I}_{\mathcal{D}}$'.

Standard subcontinua have been studied under the guise of ultracoproducts of intervals [2]. This perspective makes certain properties more transparent. For example every standard-subcontinuum $\mathbb{I}_{\mathcal{D}}$ is uniquely irreducible between the regular points $a_{\mathcal{D}}$ and $b_{\mathcal{D}}$. We call these the *end points* of $\mathbb{I}_{\mathcal{D}}$ and denote them by a and b when there is no confusion. The set $\mathbb{I}_{\mathcal{D}} - \{a, b\}$ is called the interior of $\mathbb{I}_{\mathcal{D}}$.

There exists a natural preorder on $\mathbb{I}_{\mathcal{D}}$, where $x \sqsubseteq y$ means y is between x and b, or that every subcontinuum of $\mathbb{I}_{\mathcal{D}}$ that includes b and x must also include y. As per convention we write $x \sqsubset y$ to mean $x \sqsubseteq y$ but $x \neq y$. The equivalence classes of this preorder are linearly ordered and called the *layers* of $\mathbb{I}_\mathcal{D}$. Layers are indecomposable subcontinua. The layer of each regular point of $\mathbb{I}_\mathcal{D}$ is a singleton, and the set of these singletons is dense in $\mathbb{I}_\mathcal{D}$ in both the topological and order theoretic sense. For points $x, y \in \mathbb{I}_\mathcal{D}$ we write L^x and L^y for their layers, and write such things as $[x, y)$ to mean $\{z \in \mathbb{I}_\mathcal{D} \colon L^x \sqsubseteq L^z \sqsubset L^y\}$.

We define each $[x, z]$ to be intersection of all subcontinua that include the points $x, z \in \mathbb{I}_\mathcal{D}$. By hereditary unicoherence each $[x, z]$ is a subcontinuum, called a section of $\mathbb{I}_\mathcal{D}$. In case $x = x_\mathcal{D}$ and $y = y_\mathcal{D}$ are regular points then $[x, y]$ is just the standard subcontinuum $\mathbb{J}_\mathcal{D}$ where each $J_n = [x_n, y_n]$. By writing $[x, y)$ as the union of all segments $[x, z]$ for $x \sqsubseteq z \sqsubset y$ we see that $[x, y)$ is a semicontinuum.

For any function $g \colon \omega \to \omega$ and ultrafilter $\mathcal{D}$ on ω define the image $g(\mathcal{D}) = \{E \subset \omega \colon g^{-1}(E) \in \mathcal{D}\}$. It can be shown $g(\mathcal{D})$ is the ultrafilter generated by $\{g(D) \colon D \in \mathcal{D}\}$. Suppose $\mathcal{D}$ and $\mathcal{E}$ are ultrafilters and $f \colon \omega \to \omega$ is a finite-to-one function such that $f(\mathcal{D}) = \mathcal{E}$. Then we write $\mathcal{E} \lesssim \mathcal{D}$. If in addition we can choose f to be monotone we write $\mathcal{E} \leq \mathcal{D}$. The equivalence classes of $\leq$ are called *shapes* of ultrafilters. Lemma 2.3.8 is the referee's and illustrates how the partition into shapes is strictly finer than the partition into types.

Two free ultrafilters $\mathcal{D}$ and $\mathcal{E}$ are said to *nearly cohere* if they have a common lower bound relative to $\leq$. The principle Near Coherence of Filters (NCF) states that every two free ultrafilters nearly cohere. Blass and Shelah showed this assertion is consistent relative to ZFC [7] and Mioduszewski showed that NCF is equivalent to $\mathbb{H}^*$ being a Bellamy continuum [15]. Indeed it follows from Section 4 of Blass' [6] that the following correspondence is a bijecton between the composants of $\mathbb{H}^*$ and the near-coherence classes of ω^*: Given a composant $E \subset \mathbb{H}^*$ we can define the subset $\mathscr{E} = \{\mathcal{D} \in \omega^* \colon \text{ some } \mathbb{I}_\mathcal{D} \text{ is contained in } E\}$ of ω^*. Likewise for each near-coherence class $\mathscr{E} \subset \omega^*$ we can define the subset $E = \bigcup\{\mathbb{I}_\mathcal{D} \colon \mathbb{I} \text{ is a simple sequence and } \mathcal{D} \in \mathscr{E}\}$ of $\mathbb{H}^*$.

2.3 The Betweenness Structure of $\mathbb{H}^*$

This section establishes some tools concerning the subcontinua of $\mathbb{H}^*$ for later use. Our first concerns the representation of standard subcontinua. We would like to define the shape of $\mathbb{I}_\mathcal{D}$ to be the shape of $\mathcal{D}$. To prove that this makes sense we need the following result that follows from [10] Theorem 2.11 and the proof given for Theorem 5.3.

Theorem 2.3.1. Suppose that $\mathbb{J}_\mathcal{E} \subset \mathbb{I}_\mathcal{D}$. Then $\mathcal{D} \leq \mathcal{E}$. Moreover if $\mathcal{D} < \mathcal{E}$ as well then $\mathbb{J}_\mathcal{E}$ is contained in a layer of $\mathbb{I}_\mathcal{D}$.

Lemma 2.3.2. Each standard subcontinuum has a well-defined shape.

Proof. Suppose $\mathbb{I}_\mathcal{D}$ and $\mathbb{J}_\mathcal{E}$ are two representations of the same standard subcontinuum. That is to say $\mathbb{I}_\mathcal{D} = \mathbb{J}_\mathcal{E}$. Then we have $\mathbb{J}_\mathcal{E} \subset \mathbb{I}_\mathcal{D}$ and Theorem 2.3.1 says $\mathcal{D} \leq \mathcal{E}$. Applying the same theorem to how $\mathbb{I}_\mathcal{D} \subset \mathbb{J}_\mathcal{E}$ shows that $\mathcal{E} \leq \mathcal{D}$ as well. This is the definition of $\mathcal{D}$ and $\mathcal{E}$ having the same shape. □

Theorem 2.3.1 relates the $\leq$ ordering to the interplay between different standard subcontinua. It will be helpful to know something about $\leq$-minimal elements and hence about shapes of standard subcontinua that are maximal with respect to inclusion. Here the direction of the ordering is unfortunate. We mean the standard subcontinua $\mathbb{I}_\mathcal{D}$ with $\mathbb{I}_\mathcal{D} \subset \mathbb{I}_\mathcal{E}$ only for $\mathcal{D}$ and $\mathcal{E}$ with the same shape. It turns out the $\leq$-minimal elements are already well-studied. These ultrafilters are called Q-points and are usually defined as minimal elements of the $\lesssim$ ordering. Theorem 9.2 (b) of [9] can be used to prove the following two characterisations are equivalent.

Definition 2.3.3. We call $\mathcal{D} \in \omega^*$ a Q-point to mean it satisfies either (and therefore both) of the properties below.

(1) Every finite-to-one function $f \colon \omega \to \omega$ is constant or bijective when restricted to some element of $\mathcal{D}$.

(2) $\mathcal{D}$ is $\lesssim$-minimal. That means ($\mathcal{E} \lesssim \mathcal{D} \iff \mathcal{D} \lesssim \mathcal{E}$) for all $\mathcal{E} \in \omega^*$.

Condition (1) shows that when $\mathcal{D}$ is a Q-point and $f \colon \omega \to \omega$ finite-to-one then $f(\mathcal{D})$ is either principal or is a permutation of $\mathcal{D}$. The next lemma proves our assertion that Q-points are the $\leq$-minimal ultrafilters.

Lemma 2.3.4. The Q-points are precisely the $\leq$-minimal elements.

Proof. First suppose $\mathcal{D}$ is a Q-point and that $\mathcal{E} \leq \mathcal{D}$ for some $\mathcal{E} \in \omega^*$. That means $\mathcal{E} = f(\mathcal{D})$ for some $f \colon \omega \to \omega$ monotone finite-to-one. There exists an element $D \in \mathcal{D}$ over which f is bijective. The inverse $f^{-1} \colon f(D) \to \omega$ is bijective monotone and can be extended to a finite-to-one function on ω that maps $\mathcal{E}$ to $\mathcal{D}$. Therefore $\mathcal{D} \leq \mathcal{E}$ as required.

Now let $\mathcal{D}$ be $\leq$-minimal. We will show it is $\lesssim$-minimal as well. Suppose $\mathcal{E} \lesssim \mathcal{D}$ meaning $\mathcal{E} = f(\mathcal{D})$ where $f \colon \omega \to \omega$ is finite-to-one. Lemma 2.3 (2) of [11] shows how to construct finite-to-one monotone functions g and h with $g(\mathcal{D}) = h(\mathcal{E})$. By definition $\mathcal{E} \geq h(\mathcal{E})$ and $g(\mathcal{D}) \leq \mathcal{D}$. By $\leq$-minimality the second inequality implies $g(\mathcal{D}) \geq \mathcal{D}$. Then we have $\mathcal{E} \geq h(\mathcal{E}) = g(\mathcal{D}) \geq \mathcal{D}$ and therefore $\mathcal{E} \geq \mathcal{D}$ which implies $\mathcal{E} \gtrsim \mathcal{D}$ as required. $\square$

We will use the following result again and again to slightly expand a proper subcontinuum of $\mathbb{H}^*$.

Lemma 2.3.5. Each proper subcontinuum $K \subset \mathbb{H}^*$ is contained in $\mathbb{I}_{\mathcal{D}} - \{a, b\}$ for some standard subcontinuum $\mathbb{I}_{\mathcal{D}}$. Moreover if $\mathcal{E}$ is a Q-point in the near-coherence class corresponding to the composant containing K we may assume without loss of generality that $\mathcal{D} = \mathcal{E}$.

Proof. By Theorem 5.1 of [10] we know K is included in some standard subcontinuum $\mathbb{I}_{\mathcal{D}}$. For any positive constants $\varepsilon_1, \varepsilon_2, \ldots$ define the slightly larger intervals $I'_1 = [a_1, b_1 + \varepsilon_0]$ and $I'_n = [a_n - \varepsilon_n, b_n + \varepsilon_n]$ for each $n > 1$. The constants $\varepsilon_1, \varepsilon_2, \ldots$ may be chosen such that $\mathbb{I}' = \{I'_0, I'_1, \ldots\}$ is still a simple sequence. Then the end points of $\mathbb{I}'_{\mathcal{D}}$ are not elements of $\mathbb{I}_{\mathcal{D}}$ and therefore not elements of K, as required.

Suppose $\mathcal{E}$ shares a near-coherence class with $\mathcal{D}$. Then the proof of Theorem 4.1 of [6] shows $\mathbb{I}_{\mathcal{D}}$ is contained in some standard subcontinuum $\mathbb{J}_{f(\mathcal{D})} = \mathbb{J}_{f(\mathcal{E})}$ for $f \colon \omega \to \omega$ monotone finite-to-one. But $\mathcal{E}$ being a Q-point implies $f(\mathcal{E}) = \mathcal{E}$ and thus $\mathbb{I}_{\mathcal{D}} \subset \mathbb{J}_{\mathcal{E}}$. Then we may rename $\mathbb{J}$ to $\mathbb{I}$ and expand each interval slightly as before. $\square$

We show how the ordering of layers of $\mathbb{I}_{\mathcal{D}}$ relates to the weak cut point structure of $\mathbb{H}^*$.

Lemma 2.3.6. Suppose $p \in \mathbb{I}_{\mathcal{D}}$ is not an end point. Then $[a, p)$ and $(p, b]$ are continuum components of $\mathbb{I}_{\mathcal{D}} - p$.

Proof. Since $[a, p)$ and $(p, b]$ are semicontinua each is contained in a continuum component of $\mathbb{I}_{\mathcal{D}} - p$. Moreover if a and b share a continuum component of $\mathbb{I}_{\mathcal{D}} - p$ we would have $\{a, b\} \subset R \subset \mathbb{I}_{\mathcal{D}} - p$ for some subcontinuum $R \subset \mathbb{H}^*$. But this contradicts how $\mathbb{I}_{\mathcal{D}}$ is irreducible between its endpoints. Finally observe that, by how layers are defined, every subcontinuum of $\mathbb{I}_{\mathcal{D}}$ that joins a to an element of L^p must contain L^p and thus p. Therefore the continuum component of a is contained in $\mathbb{I}_{\mathcal{D}} - L^p \cup (p, b] = [a, p)$ as required. Likewise for $(p, b]$. $\square$

Combining Lemmas 2.3.5 and 6.5.2 gives the following.

Lemma 2.3.7. Every point of $\mathbb{H}^*$ is between two points of its composant. In particular suppose $p \in \mathbb{I}_{\mathcal{D}}$ is not an end point. Then p is between a and b.

Proof. Let $p \in \mathbb{H}^*$ be arbitrary. By Lemma 2.3.5 we have $p \in \mathbb{I}_{\mathcal{D}} - \{a, b\}$ for some standard subcontinuum $\mathbb{I}_{\mathcal{D}}$. By Lemma 6.5.2 we know p is between a and b in $\mathbb{I}_{\mathcal{D}}$. But since $\mathbb{H}^*$ is hereditarily unicoherent this implies that p is between a and b in $\mathbb{H}^*$ as well. Finally observe that since $\mathbb{I}_{\mathcal{D}} \subset \mathbb{H}^*$ is a proper subcontinuum the points a, b and p share a composant. $\square$

Of course if $\mathbb{H}^*$ has more than one composant, every point is readily seen to be a weak cut point. But even then, it is not obvious that every point is between two points in its composant. Nor should we expect this to be true for all indecomposable continua. The result fails for the Knaster Buckethandle when removing its single end point – what is left of its composant even remains arcwise connected.

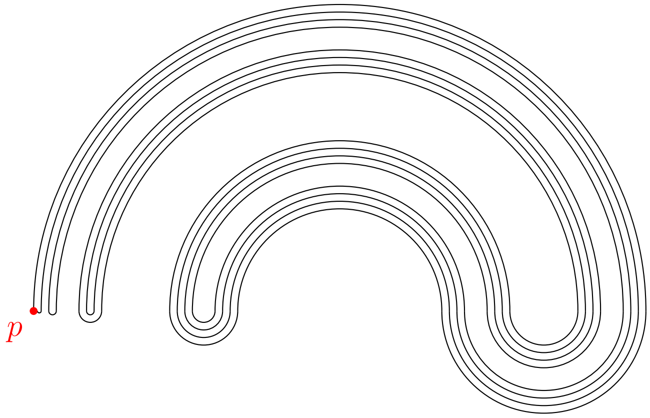

Figure 2.1: The end point p does not cut its composant.

We finish by giving the referee's example of two ultrafilters $\mathcal{D}$ and $\mathcal{E}$ such that $\mathcal{E} \lesssim \mathcal{D}$ but not $\mathcal{E} \leq \mathcal{D}$. This demonstrates how the partition of ω^* into shapes is strictly finer than the partition into types.

Lemma 2.3.8. The partition of ω^* into shapes is strictly finer than the partition into types.

Proof. Partition ω into intervals $I_0 = \{0\}$ and $I_n = [2^{n-1}, 2^n)$ for all $n > 0$. Define the filter $\mathcal{F}$ by letting $F \in \mathcal{F}$ exactly if $\big\{|I_n - F| \colon n \in \omega\big\}$ is bounded. Choose $\mathcal{D}$ as any ultrafilter extending $\mathcal{F}$. Observe each $D \in \mathcal{D}$ contains more than one element of some I_k; otherwise $F = D^c$ is an element of $\mathcal{F}$ because each $|I_n - F|$ is bounded above by 2, and this contradicts how D is an ultrafilter.

Let the permutation σ reverse the order of elements in each I_n and define $\mathcal{E} = \sigma(\mathcal{D})$. Theorem 9.2 (a) of [9] says $\mathcal{D} \neq \mathcal{E}$ and Theorem 9.3 says $\mathcal{D}$ and $\mathcal{E}$ have the same type. It remains to show they have different shapes.

By definition $f(\mathcal{D}) = \mathcal{E}$ implies $(\sigma^{-1} \circ f)(\mathcal{D}) = \mathcal{D}$. Then Theorem 9.2 (b) of [9] says $\sigma^{-1} \circ f$ is the identity, and hence $f = \sigma$, over some set $D \in \mathcal{D}$. Now let $a, b \in D \cap I_k$ be distinct for some $k \in \omega$. It follows that f reverses the order of a and b. Therefore f is not monotone and we cannot have $\mathcal{E} \leq \mathcal{D}$. Therefore the shapes are different. $\square$

2.4 The Presence of Q-Points

The number of near-coherence classes of ω^* (and hence composants of $\mathbb{H}^*$) is axiom-sensitive. Likewise for the distribution of Q-points in ω^*. It is true in ZFC that there always exists a class without Q-points [11]. At the same time it follows from Theorem 9.23 of [22] that under CH there are also $2^{\mathfrak{c}}$ classes with Q-points. Conversely NCF implies there exists a single class and no Q-points [5]. Moreover it was recently shown [14] that for each $n \in \mathbb{N}$ there may exist exactly n classes with Q-points and one class without.

Under the assumption that $\mathbb{H}^*$ has more than one composant each point is non-block and coastal for trivial reasons: Every composant $E \subset \mathbb{H}^*$ is a proper dense semicontinuum that witness how each $x \in E$ is coastal and how each $x \notin E$ is non-block. We are interested in whether this is the only reason a point can be non-block or coastal. This leads to the following definition.

Definition 2.4.1. The subset $P \subset X$ is called a *proper non-block set* to mean that P is contained in some composant E of X, and that some continuum component of $E - P$ is dense in X. The subset $P \subset X$ is called a *proper coastal set* to mean that P is contained in a dense semicontinuum that is not a composant of X. Supposing the singleton $\{p\}$ is a proper non-block (coastal) set we call p a proper non-block (coastal) point.

It turns out the existence of proper coastal and non-block sets in a composant depends on whether the corresponding near coherence class has a Q-point or not. We will examine the two possibilities separately. Henceforth $\mathcal{D}$ is assumed to be a Q-point whose near coherence class corresponds to the composant $A \subset \mathbb{H}^*$. We are grateful to the referee for correcting our earlier misconception about this case, and for providing the proof of the following lemma.

Lemma 2.4.2. For any $p \in \mathbb{I}_{\mathcal{D}} - \{a, b\}$ the semicontinuua $\kappa(b; p)$ and $\kappa(a; p)$ are dense.

Proof. We only consider $\kappa(b;p)$ because the other case is similar. There is a regular point q of $\mathbb{I}_{\mathcal{D}}$ such that $p \sqsubset q \sqsubset b$. We showed in Lemma 2.3.7 that $[p,q,b]$ which implies $\kappa(b;q) \subset \kappa(b;p)$. Therefore it suffices to show $\kappa(b;q)$ is dense.

Assuming $\kappa(b;q)$ is not dense it must be nowhere dense since $\mathbb{H}^*$ is indecomposable. Then $\overline{\kappa(b;q)}$ is a proper subcontinuum and thus by Lemma 2.3.5 is contained in the interior of some standard subcontinuum $\mathbb{J}_{\mathcal{D}}$ where each $J_n = [c_n, d_n]$.

It follows that q and b are regular points of $\mathbb{J}_{\mathcal{D}}$ and moreover $q \sqsubset b \sqsubset d$ in $\mathbb{J}_{\mathcal{D}}$. But then the interval $[b,d]$ of $\mathbb{J}_{\mathcal{D}}$ witnesses how $d \in \kappa(b;q)$. But by construction $d \notin \kappa(b;q)$, a contradiction. □

We can use the semicontinua constructed in Lemma 2.4.2 to show any countable subset of A is both proper coastal and proper non-block.

Theorem 2.4.3. Every countable $P \subset A$ is a proper non-block set and a proper coastal set.

Proof. Let $P = \{p_1, p_2, \ldots\}$. Since all p_i share a composant we can use Lemma 2.3.5 to form an increasing chain $K_1 \subset K_2 \subset \ldots$ of standard subcontina of shape $\mathcal{D}$ such that each $\{p_1, p_2, \ldots, p_n\} \subset K_n$. By the Baire Category theorem the union $\bigcup K_n$ is proper. The complement of $\bigcup K_n$ is a nonempty G_δ set. Section 1 of [13] proves the complement has nonvoid interior. This implies $\bigcup K_n$ cannot be dense, and so $\overline{\left(\bigcup K_n\right)}$ is contained in the interior of some further standard subcontinuum $\mathbb{I}_{\mathcal{D}}$. Observe that $\mathbb{I}_{\mathcal{D}} \subset A$.

There exists a regular point q of $\mathbb{I}_{\mathcal{D}}$ such that $a \sqsubset x \sqsubset q \sqsubset b$ for each $x \in \overline{\left(\bigcup K_n\right)}$. Moreover Lemma 6.5.2 implies that $x \notin \kappa(b;q)$ and $b \notin \kappa(x;q)$. Lemma 2.4.2 says the semicontinuua $\kappa(b;q)$ and $\kappa(x;q) = \kappa(a;q)$ are both dense. Moreover the subcontinuum $\overline{\left(\bigcup K_n\right)}$ witnesses how all $\kappa(p_i;q)$ coincide with each other and with $\kappa(x;q)$. Therefore $\kappa(b;q)$ witnesses how P is a proper non-block set and $\kappa(p_1;q)$ witnesses how P is a proper coastal set. $\square$

One can ask whether Theorem 2.4.3 can be strengthened by allowing the set P to have some larger cardinality. In particular we might look for the least cardinal $\eta(A)$ such that every element of $\{P \subset A \colon |P| < \eta(A)\}$ is a proper coastal set and a proper non-block set.

To see the number $\eta(A)$ is at most $2^{\aleph_0}$ consider the family P of all standard points $a_{\mathcal{D}}$ for each a_n rational. The family has cardinality $|\mathbb{Q}^{\mathbb{N}}| = {\aleph_0}^{\aleph_0} = 2^{\aleph_0}$ and is easily seen to be dense in A. Therefore P cannot be a proper coastal set. Moreover under the Continuum Hypothesis $\aleph_0$ has successor $2^{\aleph_0}$ and so it is consistent that $\eta(A) = 2^{\aleph_0}$.

Question 2.4.4. Is it consistent that there exists a composant $A \subset \mathbb{H}^*$ where the corresponding near coherence class has Q-points and $\eta(A) < 2^{\aleph_0}$?

Question 2.4.5. Is it consistent that there exist composants $A, A' \subset \mathbb{H}^*$ where the corresponding near coherence classes have Q-points and $\eta(A) \neq \eta(A')$?

Next we will treat the case when the composant $B \subset \mathbb{H}^*$ corresponds to a near coherence class without a Q-point. We remark it is consistent for the two composants A and B to exist simultaneously. For example in the model presented in [14] or indeed any model where the Continuum Hypothesis holds.

The outcome for B is the complete opposite to that for A – the composant B has neither proper coastal points nor proper non-block points. Our main tool to prove this is the following lemma, which is alluded to in the literature – for example in [11] – but for which we have been unable to find a complete proof.

Lemma 2.4.6. B is the union of an increasing chain of proper indecomposable subcontinua.

Proof. By Zorn's lemma there exists a maximal increasing chain $\mathcal{P}$ of proper indecomposable subcontinua in B. We claim the union $\bigcup \mathcal{P}$ is dense. For otherwise by Lemma 2.3.5 we have $\bigcup \mathcal{P} \subset \mathbb{I}_\mathcal{D} \subset B$ for some standard subcontinuum $\mathbb{I}_\mathcal{D}$. Then since $\mathcal{D}$ is not a Q-point it is not $\leq$-minimal by Lemma 2.3.4. Therefore we have $\mathcal{E} < \mathcal{D}$ for some $\mathcal{E} \in \omega^*$. It follows from Theorem 2.3.1 that $\mathbb{I}_\mathcal{D}$ is contained in a layer of $\mathbb{I}_\mathcal{E}$. But that layer is a proper indecomposable subcontinuum and so can be added as the new top element of $\bigcup \mathcal{P}$, contradicting how the chain is maximal. We conclude $\bigcup \mathcal{P}$ is dense.

Compactness implies there is some $x \in \bigcap \mathcal{P}$. To prove $\bigcup \mathcal{P} = B$ we take $b \in B$ to be arbitrary. There exists a standard subcontinuum L with $\{x, b\} \subset L$. We have already shown $\bigcup \mathcal{P}$ is dense. That means some $P \in \mathcal{P}$ is not contained in L. By Theorem 5.9 of [10] we have $L \subset P$. This implies $b \in P \subset \bigcup \mathcal{P}$ as required. □

Theorem 2.4.7. *B has no proper coastal points and no proper non-block points.*

Proof. Let $\mathcal{P}$ be an increasing chain of proper indecomposable subcontinua with union B. Recall each $P \in \mathcal{P}$ is nowhere-dense. Now let $S \subset B$ be an arbitrary proper semicontinuum. That means we can fix a point $y \in S$ and write $S = \bigcup_{x \in S} S(x)$ where each $S(x)$ is a subcontinuum containing $\{x, y\}$.

Choose any point $b \in (B - S)$. There exists P such that $\{b, y\} \in P \subset \mathcal{P}$. The point b witnesses how $P \not\subset S(x)$ and so Theorem 5.9 of [10] implies that $S(x) \subset P$. But since $x \in S$ is arbitrary this implies $S \subset P$. Therefore S is nowhere-dense.

We conclude that B contains no proper dense semicontinuum. It follows B has no proper coastal points and therefore no proper non-block points. □

Under NCF there are no Q-points. In this case Theorem 2.4.7 tells us $\mathbb{H}^*$ has no proper non-block points. But NCF is also equivalent to $\mathbb{H}^*$ having exactly one composant, and that implies any non-block points that exist must be proper. So we have the stronger result.

Theorem 2.4.8. (NCF) *$\mathbb{H}^*$ lacks coastal points and non-block points.*

Every separable continuum and *a fortiori* every metric continuum has two or more non-block points. The author has asked whether the separability assumption can be dropped. Theorem 2.4.8 gives an axiom-contingent answer.

Corollary 2.4.9. There consistently exists a continuum without non-block points.

Whether Corollary 2.4.9 can be proved in ZFC alone is currently unresolved. One possible line-of-attack to the problem is as follows: Observe that every composant of an hereditarily indecomposable continuum is the union of the same sort of chain as described in Lemma 26. From here a similar proof to Theorem 2.4.7 shows hereditarily indecomposable continua lack proper non-block points. Thus an hereditarily indecomposable Bellamy continuum would be an example of a continuum without non-block points. Smith has found some obstacles to constructing such a beast [18, 19, 20, 21]. But we have so far no reason to believe one exists.

We can combine the main results of this section into two sets of equivalences. The first set looks at each composant separately.

Corollary 2.4.10. The following are equivalent for any composant $E \subset \mathbb{H}^*$ and corresponding near coherence class $\mathscr{E} \subset \omega^*$.

1. Some point of E is proper non-block (coastal).
2. Every point of E is proper non-block (coastal).
3. Some countable subset of E is proper non-block (coastal).
4. Every countable subset of E is proper non-block (coastal).
5. $\mathcal{E}$ has a Q-point.
6. $\mathcal{E}$ has a $\leq$-minimal element.

The second set looks at $\mathbb{H}^*$ as a whole.

Corollary 2.4.11. The following are equivalent.

(1) NCF

(2) $\mathbb{H}^*$ has exactly one composant

(3) $\mathbb{H}^*$ lacks coastal points and non-block points.

In particular we have that – regardless of the model – it can only be the case that either every point of $\mathbb{H}^*$ is non-block or none are. This observation motivates the next and final section.

2.5 Partially Coastal Bellamy Continua

Thus far every Bellamy continuum has proved to be either coastal at every point or at no points. This section examines the remaining case. Henceforth H is some fixed Bellamy continuum. We will make frequent use of the fact that every semicontinuum $S \subset H$ is either dense or nowhere-dense. Under the assumption that there is a coastal point $x \in H$ and a non-coastal point $y \in H$, this section investigates how badly behaved H must be.

Our description is in terms of thick semicontinua. Recall the semicontinuum $S \subset H$ is called thick to mean it is proper and has nonvoid interior. Every indecomposable metric continuum has more than one composant and so cannot contain a thick semicontinuum. It is unknown whether the result generalises – no known Bellamy continuum contains a thick semicontinuum. Thus the following lemma explains our failure to provide a concrete example for the continuum H.

Lemma 2.5.1. H contains a thick semicontinuum. Moreover every dense proper semicontinuum in H is thick.

Proof. Let S be an arbitrary dense proper semicontinuum. At least one exists to witness how the point $x \in H$ is coastal. Since H has one composant there is a proper subcontinuum $K \subset H$ with $\{x, y\} \subset K$. Then $K \cup S$ is also a dense semicontinuum. But since y is non-coastal we must have $K \cup S = H$. Therefore S contains the open set $H - K$ and hence has nonvoid interior. □

Lemma 2.5.2. H has a thick semicontinuum that contains every other thick semicontinuum.

Proof. Let S be the thick semicontinuum found in Lemma 2.5.1. Every other thick semicontinuum $M \subset H$ is dense, and since S has interior this implies $S \cup M$ is a semicontinuum. Moreover $y \notin M$ since y is non-coastal and hence $S \cup M$ is proper. It follows the union of S with all possible choices for M is the maximum among thick semicontinua of H. □

Henceforth we will fix $S \subset H$ to be the maximum thick semicontinuum.

Lemma 2.5.3. S is a continuum component of $H - p$ for each $p \in H - S$.

Proof. We know S is contained in some continuum component C of $H - p$. But since a continuum component is a semicontinuum this implies $C \subset S$ by maximality of S and therefore $C = S$. $\square$

Lemma 2.5.4. $H - S^\circ$ is a subcontinuum and one of two things happens.

(1) The thick semicontinuum S is open

(2) $H - S^\circ$ is indecomposable with more than one composant

Proof. Observe that by boundary-bumping the arbitrary point p is in the closure of every continuum component of $H - p$. Therefore any union of continuum components of $H - p$ has connected closure. Lemma 2.5.3 says S is a continuum component of $H - p$. Therefore $\overline{(H - p - S)} = H - S^\circ$ is connected and hence a continuum. Call this continuum B.

There is a partition $B = A \cup C$ where $A = H - S$ is the complement of S and $C = B \cap S$ consists of the points of S outside its interior. Since S is proper A is nonempty. If we assume S is not open then C is nonempty as well.

We will demonstrate that B is irreducible between each $a \in A$ and each $c \in C$. Since A and C form a partition this will imply they are both unions of composants. In particular B will have two disjoint composants making it indecomposable.

Now suppose $E \subset B$ is a subcontinuum that meets each of A and C. Since E meets $C = B \cap S$ we know that $S \cup E$ is continuumwise connected. But since E meets $A = H - S$ we know $S \cup E$ is strictly larger than S. By assumption S is a maximal thick semicontinuum. So the only option is that $S \cup E = H$.

In particular $S \cup E$ contains $A = H - S$. This implies $A \subset E$ and since E is closed $\overline{A} \subset E$. But by definition $\overline{A} = \overline{(H - S)} = H - S^\circ = B$. We conclude $E = B$ as required. $\square$

The previous lemma showed that $H - S^\circ$ is a subcontinuum. Recall that, since H is indecomposable, its every subcontinuum has void interior. This gives us the corollary.

Corollary 2.5.5. S has dense interior.

Now we are ready to identify the coastal points of H.

Lemma 2.5.6. H is coastal exactly at the points of S.

Proof. Since S is the maximum dense proper semicontinuum, each of its points is coastal. Now let $x \in H$ be an arbitrary coastal point. That means x is an element of some proper dense semicontinuum which is, by definition, contained in S. This implies $x \in S$ as required. $\square$

We can summarise the progress made in this section in the theorem.

Theorem 2.5.7. Suppose S is the set of coastal points of the Bellamy continuum H and $\varnothing \neq S \neq H$. Then S is a semicontinuum with dense interior and contains every semicontinuum of H with nonempty interior. Moreover $H - S^\circ$ is a subcontinuum and one of the below holds.

(1) The semicontinuum S is open

(2) $H - S^\circ$ is an indecomposable continuum with more than one composant

We conclude this section with a remark on the three classes of continua. Continua of the first class are coastal at every point – for example all metric or separable continua [1]. Continua of the second class have no coastal points – for example $\mathbb{H}^*$ under NCF.

Continua of the third (and possibly empty) class are coastal only at the points of some proper subset. However, as Theorem 2.5.7 tells, these continua have the extra property of being *simultaneously coastal*. That means the set of coastal points knits together into a dense proper semicontinuum that simultaneously witnesses the coastal property for each of its points.

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Chapter 3

The Shore Point Existence Problem is Equivalent to the Non-Block Point Existence Problem

Abstract

We prove the three propositions are equivalent: (a) Every Hausdorff continuum has two or more shore points. (b) Every Hausdorff continuum has two or more non-block points. (c) Every Hausdorff continuum is coastal at each point. Thus it is consistent that all three properties fail. We also give the following characterisation of shore points: The point p of the continuum X is a shore point if and only if there is a net of subcontinua in $\{K \in C(X) : K \subset \kappa(p) - p\}$ tending to X in the Vietoris topology. In addition we prove every point of an indecomposable continuum is a shore point.

3.1 Introduction

Leonel [8] has improved the classic non-cut point theorem of Moore [10] by showing every metric continuum has two or more shore points. Bobok, Pyrih and Vejnar [5] observed Leonel's two shore points have the stronger property of being non-block points.

In [2] the author proved it is consistent the result fails to generalise to Hausdorff continua. Under Near Coherence of Filters (NCF) the Stone-Čech remainder $\mathbb{H}^*$ of the half-line lacks non-block points and hence lacks coastal points.

This left open the question of whether there is a consistent example of a Hausdorff continuum without shore points. This paper gives a positive answer. Indeed we show the shore point and non-block point existence problems are equivalent. They are also equivalent to a number of other problems involving shore, non-block, and coastal points of Hausdorff continua.

We also prove every shore point $p \in X$ has the stronger property of being a *proper shore point*. That means there is a net of subcontinua in the hyperspace $\{K \in C(X) : K \subset \kappa(p) - p\}$ tending to X in the Vietoris topology. This is not apparent from the definition of a shore point, which only requires the net elements be contained in $X - p$.

3.2 Terminology and Notation

For sets A and B define $A - B = \{a \in A \colon a \notin B\}$. For $B = \{b\}$ we write $A - b$ without confusion. For $A \subset B$ we do not presume A is a proper subset of B. For a subset $S \subset X$ denote by S° and $\overline{S}$ the interior and closure of S respectively. The *boundary* of S means the set $\partial S = \overline{S} \cap \overline{(X - S)}$.

Throughout X is a continuum. That is to say a nondegenerate compact connected Hausdorff space. For background on metric continua see [7] and [11]. The results cited here have analagous proofs for non-metric continua.

Throughout all maps are assumed to be continuous. The map $f : X \to Y$ of continua is called *monotone* to mean $f^{-1}(y) \subset X$ is connected for each $y \in Y$. Theorem 6.1.28 of [6] says moreover $f^{-1}(K) \subset X$ is a continuum for each subcontinuum $K \subset Y$.

For $a, b \in X$ we call X *irreducible* about $\{a, b\}$ to mean $\{a, b\}$ is not contained in a proper subcontinuum of X. The subspace $A \subset X$ is called a *semicontinuum* to mean for each $a, c \in A$ some subcontinuum $K \subset A$ has $\{a, c\} \subset K$. Every subspace $A \subset X$ is partitioned into maximal semicontinua called the *continuum components* of A.

For $N \geq 2$ we say the subcontinua $X_1, \ldots, X_N \subset X$ form a *decomposition* and write $X_1 \oplus \ldots \oplus X_N$ to mean $X_1 \cup \ldots \cup X_N = X$ and no X_n is contained in the union of the others. We call X *decomposable* to mean it admits a decomposition and *indecomposable* otherwise. The latter is equivalent to admitting no decomposition with $N = 2$ and equivalent to each proper subcontinuum being nowhere dense. We say X is *hereditarily indecomposable* to mean its every subcontinuum is indecomposable. Equivalently each pair of subcontinua are either disjoint or nested.

The *composant* $\kappa(x)$ of the point $x \in X$ is the union of all proper subcontinua that have x as an element. Indecomposable metric continua are partitioned into $\mathfrak{c}$ many pairwise disjoint composants [9]. In case $\kappa(x) \neq \kappa(y)$ then X is irreducible about $\{x, y\}$.

By *boundary bumping* we mean the principle that, for each proper closed $E \subset X$, each component C of E meets the boundary $\partial E = \overline{E} \cap \overline{X - E}$. For the non-metric proof see §47, III Theorem 2 of [7]. One corollary of boundary bumping is that any $p \in X$ is in the closure of each continuum component of $X - p$.

Throughout $C(X)$ is the set of subcontinua of X. We call $p \in X$ a *shore point* to mean for each finite collection of open sets $U_1, U_2, \ldots U_n \subset X$ some subcontinuum $K \subset X - p$ meets each U_m. This is equivalent to some net in $\{K \in C(X) : K \subset X - p\}$ tending to X in the Vietoris topology. We do not need the full definition of the Vietoris topology here.

We call $p \in X$ a *cut point* to mean $X - p$ is disconnected and a *non-cut point* otherwise. Clearly each shore point is non-cut. We call $p \in X$ a *non-block point* to mean $X - p$ has a dense continuum component. Every non-block point is a shore point but the converse fails in general [5]. We call $x \in X$ a *coastal point* to mean x is an element of some proper dense semicontinuum. Clearly X has a non-block point if and only if it has a coastal point.

Theorem 5 of [3] says every point of a metric continuum is coastal. The result generalises to separable continua [1] but not to Hausdorff continua. Under the set-theoretic axiom Near Coherence of Filters the non-metric continuum $\mathbb{H}^*$ lacks non-block points and hence lacks coastal points [2].

3.3 Equivalent Problems

Definition 3.3.1. The continuum X is called *partially coastal* to mean there are points $x, y \in X$ with x coastal and y non-coastal.

Theorem 1. The following propositions are equivalent.

(1) There exists a continuum without coastal points.

(2) There exists a continuum with exactly one non-coastal point.

(3) There exists a partially coastal continuum.

(4) There exists a continuum without shore points.

(5) There exists a continuum with exactly one shore point

(6) There exists a continuum without non-block points.

(7) There exists a continuum with exactly one non-block point.

Proof. $(6) \implies (1)$ because X has a non-block point if and only if it has a coastal point. $(2) \implies (3)$ follows from how every continuum has more than one point. $(4) \implies (6)$ follows from how every non-block point is a shore point.

$(1) \implies (2)$: Suppose X has no coastal points. Let $[0,1]$ be an arc and the continuum Y be obtained by gluing $1 \in [0,1]$ to any fixed $p \in X$. Identify X and $[0,1]$ with their images in Y. The fact each $x \in Y - 0$ is coastal is witnessed by the dense semicontinuum $S = \big(\bigcup\{(1/n, 1] : n \in \mathbb{N}\}\big) \cup X = Y - 0$.

To see $0 \in Y$ is non-coastal suppose otherwise. That means $0 \in S \subset Y - q$ for some dense proper semicontinuum $S \subset Y$ and point $q \in Y$. It is easy to see $[0,1] \subset S$. Hence $q \notin [0,1]$ and $q \in X - p$.

Let $Q : Y \to X$ be the monotone map that collapses $[0,1]$ to the point $p \in X$ and leaves the points of X fixed. Then $Q(S) \subset X$ is a dense semicontinuum with $p \in Q(S) \subset X - q$ contradicting the assumption that $p \in X$ is non-coastal.

$(1) \implies (5)$: Suppose X has no coastal points and define Y, Q and S as before. The dense semicontinuum S witnesses how $0 \in Y$ is a non-block point hence a shore point.

To see 0 is the only shore point first observe each $q \in (0, 1]$ is a cut point hence not a shore point. Now suppose some $q \in Y - [0, 1]$ is a shore point. Fix the open set $U = (0, 1)$. By assumption, for every collection of open sets $U_1, \ldots, U_n \subset X - 1$, some subcontinuum $K \subset Y - q$ meets U and all U_m. Since K meets U and U_1 we have $1 \in K$ and hence $p \in Q(K)$.

Now allow $\{U_1, \ldots, U_n\}$ to range over all finite collections of open subsets of X. The subcontinua $Q(K) \subset X - q$ constitute a dense proper semicontinuum of X. Therefore $p \in X$ is coastal contrary to assumption. We conclude q is not a shore point as required. The proof of $(1) \implies (7)$ is identical.

$(3) \implies (4)$: Suppose some $p \in X$ is non-coastal. Take two disjoint copies X_1 and X_2 of X. Let Y be the continuum obtained from $X_1 \sqcup X_2$ by identifying the points $a \in X_1$ and $b \in X_2$ corresponding to $p \in X$. Identify X_1 and X_2 with their images in Y and write $p \in Y$ for the shared image of a and b. We claim Y has no shore points.

Observe $Y - p$ is homeomorphic to the disjoint union of $X_1 - a$ and $X_2 - b$. Hence p is a cut point and not a shore point. Now suppose $q \in Y - p$ is a shore point. Without loss of generality $q \in X_2$. Fix some open set $U \subset X_1 - p$. For every collection of open sets $U_1, \ldots, U_n \subset X_2 - p$ some subcontinuum $K \subset Y - q$ meets U and all U_m. Since K meets U and U_1 we must have $p \in K$. Moreover $K - p$ is disconnected and each component lies in one of X_1 or X_2.

Let $\mathcal{K}$ be the family of components that lie in X_2. Boundary bumping says $L \cup \{p\}$ is a continuum for each $L \in \mathcal{K}$. It follows $K \cap X_2 = \bigcup \big\{ \{p\} \cup L : L \in \mathcal{K} \big\}$ is a subcontinuum of $X_2 - q$ that meets all $U_1, \ldots, U_n$. Now allow $\{U_1, \ldots, U_n\}$ to range over all finite collections of open subsets of X_2. Then the union M of all $K \cap X_2$ is a dense semicontinuum of $X_2 - q$. Let $Q : Y \to X_2$ be that map that compresses X_1 to the point $b \in X_2$. Then the semicontinuum $Q(M)$ contradict how X_2 is not coastal at $b \in X_2$. We conclude no $q \in Y$ is a shore point as required.

$(5) \implies (1)$: Suppose $p \in X$ is the unique shore point. We claim p is not coastal, For then $p \in S \subset X - q$ for some $q \in X$ and dense proper semicontinuum S. This implies q is a non-block point and hence a shore point contrary to assumption. This implies either (1) or (3) which we have shown are equivalent.

$(7) \implies (3)$: Suppose $p \in X$ is the unique non-block point. Since there is a non-block point there is also a coastal point. But $p \in X$ cannot be coastal as this would imply some $q \in X - p$ is a non-block point. hence X is partially coastal.

□

Under the set-theoretic axiom Near Coherence of Filters (NCF) the Stone-Čech remainder $\mathbb{H}^*$ of the half-line is a continuum with no coastal points [2]. From this we get the corollary.

Corollary 3.3.2. Propositions $(1) - (7)$ from Theorem 1 are consistent.

The author has shown for each non-coastal $p \in X$ there exists a subcontinuum $M \subset X$ with $p \in M^\circ$ and X/M indecomposable and non- coastal at M [1]. This raises the question whether $(1) - (7)$ are equivalent if we demand the continuum in question be indecomposable.

Section 5 shows (4) and (5) never hold for indecomposable continua. In particular Theorem 3 says every point of an indecomposable continua is a shore point.

For X indecomposable the remaining propositions seem more difficult and the methods of Theorem 1 no longer apply. For example if X is indecomposable with (1) we cannot simply attach an arc to prove (2) as the quotient space is manifestly decomposable.

In the other direction X having (7) does not imply the same for X/M. For example assume NCF and glue any point $p \in \mathbb{H}^*$ to the endpoint $1 \in [0, 1]$. Let

X be the quotient space. Clearly for X/M to be indecomposable M is the union of the arc and some proper subcontinuum $K \subset \mathbb{H}^*$ with $p \in K$. Since the map $\mathbb{H}^* \to \mathbb{H}^*/K$ is monotone it follows $X/M \cong \mathbb{H}^*/K$ lacks non-block points and fails (7).

Before proceeding to Section 5 we take a diversion to prove all shore points are proper shore points. Some of the terminology and results from Section 4 will be needed for Section 5.

3.4 Proper Shore and Non-Block Points

Definition 3.4.1. Recall $p \in X$ is called a *shore point* to mean for each finite collection of open sets $U_1, U_2, \ldots, U_N \subset X$ some subcontinuum $K \subset X - p$ meets each U_n. We call p a *proper shore point* to mean K can always be chosen as a subset of $\kappa(p) - p$. Otherwise we call p a *trivial shore point*.

Definition 3.4.2. Recall $p \in X$ is called a *non-block point* to mean $X - p$ has a dense continuum component. We call p a *proper non-block* point to mean $\kappa(p)-p$ has a dense continuum component. Otherwise we call p a *trivial non-block point*.

This section shows there exist both proper and trivial non-block points but there is no such thing as a trivial shore point. Thus being a proper non-block point is a meaningful notion but being a proper shore point is not. Hereditarily indecomposable metric continua provide an example of the former.

Lemma 3.4.3. Suppose M is an hereditarily indecomposable metric continuum. Each $p \in M$ is a trivial non-block point.

Proof. Recall M has $\mathfrak{c}$ many pairwise disjoint composants each of which is a dense semicontinuum [9]. Each composant other than $\kappa(p)$ witnesses how p is a non-block point. It remains to show each continuum component C of $\kappa(p) - p$ is nowhere dense.

Let $q \in C$ be arbitrary. We can write $C = \bigcup \mathcal{C}$ as a union of proper subcontinua with $q \in D$ for each $D \in \mathcal{C}$. Since $q \in \kappa(p)$ there exists a proper subcontinuum K with $\{p, q\} \subset K$. Clearly K meets but is not contained in each $D \in \mathcal{C}$. Hereditary indecomposability implies each $D \subset K$ and therefore $C = \bigcup \mathcal{C} \subset K$. Since X is indecomposable K and thus C is nowhere dense as required. $\square$

Observe we only required M to have more than one composant. The problem is open whether there exists an hereditarily indecomposable Hausdorff continuum with exactly one composant. For obstructions to finding such a space see Smith [12, 13, 14, 15].

For $\kappa(p) = X$ clearly the shore point p is proper. Henceforth assume X is irreducible about some $\{p, q\}$. We first treat the case when X is decomposable.

Lemma 3.4.4. Suppose X is decomposable and irreducible about $\{p, q\}$. Then p and q are proper non-block (shore) points.

Proof. By assumption we can write $X = A \cup B$ as the union of two proper subcontinua. Since X is irreducible we have without loss of generality $p \in A \subset X - q$ and and $q \in B \subset X - p$.

Choose any $x \in A \cap B$. It follows from boundary bumping that p is in the closure of the continuum component C of x in $A - p$. Likewise q is in the closure of the continuum component D of x in $B - q$. Then $x \in C \cup D \subset X - \{p, q\}$ and $C \cup D$ is continuumwise connected.

Observe the subcontinuum $\overline{C \cup D}$ contains $\{p, q\}$. By irreducibility we have $\overline{C \cup D} = X$ hence $C \cup D$ is dense. To see $C \cup D \subset \kappa(p)$ recall $p \in A \subset X - q$ hence $A \cup D \subset \kappa(p)$. Therefore the subset $C \cup D$ of $A \cup D$ witnesses how p is a proper non-block point. By symmetry the same argument applies to q. $\square$

We now deal with indecomposable X. The definition allows finer control over why a given $p \in X$ fails to be a shore point.

Definition 3.4.5. Suppose $p \in X$ and $\mathcal{U} = \{U_1, \ldots, U_{n+1}\}$ are open subsets of X. We say p *disrupts* $\mathcal{U}$ to mean no continuum component of $\kappa(p) - p$ meets all $U_1, \ldots, U_{n+1}$. We say p *trivially disrupts* $\mathcal{U}$ to mean there are distinct elements $U^1, U^2, \ldots, U^n \in \mathcal{U}$ and nonempty open sets $V_i \subset U^i$ such that p disrupts $\{V_1, \ldots V_n\}$. Otherwise we say p *properly disrupts* $\mathcal{U}$.

For p to properly disrupt $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ the definition requires $\mathcal{U}$ be pairwise disjoint. It is also clear the shore point $p \in X$ being trivial is equivalent to disrupting some nondegenerate family. By boundary bumping no $p \in X$ can disrupt a family with only one element. So if p disrupts $\mathcal{U} = \{U_1, U_2\}$ then p properly disrupts $\mathcal{U}$.

By induction it follows if p disrupts $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ there exist $m \leq n$ and elements $U^1, U^2, \ldots, U^m \in \mathcal{U}$ and open sets $V_i \subset U^i$ such that p properly disrupts $\{V_1, \ldots V_m\}$. The next lemmas take advantage of that fact.

Lemma 3.4.6. Suppose $p \in X$ properly disrupts $\{U_1, \ldots, U_n\}$ and let $r \leq n$ be fixed. There exists a family $\mathcal{K}(r)$ of subcontinua of $\kappa(p) - p$ that satisfies both properties below.

(1) Each $K \in \mathcal{K}(r)$ meets U_m for each $m \neq r$.

(2) $\bigcup \mathcal{K}(r)$ is dense in U_m for each $m \neq r$.

Proof. Without loss of generality $r = 1$. For each $m \neq 1$ let $V_m \subset U_m$ be an arbitrary open subset. Since p properly disrupts $\{U_1, \ldots, U_n\}$ it cannot disrupt the family $\{V_2, \ldots, V_n\}$. That means there is a subcontinuum $K \subset \kappa(p) - p$ that meets each of $V_2, \ldots, V_n$. Now let $V_2, \ldots, V_n$ range over the open subsets of $U_2, \ldots, U_n$. The union of all continua K is dense in each U_m. □

Lemma 3.4.7. Suppose X is indecomposable. Each shore point $p \in X$ is a proper shore point.

Proof. Suppose to the contrary that $p \in X$ is a trivial shore point. That means p disrupts some family $\mathcal{U} = \{U_1, \ldots U_n\}$ of open subsets. By induction we can assume p properly disrupts $\mathcal{U}$. Let $\mathcal{K}(1)$ be a family of subcontinua of $\kappa(p) - p$ as described in Lemma 3.4.6.

Each $K \in \mathcal{K}(1)$ has a continuum component $C(K)$ in $\kappa(p) - p$. Since p disrupts $\mathcal{U}$ we know $C(K)$ is disjoint from U_1. Recall $\overline{C(K)}$ is a subcontinuum. It follows from boundary bumping $p \in \overline{C(K)} \subset X - U_1$.

As K ranges over the elements of $\mathcal{K}(1)$ the set $S = \bigcup \left\{ \overline{C(K)} : K \in \mathcal{K}(1) \right\}$ constitutes a semicontinuum with $p \in S \subset X - U_1$. Therefore $\overline{S} \subset X - U_1$ is a proper subcontinuum. But Property (2) of Lemma 3.4.6 says $U_2 \subset \overline{S}$. Thus

$\overline{S} \subset X$ is a proper subcontinuum with nonvoid interior. Since X is indecomposable this cannot occur. We conclude p is a proper shore point. $\square$

Theorem 2 follows from Lemmas 3.4.4 and 3.4.7.

Theorem 2. Shore points are the same as proper shore points.

3.5 Continua Without Shore Points

The proof of Theorem 1 says we can build a continuum X without shore points by assuming NCF and gluing two copies of $\mathbb{H}^*$ together at a single point. This section is about whether arbitrary continua X without shore points come about this way – by joining together continua without non-block points. We first show X is decomposable.

Lemma 3.5.1. Suppose $p \in X$ is not a shore point. There is a decomposition $X = X_1 \oplus \ldots \oplus X_n$ with $p \in X_1 \cap \ldots \cap X_n$.

Proof. Since p is not a shore point it disrupts some family $\mathcal{U} = \{U_1, \ldots U_n\}$ of open subsets. By induction we can assume p properly disrupts $\mathcal{U}$. For each $r \leq n$ let $\mathcal{K}(r)$ be a family of subcontinua of $\kappa(p) - p$ as described in Lemma 3.4.6.

The proof of Lemma 3.4.7 shows each $K(r) = \overline{\bigcup \mathcal{K}(r)}$ is a subcontinuum with $p \in K(r) \subset X - U_r$ and $U_m \subset K(r)$ for each $m \neq r$. Let $\mathcal{L}(r)$ be the family of all subcontinua with both properties. Without loss of generality we can assume $K(r) = \overline{\bigcup \mathcal{L}(r)}$ is the largest subcontinuum with both properties.

Define the subcontinuum $Y = K(1) \cup K(2) \cup \ldots \cup K(n)$ and write $C(x)$ for the continuum component of each $x \in X - Y$. We claim each $C(x)$ meets no U_r. For suppose otherwise. Then without loss of generality some $C(x)$ meets U_1. Since p disrupts $\{U_1, \ldots U_n\}$ we know $C(x)$ fails to meet at least one U_m. Without loss of generality $m = 2$. Then $\overline{C(x)}$ is a subcontinuum with $x \in \overline{C(x)} \subset X - U_2$.

Now consider the subcontinuum $K(2) \cup \overline{C(x)}$. We have $p \in K(2) \cup \overline{C(x)} \subset X - U_2$ and $U_m \subset K(2) \cup \overline{C(x)}$ for each $m \neq 2$. Since we chose $K(2)$ to be maximal with both properties we must have $K(2) \cup \overline{C(x)} = K(2)$. Hence $\overline{C(x)} \subset K(2)$ and $x \in K(2) \subset Y$ contrary to the assumption that $x \in X - Y$. We conclude $C(x)$ meets no U_r.

It follows for each $x \in X - Y$ that $\overline{C(x)}$ is a subcontinuum with $\{p, x\} \subset$

$\overline{C(x)} \subset X-(U_1\cup\ldots\cup U_n)$. Define the subcontinuum $C = \overline{\bigcup\{C(x) : x \in X - Y\}}$. Then $\{p, x\} \subset C \subset X - (U_1 \cup \ldots \cup U_n)$.

It follows $X = C \cup K(1) \cup K(2) \cup \ldots \cup K(n)$ and p is an element of each element of the covering. By discarding any covering element contained in the union of the others we get the required decomposition. □

The theorem follows.

Theorem 3. Every point of an indecomposable continuum is a proper shore point.

The proof of Theorem 3 for the special case of $\mathbb{H}^*$ is straightforward and enlightening. An earlier paper of ours gave a lengthy argument, that considered separately two types of composants of $\mathbb{H}^*$ and did not use Theorem 2. That paper was never published because the anonymous referee was able to give the much simpler proof we include here as Lemma 3.5.2.

Lemma 3.5.2. Let $p \in \mathbb{H}^*$ be arbitrary and $\mathcal{U} = \{U_1, \ldots, U_n\}$ a family of open sets. For each standard subcontinuum $\mathbb{I}_\mathcal{D}$ with $p \in \mathbb{I}_\mathcal{D}$ some other standard subcontinuum $\mathbb{J}_\mathcal{D} \subset \kappa(p)$ has $p \notin \mathbb{J}_\mathcal{D}$ but $\mathbb{J}_\mathcal{D}$ meets each U_m. In particular each point of $\mathbb{H}^*$ is a shore point.

Proof. We only consider $n = 2$ and $\mathcal{U} = \{U_1, U_2\}$ as the general case is similar. First choose disjoint open $U, V \subset \mathbb{H}$ with the basic open sets $U^* \subset U_1$ and $V^* \subset U_2$. Then observe $U = A_1 \cup A_2 \cup \ldots$ is a disjoint union of open intervals and likewise for $V = B_1 \cup B_2 \cup \ldots$.

Choose an increasing sequence $c_1 < c_2 < \ldots$ in $\mathbb{H}$ so each $[c_n, c_{n+1}]$ contains at least one A_i and B_j. It follows $c_n \to \infty$. Define simple sequences $R_n = [c_{4n}, c_{4n+1}]$ and $L_n = [c_{4n+2}, c_{4n+3}]$. By construction $\mathbb{R}_\mathcal{D}$ and $\mathbb{L}_\mathcal{D}$ are disjoint and each meets U^* and V^*. Therefore p is an element of at most one of $\mathbb{R}_\mathcal{D}$ and $\mathbb{L}_\mathcal{D}$. Without loss of generality $p \in \mathbb{R}_\mathcal{D}$. Since $p \in \mathbb{I}_\mathcal{D}$ Theorem 4 of [4] says $\mathbb{L}_\mathcal{D} \subset \kappa(p)$. Thus we can take $\mathbb{J}_\mathcal{D} = \mathbb{L}_\mathcal{D}$. □

We would like to show our example of *spot-welding* two copies of $\mathbb{H}^*$ is generic in the following sense.

Conjecture 4. Suppose X has no shore points and $p \in X$. There is a decomposition $X = X_1 \oplus \ldots \oplus X_N$ with $p \in X_1 \cap \ldots \cap X_N$ and p non-coastal when treated as a point of each X_n.

Conjecture 4 asks for a particularly *nice* decomposition of X. One variant of the conjecture is that every decomposition with $p \in X_1 \cap \ldots \cap X_N$ is *nice*. The stronger conjecture however is false.

For a counterexample assume NCF and take two copies H_1 and H_2 of $\mathbb{H}^*$ and identify points $x_1 \in H_1$ and $x_2 \in H_2$ respectively with the endpoints 0 and 1 of the arc. Denote by X the quotient space.

Observe for $X_1 = H_1 \cup [0,1]$ and $X_2 = [0,1] \cup H_2$ we have the decomposition $X = X_1 \oplus X_2$ but the semicontinua $H_1 \cup [0,1)$ and $(0,1] \cup H_2$ witness how $p = 1/2$ is a coastal point of each element. Thus the stronger conjecture fails.

To see the weaker conjecture holds consider the second choice of decomposition $X_1 = H_1 \cup [0,1/2]$ and $X_2 = [1/2,1] \cup H_2$. By the same reasoning as in Theorem 1 we see $p = 1/2$ is a coastal point of neither element.

The second decomposition above is *minimal* in the following sense.

Definition 3.5.3. Suppose $X = X_1 \oplus \ldots \oplus X_N$ and $X = Y_1 \oplus \ldots \oplus Y_N$ are decompositions. We write $Y_1 \oplus \ldots \oplus Y_N \leq X_1 \oplus \ldots \oplus X_N$ to mean there is a permutation σ of $\{1, 2, \ldots, N\}$ with each $Y_n \subset X_{\sigma(n)}$. We call a decomposition *minimal* to mean it is minimal with respect to this partial order.

For example, each minimal decomposition $X_1 \oplus X_2$ of the arc has $X_1 \cap X_2$ a singleton. Each minimal decomposition $X_1 \oplus X_2$ of the circle has $X_1 \cap X_2$ a doubleton. For X formed by *spot-welding* finitely many indecomposable continua the natural decomposition is minimal. It may prove useful that minimal decompositions always exist.

Lemma 3.5.4. Each decomposition is $\leq$-above some minimal decomposition.

Proof. The proof uses Zorn's lemma. Suppose $\{X^i(1) \oplus \ldots \oplus X^i(N) : i \in \mathcal{I}\}$ is a chain of decompositions. Without loss of generality $\mathcal{I}$ has top element $1 \in \mathcal{I}$ and no bottom element. Since $X^1(1) \oplus \ldots \oplus X^1(N)$ is a decomposition there are points $x_n \in X^1(n) - \bigcup\{X^1(m) : m \neq n\}$.

Let $X^i(1) \oplus \ldots \oplus X^i(N)$ be arbitrary. Since $i \leq 1$ some permutation σ_i has each $X^i\big(\sigma_i(n)\big) \subset X^1(n)$. Thus $X^i\big(\sigma_i(n)\big)$ includes at most the element x_n of $\{x_1, x_2, \ldots, x_N\}$. But since $X^i(1) \cup \ldots \cup X^i(N) = X$ each x_n is an element of some $X^i(m)$. We conclude each $x_n \in X^i\big(\sigma_i(n)\big) - \bigcup\big\{X^i\big(\sigma_i(m)\big) : m \neq n\big\}$.

For $j \leq i$ we know $X^j\big(\sigma_j(n)\big)$ is contained in some $X^i\big(\sigma_i(m)\big)$. The point x_n witnesses how $m = n$. We conclude each $X^j\big(\sigma_j(m)\big) \subset X^i\big(\sigma_i(m)\big)$. Then [11] Proposition 1.7 says the intersection $X_n = \bigcap\big\{X^i\big(\sigma_i(n)\big) : i \in I\big\}$ is a subcontinuum. We claim $X_1, \ldots, X_N$ form a decomposition which is clearly a lower bound for the chain. First observe each x_n witnesses how X_n is not contained in $\bigcup\{X(m) : m \neq n\}$.

To show $X_1 \cup \ldots \cup X_N = X$ let $x \in X$ be arbitrary. For each $i \in \mathcal{I}$ there is $n_i \leq N$ with $x \in X^i\big(\sigma_i(n_i)\big)$. Write $A(n) = \big\{i \in \mathcal{I} : x \in X^i\big(\sigma_i(n)\big)\big\}$ for $n = 1, 2, \ldots, N$. Since $\mathcal{I}$ has no bottom element one of $A(n)$ is cofinal. Thus $x \in \bigcap\big\{X^i\big(\sigma_i(n)\big) : i \in A(n)\big\}$ which equals $\bigcap\big\{X^i\big(\sigma_i(n_i)\big) : i \in \mathcal{I}\big\}$ by cofinality and thus $x \in X_n$. Since x is arbitrary we see X_n cover X.

We conclude the chain $\{X^i(1) \oplus \ldots \oplus X^i(N) : i \in \mathcal{I}\}$ has a lower bound $X_1 \oplus \ldots \oplus X_N$. Zorn's lemma then implies each decomposition is above a minimal decomposition. □

Conjecture 5. Suppose X has no shore points and $X = X_1 \oplus \ldots \oplus X_N$ is a minimal decomposition with $p \in X_1 \cap \ldots \cap X_N$. Then p is non-coastal when treated as a point of each X_n.

Thus far we have only the partial results Lemmas 3.5.5, 3.5.8 and 3.5.9. Henceforth assume X has no shore points. Fix $p \in X$ and let $X = X_1 \oplus X_2$ be a decomposition with $p \in X_1 \cap X_2$. The case for $n > 2$ is similar.

Lemma 3.5.5. Each dense semicontinuum of X_1 (resp. X_2) at p contains $X_1 - X_2$ (resp. $X_2 - X_1$).

Proof. Suppose for example $p \in S \subset X_1 - q$ for some dense semicontinuum $S \subset X_1$ and $q \in X_1 - X_2$. Then $X_2 \cup S \subset X - q$ is a dense semicontinuum of X. This implies $q \in X$ is a non-block point hence a shore point contrary to assumption. □

Corollary 3.5.6. Suppose $X_1 \cap X_2 = \{p\}$. Then p is non-coastal as an element of X_1 and X_2.

The following notation is part of Lemma 3.5.8

Notation 3.5.7. Suppose $X = X_1 \oplus X_2$. Write $\mathcal{S}(X_1)$ (resp. $\mathcal{S}(X_2)$) for the collections of proper dense semicontinua of X_1 (resp. X_2) that meet both $X_1 \cap X_2$ and $X_1 - X_2$ (resp. $X_2 - X_1$). Define two subsets of $X_1 \cap X_2$.

$$C_1 = \{x \in X_1 \cap X_2 : x \in S \text{ for each } S \in \mathcal{S}(X_1)\}.$$

$$C_2 = \{x \in X_1 \cap X_2 : x \in S \text{ for each } S \in \mathcal{S}(X_2)\}.$$

Lemma 3.5.8. Suppose p is coastal as an element of both X_1 and X_2. Then one of $C_1 \cap C_2 = \varnothing$ or $C_1 \cup C_2 = X_1 \cap X_2$ holds.

Proof. Since $p \in X_1$ is coastal $\mathcal{S}(X_1)$ is nonempty and likewise for $\mathcal{S}(X_2)$. Suppose $C_1 \cap C_2 \neq \varnothing$ and $C_1 \cup C_2 \neq X_1 \cap X_2$. That means there are $x \in C_1 \cap C_2$ and $y \in X_1 \cap X_2 - C_1 \cup C_2$. Select $S_1 \in \mathcal{S}(X_1)$ and $S_2 \in \mathcal{S}(X_2)$ with $y \notin S_1$ and $y \notin S_2$.

By definition we have $x \in S_1$ and $x \in S_2$. Thus $S_1 \cup S_2 \subset X$ is a dense semicontinuum that excludes the point $y \in X$. This contradicts how X has no shore points. We conclude $C_1 \cap C_2 \neq \varnothing$ and so $C_1 \cup C_2 = X_1 \cap X_2$. □

In the first case of Lemma 3.5.8 we can say more.

Lemma 3.5.9. Suppose p is coastal as an element of both X_1 and X_2 and $C_1 \cap C_2 = \varnothing$. Then each subcontinuum of X that meets C_1 and C_2 also contains $X_1 \cap X_2 - C_1 \cup C_2$.

Proof. Suppose the subcontinuum $K \subset X$ meets C_1 and C_2. Let $x \in X_1 \cap X_2 - C_1 \cup C_2$ be arbitrary. Select $S_1 \in \mathcal{S}(X_1)$ and $S_2 \in \mathcal{S}(X_2)$ with $x \notin S_1$ and $x \notin S_2$. It follows $S_1 \cup K \cup S_2 \subset X$ is a dense semicontinuum. Since X has no shore points $S_1 \cup K \cup S_2 = X$ and so $x \in K$. Since $x \in X_1 \cap X_2 - C_1 \cup C_2$ is arbitrary we conclude $X_1 \cap X_2 - C_1 \cup C_2 \subset K$. □

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Chapter 4

Indecomposable Continuum with a Strong Non-Cut Point

Abstract

We construct an indecomposable continuum with exactly one strong non-cut point. The method is an adaptation of Bellamy [1]. We start with an ω_1-chain of indecomposable metric continua and retractions. The inverse limit is an indecomposable continuum with exactly two composants. Our example is formed by identifying a point in each composant.

4.1 Introduction

Every point p of an indecomposable metric continuum M is a weak cut point. That means there are distinct $x, y \in M - p$ such that each subcontinuum $K \subset M$ with $\{x, y\} \subset K$ has $p \in K$. The proof follows from M having more than one composant; and the composant-by-composant version of the result fails. Namely some $q \in M$ might fail to weakly cut its composant $\kappa(q)$. For example consider the endpoint c of the Knaster buckethandle. It is easy to see $\kappa(c) - c$ is even arcwise connected. Hence c has only trivial reasons for being a weak cut point.

There exist indecomposable non-metric continua with exactly one composant $-$ henceforth called *Bellamy continua*. Each Bellamy continuum is simultaneously an indecomposable continuum and a composant of an indecomposable continuum. Bellamy continua resemble indecomposable metric continua in being compact. In this paper we show they resemble composants of indecomposable metric continua in that they might not be composed entirely of weak cut points.

4.2 Terminology and Notation

Throughout X is a continuum. That means a nondegenerate compact connected Hausdorff space. For $a, b \in X$ we say X is *irreducible about* $\{a, b\}$ or *irreducible from* a to b to mean no proper subcontinuum of X contains $\{a, b\}$. The subspace $A \subset X$ is called a *semicontinuum* to mean for each $a, c \in A$ some subcontinuum $K \subset A$ has $\{a, c\} \subset K$. Every subspace $A \subset X$ is partitioned into maximal semicontinua called the *continuum components* of A. For background on metric continua see [4] and [6]. The results cited here have analagous proofs for non-metric continua.

For a subset $S \subset X$ denote by S° and $\overline{S}$ the interior and closure of S respectively. By *boundary bumping* we mean the principle that, for each closed $E \subset X$, each component C of E meets $\partial E = \overline{E} \cap \overline{X - E}$. For the non-metric proof see §47, III Theorem 2 of [4]. One corollary of boundary bumping is that the point $p \in X$ is in the closure of each continuum component of $X - p$.

For $b \in X$ we omit the curly braces and write $X - b$ instead of $X - \{b\}$ without confusion. For distinct $a, b, c \in X$ we say b *weakly cuts* a from c and write $[a, b, c]_X$ (or $[a, b, c]$ without confusion) to mean a and c have different continuum components in $X - b$. We say $b \in X$ *weakly cuts* the semicontinuum $A \subset X$ to mean $[a, b, c]$ for some $a, c \in A$ and call b a *weak cut point* to mean it weakly cuts X and a *strong non-cut point* otherwise.

We write $[a, c]_X$ $\big($or just $[a, c]\big)$ for $\big\{b \in X : [a, b, c]_X\big\}$. Note $[a, c]$ is not in general connected as the interval notation suggests. In case $[a, c]$ is connected and $b \in [a, c]$ we have $[a, b] \cup [b, c] = [a, c]$. Moreover $[a, b, c]_{[a,c]}$ for each $b \in [a, c] - \{a, c\}$. Clearly the weak cut structure is topologically invariant. That means $[a, b, c]_X \iff [h(a), h(b), h(c)]_Y$ for each $a, b, c \in X$ and homeomorphism $h : X \to Y$.

We say X is *indecomposable* to mean it is not the union of two proper subcontinua. Equivalently each proper subcontinuum is nowhere dense. The *composant* $\kappa(x)$ of the point $x \in X$ is the union of all proper subcontinua that have x as an element. Indecomposable metric continua are partitioned into $\mathfrak{c}$ many pairwise disjoint composants [5]. In case $\kappa(x) \neq \kappa(y)$ then X is irreducible about $\{x, y\}$. There exist indecomposable non-metric continua with exactly one composant, henceforth called *Bellamy continua*.

We call X *hereditarily unicoherent* to mean it has some (and therefore all) of the equivalent properties:

(I) The intersection of any two subcontinua of X is connected.

(II) $[a, b, c]_X \iff [a, b, c]_L$ for each subcontinuum $L \subset X$ with $a, b, c \in L$.

(III) $[a, c]_X = [a, c]_L$ for each subcontinuum $L \subset X$ with $a, c \in L$.

(IV) Whenever $a, c \in X$ the set $[a, c]$ is a subcontinuum.

The continuous function $f : X \to Y$ is called *monotone* to mean each $f^{-1}(y) \subset X$ is connected for $y \in Y$. Theorem 6.1.28 of [3] says moreover each $f^{-1}(K) \subset X$ is a continuum for $K \subset Y$ a continuum. The function f is called *proper* to mean $f(L) \subset Y$ is proper whenever $L \subset X$ is a proper subcontinuum.

The partition $\mathcal{P}$ of X into closed subsets is called *upper semicontinuous* to mean the following: For each $P \in \mathcal{P}$ and open $U \subset X$ containing P there is open $V \subset U$ with $P \subset V$ and V a union of elements of $\mathcal{P}$. Upper semicontinuity of the partition is equivalent to the quotient space $X/\mathcal{P}$ being a continuum.

Throughout $K \subset \mathbb{R}^2$ is the *quinary Cantor set*. That means the points in $[0, 1] \times \{0\}$ whose x-coordinate can be expressed in base-5 without the digits 1 or 3. We write K_1 for the middle third of K; K_2 for the leftmost two thirds of the portion of K right of K_1 ; K_3 for the leftmost two thirds of the portion of K right of K_2 and so on. Formally $K_1 = K \cap [2/5, 3/5]$, $K_2 = K \cap [4/5, 1 - 2/25]$ and $K_n = K \cap [1 - 1/5^n, 1 - 2/5^{n+1}]$ for each $n > 1$. Let each $k(n) = 1 - 2/5^n$ be the right endpoint of K_n. Let $G : K \to K$ be the unique linear order-reversing bijection and define each $P_n = G(K_{n+1})$ and $p(n) = G(k(n+1))$. Clearly $K = \{0\} \cup \left(\bigcup_n P_n\right) \cup \left(\bigcup_n K_n\right) \cup \{1\}$ is a disjoint union.

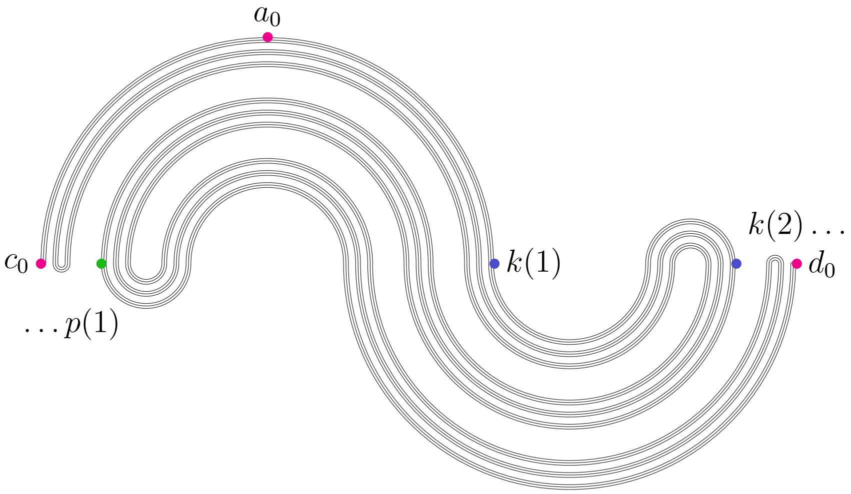

Figure 4.1: The *quinary Knaster buckethandle*

We write $Q' \subset \mathbb{R}^2$ for the union of all semicircles in the upper half-plane with centre $(1 - 7/10^n, 0)$ for some $n \in \mathbb{N}$ and endpoints in K. We write $R' \subset \mathbb{R}^2$ for the union of all semicircles in the lower half-plane with centre $(7/10^n, 0)$ for some $n \in \mathbb{N}$ and endpoints in K. We write Q for the set $\{(x, y+1) \in \mathbb{R}^2 : (x, y) \in Q'\}$ and R for the set $\{(x, y-1) \in \mathbb{R}^2 : (x, y) \in R'\}$.

Throughout $B = Q' \cup R'$ is the *quinary Knaster buckethandle*. B is a metric indecomposable hereditarily unicoherent continuum. It is easy to see $[x, y]$ is an arc whenever x, y share a composant and $[x, y] = B$ otherwise. Let a_0 be the point $(3/10, 3/10) \in B$. We write C_0 for the composant of the left endpoint $c_0 = (0, 0)$ and D_0 for the composant of the right endpoint $d_0 = (1, 0)$.

Throughout $\omega = \{0, 1, 2, \ldots\}$ is the first infinite ordinal and ω_1 is the first uncountable ordinal. Every initial segment of ω_1 is countable and every countable subset has an upper bound. Every countable ordinal has a cofinal subset order-isomorphic to ω. For the ordered set Ω we say $\Psi \subset \Omega$ is *cofinal* to mean it has no upper bound in Ω. We say $\Psi \subset \Omega$ is *terminal* to mean $\Omega - \Psi$ has an upper bound.

The poset Ω is said to be *directed* to mean for each $\gamma, \beta \in \Omega$ there is $\alpha \in \Omega$ with $\gamma, \beta \leq \alpha$. Note most authors require $\gamma, \beta < \alpha$. This prohibits maximal elements. In this paper it is convenient to allow a directed set to have maximal elements.

An *inverse system* over the directed set Ω consists of the following data: (1) a family of topological spaces $T(\alpha)$ for each $\alpha \in \Omega$ and (2) a family of continuous maps $f^\alpha_\beta : T(\alpha) \to T(\beta)$ for each $\beta \leq \alpha$ such that (3) we have $f^\beta_\gamma \circ f^\alpha_\beta = f^\alpha_\gamma$ whenever $\gamma \leq \beta \leq \alpha$. The property (3) is called *commutativity of the diagram*. The *inverse limit* T of the system is the space

$$\varprojlim\{T(\alpha); f^\alpha_\beta : \alpha, \beta \in \Omega\} = \Big\{(x_\alpha) \in \prod_{\alpha \in \Omega} T(\alpha) : f^\alpha_\beta(x_\alpha) = x_\beta \text{ for all } \beta \leq \alpha\Big\}.$$

We often suppress the index set and write for example $\varprojlim\{T(\alpha); f^\alpha_\beta\}$. The functions f^α_β are called the *bonding maps*. Write $\pi_\beta : T \to T(\beta)$ for the restriction of the projection $\prod_\alpha T(\alpha) \to T(\beta)$.

The inverse limit X of a system $\{X(\alpha); f^\alpha_\beta\}$ of continua is itself a continuum. If moreover each bonding map is surjective then so is each π_β. In that case we call the inverse system (limit) *surjective*.

For any subcontinuum $L \subset X$ we have $L = \varprojlim\{\pi_\alpha(L); f^\alpha_\beta\}$ where each f^α_β is restricted to $\pi_\alpha(L)$. Note commutativity implies f^α_β has range $\pi_\beta(L)$ hence the subsystem is well defined. For cofinal $\Psi \subset \Omega$ the map $(x_\alpha)_{\alpha \in \Omega} \mapsto (x_\alpha)_{\alpha \in \Psi}$ is an homeomorphism between X and the inverse limit $\varprojlim\{X(\alpha); f^\alpha_\beta : \alpha, \beta \in \Psi\}$ over Ψ.

4.3 The Successor Stage

We use transfinite recursion to construct the eponymous indecomposable continuum as the limit of a system $\{X(\alpha); f^\alpha_\beta : \alpha, \beta < \omega_1\}$ of metric continua and retractions. This section shows how to construct each $X(\beta+1)$ from $X(\beta)$. The following section deals with limit ordinals.

To begin let $X(0) = B$ be the quinary buckethandle. Let the composants $C_0, D_0 \subset B$ and points $a_0, c_0 \in C_0$ be as described in the Introduction. Define the following two sequences (p^n_0) and (q^n_0) in $X(0)$: Choose an homeomorphism $[0,1] \mapsto [a_0, c_0]$ with $0 \mapsto a_0$ and $1 \mapsto c_0$ and let each p^n_0 be the image of $1 - 1/n$. Let each q^n_0 be the point $k(n) \in B$ as defined in the Introduction. Observe the pair of sequences $\big((p^n_0), (q^n_0)\big)$ satisfies the following definition.

Definition 4.3.1. For $a \in X$ and $c \in \kappa(a)$ we define a *tail from a to c* as an ordered pair $T = \big((p^n), (q^n)\big)$ of sequences in $\kappa(a)$ with the properties:

(1) Each $a \in [p^n, q^n]$ and $a \in [c, q^n]$.

(2) Each $q^n \notin [c, a]$.

(3) $[p^1, a] \subsetneq [p^2, a] \subsetneq \dots$.

(4) $[a, q^1] \subsetneq [a, q^2] \subsetneq \dots$.

(5) $\bigcup \big\{[p^n, q^n] : n \in \mathbb{N}\big\} = \kappa(a) - c$.

(6) $\bigcup \big\{[p^n, a] : n \in \mathbb{N}\big\} = [c, a] - c$

(7) For each $x \notin [c, a]$ and $n \in \mathbb{N}$ either $[c, q^n] \subset [c, x]$ or $[c, x] \subset [c, q^n]$.

The notion of a tail is pivotal to our example. Indeed as part of the construction we will at stage $\alpha < \omega_1$ choose a tail $T^\alpha = \big((p^n_\alpha), (q^n_\alpha)\big)$ on $X(\alpha)$ so that the tails behave nicely with respect to the bonding maps. This is made precise below.

In the next definition and throughout when we write for example $a_\beta \mapsto a_\gamma$ the map in question is understood to be the bonding map f^β_γ. Similarly for subsets $B \subset X(\beta)$ and $C \subset X(\gamma)$ we write $B \to C$ to mean $f^\beta_\gamma(B) = C$.

Definition 4.3.2. Suppose $\{X(\beta); f^\beta_\gamma : \gamma, \beta < \alpha\}$ is an inverse system and each $X(\beta)$ has a distinguished pair of points (a_β, c_β) and pair of sequences $\big((p^n_\beta), (q^n_\beta)\big)$. We say the system is *coherent* to mean $a_\beta \mapsto a_\gamma$ and $c_\beta \mapsto c_\gamma$ and each $p^n_\beta \mapsto p^n_\gamma$, $q^n_\beta \mapsto q^n_\gamma$, $[p^n_\beta, q^n_\beta] \to [p^n_\gamma, q^n_\gamma]$, $[p^n_\beta, a_\beta] \to [p^n_\gamma, a_\gamma]$ and $[c_\beta, a_\beta] \to [c_\gamma, a_\gamma]$ for $\gamma, \beta < \alpha$.

In practice the distinguished points and sequences in Definition 5.3.3 will always come about from a tail. However in Section 4.4 we already have an inverse system, and most of the work goes into showing a given pair of sequences is indeed a tail. Thus we give the definition in slightly more generality.

At stage $\alpha < \omega_1$ we have already constructed the coherent inverse system $\{X(\beta); f^\beta_\gamma : \beta, \gamma < \alpha\}$ of indecomposable hereditarily unicoherent metric continua and retractions. We assume the following objects have been specified for each $\beta < \alpha$:

(i) Distinct composants $C(\beta), D(\beta) \subset X(\beta)$

(ii) Points $a_\beta, c_\beta \in C(\beta)$

(iii) A tail $T^\beta = \big((p^n_\beta), (q^n_\beta)\big)$ from a_β to c_β

We also assume for each $\gamma, \delta < \alpha$ the three conditions hold:

(a) $\bigcup \big\{X(\delta) : \delta < \gamma\big\} \subset D(\gamma)$.

(b) $\bigcup \big\{f^\gamma_\delta\big(X(\gamma) - X(\delta)\big) : \gamma > \delta\big\} = C(\delta)$.

(c) $\{x \in [c_\gamma, a_\gamma] : f^\gamma_\delta(x) \in [p^n_\delta, a_\delta]\} = [p^n_\gamma, a_\gamma]$ for each $n \in \mathbb{N}$.

Conditions (a) and (b) come straight from [1] and will ensure the limit has exactly two composants. Condition (c) is needed to make the resulting point (c_β) not weakly cut the composant.

For an illustration of what Condition (c) means consider the system of retractions $[0,1] \leftarrow [0,2] \leftarrow [0,3] \leftarrow \ldots$ where the bonding map $[0,n] \to [0,m]$ collapses $[m,n]$ to the point $m \in [0,m]$. For $X(\beta) = [0,\beta]$ and $a_\beta = \beta$ and each $p^n_\beta = 1/n$ we see the system has Condition (c). Indeed our initial $[c_0, a_0] \leftarrow [c_1, a_1] \leftarrow [c_2, a_2] \leftarrow \ldots$ will turn out to be a copy of this simpler system, and so the example should be kept in mind throughout.

We are ready to begin the succesor step. Suppose $\alpha = \beta + 1$ is a successor ordinal. We will construct an indecomposable hereditarily unicoherent metric continuum $X(\beta+1)$ and retraction $f^{\beta+1}_\beta : X(\beta+1) \to X(\beta)$. Then we can define the bonding maps $f^{\beta+1}_\gamma = f^{\beta+1}_\beta \circ f^\beta_\gamma$. We will specify the objects (i), (ii) and (iii) when β is replaced by $\beta+1$. Finally we will check the enlarged system is coherent and Condition (a), (b) and (c) hold for all $\gamma, \delta \leq \beta + 1$.

We use terminology from the Introduction. Identify $Q \cup R \subset \mathbb{R}^2$ with the subspace $(Q \cup R) \times \{a_\beta\}$ of $\mathbb{R}^2 \times X(\beta)$. Define $M \subset \mathbb{R}^2 \times X(\beta)$ as follows.

$$M = \Big(\bigcup_{n\in\mathbb{N}} \big(P_n \times \{-1,1\} \times [p^n_\beta, a_\beta]\big)\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} \big(K_n \times \{-1,1\} \times [a_\beta, q^n_\beta]\big)\Big)$$

Properties (4) and (5) of the tail imply the closure of M is equal to

$$\big(\{0\} \times \{-1,1\} \times [c_\beta, a_\beta]\big) \cup M \cup \big(\{1\} \times \{-1,1\} \times X(\beta)\big).$$

The set $\overline{M} \cup (Q \cup R)$ is compact since $(Q \cup R)$ is closed and bounded and $\overline{M}$ is contained in the product $K \times \{-1,1\} \times X(\beta)$ of compact sets. It is also metric since $\mathbb{R}^2$, K, $\{-1,1\}$ and by assumption $X(\beta)$ are metric spaces.

To obtain $X(\beta+1)$ first make for each $n \in \mathbb{N}$ and $k \in \bigcup P_n$ the identification $(k,-1,p^n_\beta) \sim (k,1,p^n_\beta)$ and for each $k \in \bigcup K_n$ make the identification $(k,-1,q^n_\beta) \sim (k,1,q^n_\beta)$. Then for each $x \in [c_\beta, a_\beta]$ make the identification $(0,-1,x) \sim (0,1,x)$ and for each $x \in X(\beta)$ make the identification $(1,-1,x) \sim (1,1,x)$. It is straightforward to verify the decomposition is upper semicontinuous. Hence $X(\beta+1)$ is a continuum and [6] Lemma 3.2 says $X(\beta+1)$ is metric.

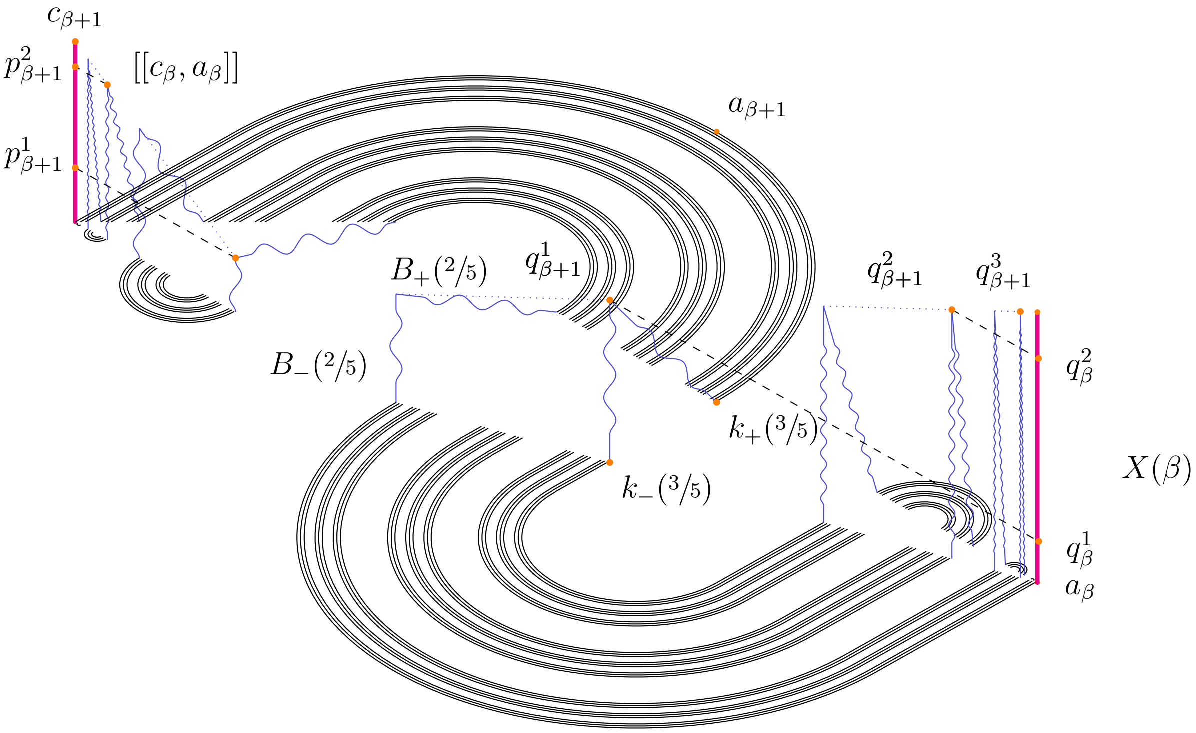

Figure 4.2: Schematic for $X(\beta+1)$

For $k \in K$ write $B(k)$ for the quotient space of $\{(x,y,z) \in M : x = k\}$. For $k \notin \{0,1\}$ clearly $B(k)$ is homeomorphic to two copies of some $[p^n_\beta, a_\beta]$ or $[a_\beta, q^n_\beta]$ joined at the points corresponding to p^n_β or q^n_β respectively. Hence $B(k)$ is a continuum irreducible from $(k,-1,a_\beta)$ to $(k,1,a_\beta)$. $B(1)$ is a copy of $X(\beta)$. Henceforth identify that copy with $X(\beta)$. $B(0)$ is a copy of $[c_\beta, a_\beta]$. Denote that copy by $[[c_\beta, a_\beta]]$ and write $\widetilde{x} \in [[c_\beta, a_\beta]]$ for the point corresponding to $x \in [c_\beta, a_\beta]$.

Define a surjection $g : X(\beta+1) \to B$ onto the buckethandle.

$$g(x,y,z) = \begin{cases} (x, y-1) & \text{for } x \in Q \\ (x, y+1) & \text{for } x \in R \\ (k, 0) & \text{for } x \in B(k) \\ (1, 0) & \text{for } x \in X(\beta) \\ (0, 0) & \text{for } x \in [[c_\beta, a_\beta]] \end{cases}$$

For each $x \in B$ the fibre $g^{-1}(x)$ is either a singleton, $X(\beta)$, $[c_\beta, a_\beta]$ or some $B(k)$. Thus all fibres are subcontinua and g is monotone. Therefore subcontinua pull back to subcontinua. In particular $g^{-1}(B) = X(\beta+1)$ is a continuum. Define the points $c_{\beta+1} = \widetilde{c_\beta}$ and $a_{\beta+1} = g^{-1}(a_0)$ and each $p^n_{\beta+1} = \widetilde{p^n_\beta}$. The definition of $q^n_{\beta+1}$ will be given later in the construction.

Claim 1. The function $g : X(\beta+1) \to B$ is proper.

Proof. Identify the plane $\mathbb{R}^2$ with the subspace $\mathbb{R}^2 \times \{a_\beta\}$ of $\mathbb{R}^2 \times X(\beta)$. For each $n \in \mathbb{N}$ let $K(n)$ be the nth stage of construction of the quinary Cantor set. Let $Q(n) \subset \mathbb{R}^2$ be the union of all upper semicircles with centre $(1 - 7/10^m, 1)$ for some $m \in \mathbb{N}$ and endpoints in $K(n) \times \{1\}$. Let $R(n) \subset \mathbb{R}^2$ be the union of all lower semicircles with centre $(7/10^m, -1)$ for some $m \in \mathbb{N}$ and endpoints in $K(n) \times \{-1\}$. Define the continuum $Y(n) = Q(n) \cup R(n) \cup X(\beta+1)$.

Recall $K(n)$ consists of 3^n intervals of length $1/5^n$. For any such interval I with $0, 1 \notin I$ it is a straightforward exercise to verify the closure of $Y(n) - \bigcup\{B(k) : k \in I\}$ has exactly two components A_1 containing $I \times \{1\} \times \{a_\beta\}$ and A_2 containing $I \times \{-1\} \times \{a_\beta\}$.

Suppose the subcontinuum $L \subset X(\beta+1)$ has $g(L) = B$. Clearly L contains the interior of $Q \cup R$. Now assume L is proper. Then $X(\beta+1) - L$ contains a basic open subset of the quotient space of some $P_m \times \{\pm 1\} \times [p^m_\beta, a_\beta]$ or

$K_m \times \{\pm 1\} \times [a_\beta, q_\beta^m]$. Without loss of generality that set is $V = I \times \{1\} \times U$ for some open $I \subset K_m$ and $U \subset [a_\beta, q_\beta^m]$. We can shrink I further to make it an interval of some $K(n)$ and shrink U to get $a_\beta \notin U$.

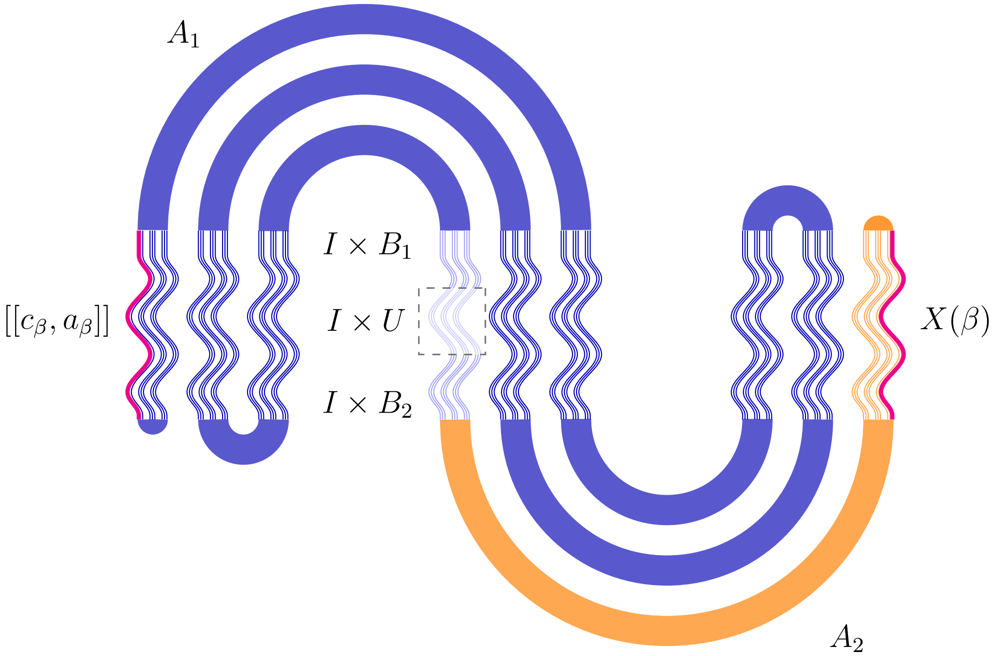

Figure 4.3: Schematic for I an interval of $K(2)$

The subcontinua $B(k)$ are homeomorphic for $k \in I$ and $\bigcup\{B(k) : k \in I\}$ is just $I \times B(k)$. Treat U as a subset of $B(k)$. Since $B(k)$ is irreducible from $(k, 1, a_\beta)$ to $(k, -1, a_\beta)$ we know $B(k) - U$ is the disjoint union $B_1 \cup B_2$ of two clopen sets that include $(k, 1, a_\beta)$ and $(k, -1, a_\beta)$ respectivly. Hence

$$\bigcup\{B(k) : k \in I\} - V = (I \times B_1) \cup (I \times B_2)$$

is the disjoint union of two clopen sets containing $I \times \{1\} \times \{a_\beta\}$ and $I \times \{-1\} \times \{a_\beta\}$ respectively. It follows that

$$Y(m) - V = \big((I \times B_1) \cup A_1\big) \cup \big((I \times B_2) \cup A_2\big)$$

is the disjoint union of two clopen sets containing $I \times \{1\} \times \{a_\beta\}$ and $I \times \{-1\} \times \{a_\beta\}$ respectively.

Recall L is contained $X(\beta+1) - V \subset Y(m) - V$. Since the closed set L contains the interior of $Q \cup R$ it meets both clopen sets which contradicts how L is connected. We conclude $g(L) = B$ if and only if $L = X(\beta+1)$. $\square$

Claim 2. The composants of $X(\beta+1)$ are $\{g^{-1}(E) : E \subset B \text{ is a composant}\}$. Hence $X(\beta+1)$ is indecomposable.

Proof. Since g is monotone and proper each $g^{-1}(E)$ is contained in some composant of $X(\beta+1)$. Now suppose the subcontinuum $L \subset X(\beta+1)$ meets two distinct $g^{-1}(E_1)$ and $g^{-1}(E_2)$. It follows $g(L) = B$. Since g is proper we must have $L = X(\beta+1)$. The result follows. $\square$

Claim 2 tells us $g^{-1}(C_0)$ and $g^{-1}(D_0)$ are distinct composants of $X(\beta+1)$. Define $C(\beta+1) = g^{-1}(C_0)$ and $D(\beta+1) = g^{-1}(D_0)$. From the construction $X(\beta) \subset D(\beta+1)$. The next claim follows.

Claim 3. Condition (a) holds for all $\gamma, \delta \leq \beta+1$.

Observe each subcontinuum of B is contained in some union $J_1 \cup J_2 \cup \ldots \cup J_N$ of arcs where $J_i = [a_i, b_i]$ are alternately contained in Q' or R' and $J_i \cap J_j = \{b_n\} \iff \{i,j\} = \{n, n+1\}$ and $J_i \cap J_j = \varnothing$ otherwise. Let each J_i be identified with the corresponding arc in Q or N.

The monotone sujection g witnesses how each subcontinuum $L \subset X(\beta+1)$ is contained in some

$$J_1 \cup B(x^1) \cup J_2 \cup B(x^2) \cup \ldots \cup J_N \cup B(x^{N+1}) \cup J_{N+1} \qquad (\dagger)$$

for each $J_n \cap B(x^n) = (x^n, \pm 1, a_\beta)$ and $B(x^n) \cap J_{n+1} = (x^n, \mp 1, a_\beta)$ and all other intersections are empty.

Observe each summand of (†) is hereditarily unicoherent and irreducible between two endpoints – and meets the previous factor at exactly one endpoint and the next factor at the other endpoint. It follows by induction the union is hereditarily unicoherent.

Claim 4. $X(\beta+1)$ is hereditarily unicoherent

Proof. Suppose the proper subcontinua $P_1, P_2 \subset X(\beta+1)$ have nonempty intersection. Then the subcontinuum $P_1 \cup P_2$ is contained in a union of the form (†). Since the union is an hereditarily unicoherent subcontinuum $P_1 \cap P_2$ is connected. $\square$

Let $\{y^0, y^1, y^2 \ldots\} \subset B$ be the set $C_0 \cap K$ linearly ordered by $x \leq y \iff [c, x] \subset [c, y]$. We must have $y^0 = c_0$. For each $n \in \mathbb{N}$ write $y^n_\pm = (y^n, \pm 1, a_\beta)$. Note for $n = 0$ we have $y^n_+ = y^n_- = c_{\beta+1}$. For each $n > 0$ write $B(n)$ instead of $B(y^n)$. Write $I(n)$ for the arc in Q or R corresponding to the arc $[y^n, y^{n+1}] \subset B$. By (†) each subcontinuum of $C(\beta+1)$ is contained in some

$$[[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(N-1) \cup B(N) \cup I(N). \qquad (\dagger\dagger)$$

Observe each point of $X(\beta+1)$ is an element of some $B(n)$ or $I(m)$. For example let $x \in B(n)$ and $y \in I(m)$ be arbitrary with $n < m$. It follows from hereditary unicoherence that

$$[x, y] = [x, y^n_\pm]_{B(n)} \cup I(n) \cup B(n+1) \cup \ldots \cup B(m) \cup [y^m_\pm, y]_{I(m)} \qquad (\ddagger)$$

for exactly one of the four choices of ± indices. The other intervals of $[x, y]$ have a similar form based on whether each endpoint is in some $B(n)$ or $I(m)$.

For each $n \in \mathbb{N}$ write $k_\pm(n) = \big(k(n), \pm 1, a_\beta\big)$. By definition $k(n)$ is the right endpoint of K_n. Thus we see $B\big(k(n)\big)$ is the quotient space of the set $\{k(n)\} \times \{-1, 1\} \times [a_\beta, q^n]$. Write $B_\pm\big(k(n)\big)$ for the quotient space of the set $\{k(n)\} \times \{\pm 1\} \times [a_\beta, q^n]$. Then the continua $B_+\big(k(n)\big)$ and $B_-\big(k(n)\big)$ meet at the point $\big(k(n), -1, q^n\big) \sim \big(k(n), 1, q^n\big)$. Define each $q^n_{\beta+1} \in X(\beta+1)$ as that point. The expression (‡) gives the equalities.

$$[p^n_{\beta+1}, q^n_{\beta+1}] = [[p^n_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(k(n)-1) \cup B_+(k(n))$$

$$[a_{\beta+1}, q^n_{\beta+1}] = [a_{\beta+1}, k_+(1)] \cup B(1) \cup \ldots \cup I(k(n)-1) \cup B_+(k(n))$$

$$[c_{\beta+1}, q^n_{\beta+1}] = [[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(k(n)-1) \cup B_+(k(n))$$

$$[p^n_{\beta+1}, a_{\beta+1}] = [[p^n_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$$

$$[c_{\beta+1}, a_{\beta+1}] = [[c_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$$

Claim 5. $T^{\beta+1} = \big((p^n_{\beta+1}), (q^n_{\beta+1})\big)$ is a tail from $a_{\beta+1}$ to $c_{\beta+1}$

Proof. Properties (1) – (4) follow from the equalities above. Property (6) follows from the equalities, the definition of $[[c_\beta, a_\beta]]$ as a copy of $[c_\beta, a_\beta]$, and how T^β has Property (6).

For Property (5) first observe each $c_{\beta+1} \notin [p^n_{\beta+1}, q^n_{\beta+1}]$. Then recall $\kappa(a_{\beta+1}) = C(\beta+1) = g^{-1}(C_0)$. First suppose $x \in B(0) = [[c_\beta, a_\beta]] \subset [c_{\beta+1}, a_{\beta+1}]$. Then Property (6) implies x is some $[p^N_{\beta+1}, a_{\beta+1}] \subset [p^N_{\beta+1}, q^N_{\beta+1}]$.

Otherwise (‡‡) says x is an element of some $B(n)$ or $I(n)$. It is easy to verify the set $\{k(n) : n \in \mathbb{N}\}$ is cofinal in $\{y^0, y^1, y^2, \ldots\}$. Thus $y^n + 1 < k(N)$ for some $N \in \mathbb{N}$. Then the above equalities say $x \in [p^N_{\beta+1}, q^N_{\beta+1}]$.

To prove Property (7) for $T^{\beta+1}$ observe each $[c_{\beta+1}, x]$ has one of the forms

$$[[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(N-1) \cup B(N) \cup [y^N_\pm, x]_{I(N)}$$

$$[[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup B(N-1) \cup I(N-1) \cup [y^N_\pm, x]_{B(N)}$$

depending on whether $x \in I(N)$ or $B(N)$ for some $N \in \mathbb{N}$. For $k(n) < N$ the expression for $[c_\beta, q^n_{\beta+1}]$ says $[c_\beta, q^n_{\beta+1}] \subset [c_\beta, x]$. Likewise $[c_\beta, x] \subset [c_\beta, q^n_{\beta+1}]$ for $N < k(n)$.

Finally assume $N = k(n)$. For $x \in I(N)$ compare the two expressions to see $[c_\beta, x] \subset [c_\beta, q^n_{\beta+1}]$. For $x \in B(N)$ we need only compare the final two summands $[y^N_\pm, x]_{B(N)}$ and $B_+\big(k(n)\big)$. Observe both summands are sub- continua of $B\big(k(n)\big)$ and include the point $k_+(n)$. For $x \in B_+\big(k(n)\big)$ clearly $[y^N_\pm, x]_{B(N)} \subset B_+\big(k(n)\big)$ and so $[c_\beta, x] \subset [c_\beta, q^n_{\beta+1}]$.

Otherwise $x \in B_-\big(k(n)\big)$. Recall that $B_+\big(k(n)\big) = [k_+(n), q^n_{\beta+1}]$ and $B_-\big(k(n)\big) = [q^n_{\beta+1}, k_-(n)]$ and $B_+\big(k(n)\big) \cap B_-\big(k(n)\big) = \{q^n_{\beta+1}\}$. It follows $[y^N_\pm, x]_{B(N)}$ includes $q^n_{\beta+1}$ hence contains $B_+\big(k(n)\big)$. We conclude that $[c_\beta, q^n_{\beta+1}] \subset [c_\beta, x]$. □

Recall we built $X(\beta+1)$ from the subset $\overline{M} \cup (Q \cup R)$ of $\mathbb{R}^2 \times X(\beta)$ by making some identifications. Since the projection $\mathbb{R}^2 \times X(\beta) \to X(\beta)$ onto the third coordinate respects those identifications, it induces a continuous map $f^{\beta+1}_\beta : X(\beta+1) \to X(\beta)$.

Claim 6. The map $f^{\beta+1}_\beta : X(\beta+1) \to X(\beta)$ is a retraction.

Proof. Write $\pi : \mathbb{R}^2 \times X(\beta) \to X(\beta)$ for the projection onto the third coordinate. Recall the subset $M_1 = \big\{(x,y,z) \in M : x = 1\big\}$ is the disjoint union $\{1\} \times \{-1, 1\} \times X(\beta)$ of two copies of $X(\beta)$. In forming $X(\beta+1)$ from $\overline{M} \cup (Q \cup R)$ we identify $(1,-1,x) \sim (1,1,x)$ for each $x \in X(\beta)$. Hence the restriction of $f^{\beta+1}_\beta$ to the quotient space of M_1 is an homeomorphism onto $X(\beta)$. But recall we have identified $X(\beta)$ with the quotient space of M_1. □

Claim 7. Condition (b) holds for all $\gamma, \delta \leq \beta + 1$.

Proof. Recall we have defined each $f^{\beta+1}_\gamma = f^{\beta+1}_\beta \circ f^\beta_\gamma$. Hence by induction and commutativity of the diagram it is enough to show $f^{\beta+1}_\beta\big(X(\beta+1) - X(\beta)\big) = C(\beta)$. To that end recall we have

$$X(\beta+1) = Q \cup R \cup [[c_\beta, a_\beta]] \cup \Big(\bigcup \big\{B(k) : k \in K - \{0,1\}\big\}\Big) \cup X(\beta).$$

Consider the image of each factor under the projection π onto the third coordinate. Both Q and R map onto the singleton $a_\beta \in C(\beta)$. By definition $[[c_\beta, a_\beta]]$ is a copy of $[c_\beta, a_\beta]$ and the restriction of π is a homeomorphism $[[c_\beta, a_\beta]] \to [c_\beta, a_\beta] \subset C(\beta)$.

For $k \in K - \{0, 1\}$ each $B(k)$ is homeomorphic to two copies of some $[p^n_\beta, a_\beta]$ or $[a_\beta, q^n_\beta]$ joined at the points corresponding to p^n_β or q^n_β respectively; and the restriction of π projects each copy onto the subset $[p^n_\beta, a_\beta]$ or $[a_\beta, q^n_\beta]$ of $C(\beta)$ respectively.

Conversely the construction ensures that for any given $n \in \mathbb{N}$ there are $k_1, k_2 \in K - \{0, 1\}$ with $B(k_1)$ and $B(k_2)$ formed from two copies of $[p^n_\beta, a_\beta]$ and $[a_\beta, q^n_\beta]$ respectively as described above. From this we see the set $f^{\beta+1}_\beta\big(X(\beta+1) - X(\beta)\big)$ equals

$$\begin{aligned}
&\{a_\beta\} \cup [c_\beta, a_\beta] \cup \Big(\textstyle\bigcup_n [p^n_\beta, a_\beta]\Big) \cup \Big(\textstyle\bigcup_n [a_\beta, q^n_\beta]\Big) \\
&= [c_\beta, a_\beta] \cup \Big(\textstyle\bigcup_n [p^n_\beta, q^n_\beta]\Big) \\
&= [c_\beta, a_\beta] \cup \big(\, \kappa(a_\beta) - c_\beta \,\big) \\
&= \kappa(a_\beta) = C(\beta).
\end{aligned}$$

The first equality follows from hereditary unicoherence and how each $a_\beta \in [p^n_\beta, q^n_\beta]$; the second from Property (5) of the tail; and the last from the definition of $C(\beta)$. $\square$

Claim 8. The system $\{X(\gamma); f^\gamma_\delta : \gamma, \delta \leq \beta + 1\}$ is coherent.

Proof. The choice of $a_{\beta+1}$ and $c_{\beta+1}$ and $f^{\beta+1}_\beta$ makes it clear $a_{\beta+1} \mapsto a_\beta$ and $c_{\beta+1} \mapsto c_\beta$. To see $f^{\beta+1}_\beta\big([c_{\beta+1}, a_{\beta+1}]\big) = [c_\beta, a_\beta]$ recall that $[c_{\beta+1}, a_{\beta+1}] = [[c_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$. The bonding map projects $[[c_\beta, a_\beta]]$ onto $[c_\beta, a_\beta]$ and sends $[\widetilde{a}_\beta, a_{\beta+1}] \subset I(0)$ to the point $a_\beta \in [c_\beta, a_\beta]$. Likewise $f_\beta^{\beta+1}$ projects each $[p^n_{\beta+1}, a_{\beta+1}] = [[p^n_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$ onto $[p^n_\beta, a_\beta]$.

Since the subcontinuum $B_+\big(k(n)\big) \subset [p^n_{\beta+1}, q^n_{\beta+1}]$ projects onto $[p^n_\beta, q^n_\beta]$ we have $[p^n_\beta, q^n_\beta] \subset f_\beta^{\beta+1}\big([p^n_{\beta+1}, q^n_{\beta+1}]\big)$. To get equality observe $[p^n_{\beta+1}, q^n_{\beta+1}] = [p^n_{\beta+1}, a_{\beta+1}] \cup [a_{\beta+1}, q^n_{\beta+1}]$. The previous paragraph shows $[p^n_{\beta+1}, a_{\beta+1}]$ maps onto $[p^n_\beta, a_\beta] \subset [p^n_\beta, q^n_\beta]$.

Now we treat the remainder $[a_{\beta+1}, q^n_{\beta+1}]$. Recall we define $\big[a_0, k(n)\big]$ as the unique arc in B with endpoints a_0 and $k(n)$. Now observe $[a_{\beta+1}, q^n_{\beta+1}] \subset g^{-1}\big(\big(a_0, k(n)\big]\big)$.

Consider the intersection $K \cap \big[a_0, k(n)\big]$ of the arc with the Cantor set. Inspecting the buckethandle we see $p(n) \le x \le k(n)$ for each $x \in K \cap \big[a_0, k(n)\big]$. Therefore $\big\{k \in K : [a_{\beta+1}, q^n_{\beta+1}] \text{ meets } B(k)\big\}$ is contained in the interval $\big[p(n), k(n)\big] \subset K$.

By construction all $f_\beta^{\beta+1}\big(B(k)\big)$ for $1/5 \le k \le k(n)$ are contained in $f_\beta^{\beta+1}\big(B(k(n)\big) = [p^n_\beta, q^n_\beta]$ and all $f_\beta^{\beta+1}\big(B(k)\big)$ for $p(n) \le k \le 1/5$ are contained in $f_\beta^{\beta+1}\big(B(p(n)\big) = [p^n_\beta, a_\beta] \subset [p^n_\beta, q^n_\beta]$. We conclude that $f_\beta^{\beta+1}\big([p^n_{\beta+1}, q^n_{\beta+1}]\big) \subset [p^n_\beta, q^n_\beta]$ as required. □

Claim 9. Condition (c) holds for all $\gamma, \delta \le \beta + 1$.

Proof. Let $h : [c_{\beta+1}, a_{\beta+1}] \to [c_\beta, a_\beta]$ be the restriction of $f_\beta^{\beta+1}$. By induction we need only show each $h^{-1}\big([p^n_\beta, a^n_\beta]\big) = [p^n_{\beta+1}, a^n_{\beta+1}]$. To that end recall $[c_{\beta+1}, a_{\beta+1}] = [[c_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$. Since $[\widetilde{a}_\beta, a_{\beta+1}] \subset Q$ the map h takes $[\widetilde{a}_\beta, a_{\beta+1}]$ to a_β and sends each $\widetilde{x} \in [[c_\beta, a_\beta]]$ to the corresponding $x \in [c_\beta, a_\beta]$. It follows $h^{-1}\big([x, a_\beta]\big) = [\widetilde{x}, a_{\beta+1}]$ for each $x \in [c_\beta, a_\beta]$. Taking $x = p^n_\beta$ we see

$h^{-1}\big([p^n_\beta, a_\beta]\big) = [\,\widetilde{p}^n_\beta, a_{\beta+1}] = [p^n_{\beta+1}, a_{\beta+1}]$ as required. $\square$

Claim 9 completes the discussion of $\alpha = \beta + 1$ a successor ordinal.

4.4 The Limit Stage

This section deals with the limit stage of our construction. Henceforth assume $\alpha \leq \omega_1$ is a limit ordinal and $\{X(\beta); f^\beta_\gamma : \beta, \gamma < \alpha\}$ a coherent system of indecomposable hereditarily unicoherent metric continua and retractions. For all $\beta, \gamma, \delta < \alpha$ we assume the objects (i), (ii) and (iii) from Section 4.3 have been specified and Conditions (a), (b) and (c) hold.

We define $X(\alpha) = \varprojlim\{X(\beta); f^\beta_\gamma\}$ and each f^α_β as the projection from the inverse limit onto its factors. For each $\gamma < \alpha$ we identify $X(\gamma)$ with the subset $\big\{x \in X(\alpha) : x_\beta = x_\gamma \text{ for all } \beta > \gamma\big\}$ of $X(\alpha)$.

Straightforward modifications of [7] Theorem 3.1 and [2] Corollary 1 show $X(\alpha)$ is both indecomposable and hereditarily unicoherent. To see $X(\alpha)$ is metric we observe by definition that α is a countable ordinal. The product $\prod_{\beta<\alpha} X(\beta)$ of countably many metric spaces is itself a metric space. The inverse limit $X(\alpha)$ is by definition a subset of that product and therefore a metric space.

It remains to show the enlarged system is coherent; to check Conditions (a), (b) and (c) hold for the enlarged system; and to specify the data (i), (ii) and (iii) for $X(\alpha)$. Much of the effort will go into proving the pair of sequences given for (iii) is indeed a tail. To that end we first prove some general facts about tails.

Recall at stage 0 we defined the tail $\big((p^n_0), (q^n_0)\big)$ on the quinary buckethandle. Observe the sequence (p^n_0) tends to c_0 and the intervals $[p^n_0, a_0]$ increase to be dense in $[c_0, a_0]$ while the sequence (q^n_0) has no limit and the intervals $[a_0, q^n_0]$ increase to be dense in the whole space. The next lemma show this holds for general tails.

Lemma 4.4.1. Suppose the indecomposable and hereditarily unicoherent continuum X has a tail $T = (p^n, q^n)$ from a to c. Then $\bigcup_n [p^n, a]$ is nowhere dense and $\bigcup_n [a, q^n]$ is dense.

Proof. For the first statement Property (6) says $\bigcup_n [p^n, a] \subset [c, a]$. By hereditary unicoherence $[c, a]$ is a subcontinuum, and it is nowhere dense by indecomposability. We conclude the first union is nowhere dense.

For the second statement Property (1) says each $a \in [p^n, q^n]$. Hence $[p^n, q^n] = [p^n, a] \cup [a, q^n]$. Now observe by Property (5) that

$$\kappa(a) - c = \bigcup_{n\in\mathbb{N}} [p^n, q^n] = \bigcup_{n\in\mathbb{N}} \Big([p^n, a] \cup [a, q^n]\Big) =$$

$$\Big(\bigcup_{n\in\mathbb{N}} [p^n, a]\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} [a, q^n]\Big) = \Big([c, a] - c\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} [a, q^n]\Big).$$

The left-hand-side is dense in X. The right hand-side is the union of two sets. We have already showed the right-hand-side is nowhere dense. It follows the second set is dense. This completes the proof. $\square$

One can observe Section 4.3 made no explicit reference to strong non-cut points. In fact these points were mentioned explicitly when we talked about tails. The next lemma is needed to obtain the strong non-cut point mentioned in the title.

Lemma 4.4.2. Suppose the indecomposable and hereditarily unicoherent continuum X has a tail $T = (p^n, q^n)$ from a to c. Then c is the only point of $\kappa(a)$ to not weakly cut the composant.

Proof. Property (5) says $\kappa(a) - c$ is the union of a chain of subcontinua hence is a semicontinuum. This is the definition of c not weakly cutting $\kappa(a)$. Now let $b \in \kappa(a) - c$ be arbitrary. Property (6) says b is an element of some $[p^n, q^n]$. Define $L = [p^n, q^n]$ and $P = [p^{n+1}, q^{n+1}]$

First suppose $b \notin \{p^n, q^n\}$. Then $[p^n, b, q^n]_L$ and thus $[p^n, b, q^n]_X$ by hereditary unicoherence. Now suppose $b = p^n$. Then Properties (2) and (3) imply $p^n \neq p^{n+1}$ and $q^n \neq p^{n+1}$ respectively. Thus $b \notin \{p^{n+1}, q^{n+1}\}$ which implies $[p^{n+1}, b, q^{n+1}]_P$

which in turn implies $[p^{n+1}, b, q^{n+1}]_X$ by hereditary unicoherence. The case for $b = q^n$ is similar. We conclude each $b \in \kappa(a) - c$ weakly cuts the composant. $\square$

We have assumed $\{X(\beta); f^{\beta}_{\gamma} : \beta, \gamma < \alpha\}$ is coherent. Hence each $a_\beta \mapsto a_\gamma$, $c_\beta \mapsto c_\gamma$, $p^n_\beta \mapsto p^n_\gamma$, $q^n_\beta \mapsto q^n_\gamma$ and it follows the inverse limit $X(\alpha)$ has points $a_\alpha = (a_\beta)_{\beta<\alpha}$, $c_\alpha = (c_\beta)_{\beta<\alpha}$, $p^n_\alpha = (p^n_\beta)_{\beta<\alpha}$ and $q^n_\alpha = (q^n_\beta)_{\beta<\alpha}$. For ease of notation we suppress the subscript and write a, c, p^n, q^n instead of $a_\alpha, c_\alpha, p^n_\alpha, q^n_\alpha$ respectively.

To see $T^\alpha = \big((p^n), (q^n)\big)$ is a tail from a to c we use Corollary 4.4.5 and the facts $[p^n_\beta, q^n_\beta] \to [p^n_\gamma, q^n_\gamma]$, $[p^n_\beta, a_\beta] \to [p^n_\gamma, a_\gamma]$ and $[c_\beta, a_\beta] \to [c_\gamma, a_\gamma]$ respectively, from the definition of coherence, to get the three expressions.

$$[p^n, q^n] = \varprojlim \big\{[p^n_\beta, q^n_\beta]; f^{\beta}_{\gamma}; \gamma, \beta < \alpha\big\} \tag{I}$$

$$[p^n, a] = \varprojlim \big\{[p^n_\beta, a_\beta]; f^{\beta}_{\gamma}; \gamma, \beta < \alpha\big\} \tag{II}$$

$$[c, a] = \varprojlim \big\{[c_\beta, a_\beta]; f^{\beta}_{\gamma}; \gamma, \beta < \alpha\big\} \tag{III}$$

The three expressions show the expanded system $\{X(\beta); f^{\beta}_{\gamma} : \beta, \gamma \le \alpha\}$ is coherent. Expressions (I), (III) and (II) respectively imply T^α has Properties (1), (2) and (3) of being a tail. Property (4) is slightly more complicated.

Claim 10. T^α has Property (4) of being a tail from a to c.

Proof. We first show each $q^{n+1}_\beta \notin [p^n_\beta, q^n_\beta]$. Property (1) for T^β says $a_\beta \in [p^n_\beta, q^n_\beta]$. Hence $[p^n_\beta, q^n_\beta] = [p^n_\beta, a_\beta] \cup [a_\beta, q^n_\beta]$. Property (6) says $[p^n_\beta, a_\beta] \subset [c_\beta, a_\beta]$ thus $q^{n+1} \notin [p^n_\beta, a_\beta]$ by Property (2). Property (4) says $q^{n+1} \notin [a_\beta, q^n_\beta]$. We conclude $q^{n+1}_\beta \notin [p^n_\beta, q^n_\beta]$.

Now we show $q^{n+1} \notin [a, q^n]$ which proves Property (4) for T^α . Just like before we have $[p^n, q^n] = [p^n, a] \cup [a, q^n]$. In particular $[a, q^n]$ is contained in

$[p^n, q^n]$. We have already shown $q_\beta^{n+1} \notin [p_\beta^n, q_\beta^n]$. Thus the expression (I) implies $q^{n+1} \notin [p^n, q^n]$ as required. $\square$

To show T^α has Properties (5) and (6) we introduce some notation to measure how far a given subcontinuum extends along the tail.

Notation 4.4.3. For each $\beta < \alpha$ and subcontinuum $L \subset X(\beta)$ define

$$\|L\| = \max\big\{n \in \mathbb{N} : [c_\beta, q_\beta^n] \subset L\big\}$$

where we allow the value $\|L\| = \infty$ in case all $[c, q^n] \subset L$

Lemma 4.4.1 says each $\bigcup_n [a_\beta, q_\beta^n]$ is dense. Therefore $\|L\|$ is well defined whenever $L \subset X(\beta)$ is proper. In case $L = X(\beta)$ we define $\|L\| = \infty$. Clearly $\|L\| \leq \|P\|$ for all $L \subset P$ and $\|L\| = 0$ when either $c_\beta \notin L$ or $L \subset X(\beta) = \kappa(a_\beta) = X(\beta) - C(\beta)$.

Claims 11 and 12 will be used to show T^α has Property (5).

Claim 11. We have $\|L\| \leq \|f_\gamma^\beta(L)\|$ for each $\gamma, \beta < \alpha$ and subcontinuum $L \subset X(\beta)$.

Proof. In case $\|L\| = 0$ the result is obvious. Hence assume $\|L\| > 0$. That means $[c_\beta, q_\beta^n] \subset L$ for some $n > 0$. Property (1) for T^β says $a_\beta \in [c_\beta, q_\beta^n]$ hence $[c_\beta, q_\beta^n] = [c_\beta, a_\beta] \cup [a_\beta, q_\beta^n]$. Property (6) says each $p_\beta^m \in [c_\beta, a_\beta]$ hence $[c_\beta, q_\beta^n] = [c_\beta, p_\beta^m] \cup [p_\beta^m, a_\beta] \cup [a_\beta, q_\beta^n]$. We conclude each $[p_\beta^m, a_\beta] \subset L$.

To show $[c_\gamma, q_\gamma^n] \subset f_\gamma^\beta(L)$ first observe $c_\beta \in L$ and so $c_\gamma \in f_\gamma^\beta(L)$ by coherence. Now suppose $x \in [c_\gamma, a_\gamma] - c_\gamma$. Property (6) for $X(\gamma)$ says $x \in [p_\gamma^m, a_\gamma]$ for some $m > 0$. We have already shown $[p_\beta^m, a_\beta] \subset L$. Therefore $f_\gamma^\beta(L)$ contains $f_\gamma^\beta\big([p_\beta^m, a_\beta]\big) = [p_\gamma^m, a_\gamma]$ by coherence hence $x \in f_\gamma^\beta(L)$.

Now suppose $x \in [c_\gamma, q_\gamma^n] - [c_\gamma, a_\gamma]$. By the first paragraph we have $[c_\gamma, q_\gamma^n] = [c_\gamma, a_\gamma] \cup [a_\gamma, q_\gamma^n]$ hence $x \in [a_\gamma, q_\gamma^n]$. The first paragraph proves $[c_\beta, q_\beta^n] = [c_\beta, p_\beta^n] \cup [p_\beta^n, a_\beta] \cup [a_\beta, q_\beta^n]$ which equals $[c_\beta, p_\beta^n] \cup [p_\beta^n, q_\beta^n]$ since $a_\beta \in [p_\beta^n, q_\beta^n]$. Therefore $[p_\beta^n, q_\beta^n] \subset [c_\beta, q_\beta^n]$. Finally observe $x \in [a_\gamma, q_\gamma^n] \subset [p_\gamma^n, q_\gamma^n] = f_\gamma^\beta\big([p_\beta^n, q_\beta^n]\big) \subset f_\gamma^\beta\big([c_\beta, q_\beta^n]\big) \subset f_\gamma^\beta(L)$. We conclude $x \in f_\gamma^\beta(L)$ as required. Taking $n = \|L\|$ gives the result. □

Claim 12. Suppose $x \in \kappa(a) - [c, a]$. Then $x \in [p^n, q^n]$ for some $n \in \mathbb{N}$.

Proof. By Lemma 4.4.4 the set $\Psi = \big\{\beta < \alpha : x_\beta \notin [c_\beta, a_\beta]\big\}$ is terminal. Replace α with Ψ and hence assume all $x_\beta \notin [c_\beta, a_\beta]$. This does not change $X(\alpha)$ or whether $x \in [p^n, q^n]$.

We deal with two cases separately. First assume $\big\{\big\|[c_\beta, x_\beta]\big\| : \beta \in \Omega\big\}$ is bounded for some terminal $\Omega \subset \alpha$. Like before we can assume without loss of generality $\alpha = \Omega$. Hence there is $N \in \mathbb{N}$ with each $[c_\beta, q_\beta^N] \not\subset [c_\beta, x_\beta]$. Property (7) for β says each $[c_\beta, x_\beta] \subset [c_\beta, q_\beta^N]$. In particular $x_\beta \in [c_\beta, q_\beta^N] = [c_\beta, a_\beta] \cup [a_\beta, q_\beta^N]$ and so $x_\beta \in [a_\beta, q_\beta^N] \subset [p_\beta^N, q_\beta^N]$. By (I) we conclude $(x_\beta) \in [p^N, q^N]$.

Now assume no such Ω exists. By induction we can select an increasing sequence $\beta(1) < \beta(2) < \ldots$ with $\big\|[c_{\beta(n)}, x_{\beta(n)}]\big\| > n$ for each $n \in \mathbb{N}$. Lemma 4.4.6 says $\pi_0\big([c, x]\big) = \overline{\bigcup\big\{f_0^\beta\big([c_\beta, x_\beta] : \beta < \alpha\big)\big\}}$. In particular $\pi_0\big([c, x]\big)$ contains each $f_0^{\beta(n)}\big([c_{\beta(n)}, x_{\beta(n)}]\big)$. Thus we have $\big\|\pi_0\big([c, x]\big)\big\| \geq \big\|f_0^{\beta(n)}\big([c_{\beta(n)}, x_{\beta(n)}]\big)\big\| \geq \big\|[c_{\beta(n)}, x_{\beta(n)}]\big\|$ by Claim 11 and so $\big\|\pi_0\big([c, x]\big)\big\| > n$.

Since $n \in \mathbb{N}$ was arbitrary we conclude $\big\|\pi_0\big([c, x]\big)\big\| = \infty$. This means $\pi_0\big([c, x]\big) = X(0)$. Replacing 0 with any $\gamma < \alpha$ we see each $\pi_\gamma\big([c, x]\big) = X(\gamma)$ hence $[c, x] = X(\alpha)$. That means $X(\alpha)$ is irreducible from c to x. In other words

$x \notin \kappa(c)$. Since $\kappa(c) = \kappa(a)$ this contradicts the assumption. $\square$

Claim 13. T^α has Property (7) of being a tail from a to c.

Proof. Suppose $x \notin [c, a]$. It follows from (III) and Lemma 4.4.4 the set $\big\{\beta < \alpha : x_\beta \notin [c_\beta, a_\beta]\big\}$ is terminal. Without loss of generality assume all $x_\beta \notin [c_\beta, a_\beta]$. Now suppose $[c, q^n] \not\subset [c, x]$. It follows from Lemma 4.4.6 we cannot have $[c_\beta, q^n_\beta] \subset [c_\beta, x_\beta]$ cofinally. Therefore $[c_\beta, q^n_\beta] \not\subset [c_\beta, x_\beta]$ terminally. Property (6) for β says $[c_\beta, x_\beta] \subset [c_\beta, q^n_\beta]$ terminally. Thus $[c, x] \subset [c, q^n]$ by Lemma 4.4.6. $\square$

To deal with Property (5) of being a tail, we use Condition (c) from the construction

Claim 14. T^α has Property (6) of being a tail from a to c.

Proof. Let $x \in [c, a] - c$ be arbitrary. By (III) each $x_\beta \in [c_\beta, a_\beta]$. Let $\gamma < \alpha$ be fixed and observe $\Psi = \{\beta < \alpha : \gamma \leq \beta\}$ is cofinal. Property (6) for γ says $x_\gamma \in [p^n, a_\gamma]$ for some $n \in \mathbb{N}$. Let $g : [c_\beta, a_\beta] \to [c_\gamma, a_\gamma]$ be the restriction of f^β_γ.

Then $x_\gamma \in [p^n, a_\gamma]$ hence $x_\beta \in g^{-1}\big([p^n_\gamma, a_\gamma]\big)$ which equals $[p^n_\beta, a_\beta]$ by Condition (c). Thus $x_\beta \in [p^n_\beta, a_\beta]$ and $x \in \varprojlim \big\{[p^n_\beta, a_\beta] : \beta \in \Psi\big\}$. The expression (II) says the set on the right-hand-side equals $[p^n, a]$ and so $x \in [p^n, a]$.

Since $x \in [c, a] - c$ is arbitrary we see $\bigcup \big\{[p^n, a] : n \in N\big\}$ contains $[c, a] - c$. Property (6) for β says each $c_\beta \notin [p^n_\beta, a_\beta]$ thus $c \notin \bigcup \big\{[p^n, a] : n \in \mathbb{N}\big\}$. We conclude $\bigcup \big\{[p^n, a] : n \in \mathbb{N}\big\} = [c, a] - c$ $\square$

Claim 15. T^α is a tail from a to c.

Proof. The expressions (I), (III) and (II) imply T^α has Properties (1), (2) and (3) respectively. Properties (4), (6) and (7) follow from Claims 10, 14 and 13 imply T^α respectively.

Claim 12 says that $\bigcup_n \big[p^n, q^n\big]$ contains each $x \in \kappa(a) - [c, a]$. Combined with Property (6) and how $[p^n, q^n] = [p^n, a] \cup [a, q^n]$ we see $\bigcup_n \big[p^n, q^n\big]$ contains $\kappa(a) - c$. Since it is disjoint from $D(\alpha)$ it is a proper semicontinuum. Hence the only possibilities are $\bigcup_n \big[p^n, q^n\big] = \kappa(a) - c$ or $\bigcup_n \big[p^n, q^n\big] = \kappa(a)$. To see the former observe Property (5) for each β says $c_\beta \notin [p^n_\beta, q^n_\beta]$ thus $c \notin \bigcup_n \big[p^n, q^n\big]$. This shows Property (5) for T^α. □

We have already chosen the points $a = a_\alpha$ and $c = c_\alpha$. Claim 15 shows the pair T^α of sequences is indeed a tail from a_α to c_α. Thus we have specified the objects (ii) and (iii) for stage α. It remains to select the composants (i) and prove Conditions (a), (b) and (c) hold for all $\gamma, \delta \leq \alpha$. Finally we must show each $X(\beta)$ occurs as a subspace of $X(\alpha)$ and how f^α_β is a retraction.

For (i) we must choose $C(\alpha) = \kappa(a_\alpha)$. Hereditary unicoherence along with (III) shows $C(\alpha) = \kappa(c_\alpha)$ as required. For each $\gamma < \alpha$ the identification $X(\gamma) = \big\{x \in X(\alpha) : x_\beta = x_\gamma \text{ for all } \beta > \gamma\big\}$ makes it clear the $X(\gamma)$ are nested and the bonding maps are retractions. Hence $\bigcup\{X(\beta) : \beta < \alpha\}$ is a semicontinuum of $X(\alpha)$ and thus contained in some composant $D(\alpha)$ of $X(\alpha)$. Then condition (a) follows from the definition of $D(\alpha)$.

Claim 16. The composants $D(\alpha)$ and $C(\alpha)$ of $X(\alpha)$ are distinct. Thus at stage α the condition on (i) holds.

Proof. Since X is indecomposable it is enough to show it is irreducible from $c \in C(\alpha)$ to some $x \in D(\alpha)$. Let $\gamma < \alpha$ and $x \in X(\gamma)$ be arbitrary and suppose $\{c, x\} \subset L$ for some subcontinuum $L \subset X(\alpha)$. We claim $\pi_\beta(L) = X(\beta)$ for all $\beta > \gamma$. Thus by surjectivity $L = X(\alpha)$.

Clearly $\{c_\beta, x_\beta\} \subset \pi_\beta(L)$. By definition of the embedding $X(\gamma) \to X(\alpha)$ we have $x_\beta = x_\gamma$ hence$\{c_\beta, x_\gamma\} \subset \pi_\beta(L)$. Since $x_\gamma = \pi_\gamma(x)$ we have $x_\gamma \in X(\gamma)$. By (a) for stage β we have $X(\gamma) \subset D(\beta)$. Hence $\pi_\beta(L)$ meets the distinct composants

$C(\beta)$ and $D(\beta)$ of $X(\beta)$ and so $\pi_\beta(L) = X(\beta)$ as required. $\square$

To prove Condition (b) for the expanded system we use the following claim.

Claim 17. Suppose $x_\gamma = x_{\gamma+1}$ for some $x \in X(\alpha)$ and $\gamma < \alpha$. Then $x \in X(\gamma)$.

Proof. We claim $x_\beta = x_{\gamma+1}$ for each $\beta > \gamma + 1$. If $x_\beta \in X(\gamma+1)$ then since $f^\beta_{\gamma+1}$ is a retraction we have $x_\beta = x_{\gamma+1}$. Otherwise $x_\beta \in X(\beta) - X(\gamma+1)$ and Condition (b) says $f^\beta_{\gamma+1}(x_\beta) \in C(\gamma+1)$. But recall $f^\beta_{\gamma+1}(x_\beta) = f^\beta_{\gamma+1} \circ f^\alpha_\beta(x) = f^\alpha_{\gamma+1}(x) = x_{\gamma+1}$. By assumption $x_{\gamma+1} = x_\gamma$ and therefore $x_\gamma \in C(\gamma+1)$. But $x_\gamma \in X(\gamma)$ and by Condition (a) we have $X(\gamma) \subset D(\gamma+1)$. This is a contradiction since the composants $C(\gamma+1)$ and $D(\gamma+1)$ are disjoint. $\square$

Claim 18. Condition (b) holds for all $\gamma, \delta \leq \alpha$.

Proof. We want to show $\bigcup \big\{ f^\gamma_\delta\big(X(\gamma) - X(\delta)\big) : \alpha \geq \gamma > \delta \big\} = C(\delta)$ for each $\delta < \alpha$. First observe the above union can be written

$$f^\alpha_\delta\big(X(\alpha) - X(\delta)\big) \cup \Big(\bigcup \big\{ f^\gamma_\delta\big(X(\gamma) - X(\delta)\big) : \alpha > \gamma > \delta \big\} \Big).$$

By induction the second factor equals $C(\delta)$. So it is enough to show the first factor equals $C(\delta)$ as well.

To see $f^\alpha_\delta\big(X(\alpha) - X(\delta)\big) \subset C(\delta)$ let $x \in X(\alpha) - X(\delta)$ be arbitrary. Claim 17 implies $x_{\delta+1} \neq x_\delta$. Since $f^{\delta+1}_\delta$ is a retraction $x_{\delta+1} \in X(\delta+1) - X(\delta)$ and so $f^{\delta+1}_\delta(x_{\delta+1}) \in C(\delta)$ by Condition (b) at stage $\delta + 1$. But $f^{\delta+1}_\delta(x_{\delta+1})$ is just $x_\delta = f^\alpha_\delta(x)$ and so $f^\alpha_\delta(x) \in C(\delta)$.

To see $f^\alpha_\delta\big(X(\alpha) - X(\delta)\big) = C(\delta)$ recall T^α is a tail from a to c. Property (5) says $\{c\} \cup \Big(\bigcup_n [p^n, q^n] \Big) = C(\alpha)$. By coherence the image under f^α_γ is $\{c_\gamma\} \cup \Big(\bigcup_n [p^n_\gamma, q^n_\gamma] \Big)$ which equals $C(\gamma)$ by Property (5) of T^γ. $\square$

Finally we prove (c) for the expanded system.

Claim 19. Condition (c) holds for all $\gamma, \delta \leq \alpha$.

Proof. By commutativity and Condition (c) at earlier stages it is enough to consider the case $\gamma = \alpha$. We must show $\big\{x \in [c, a] : x_\delta \in [p^n_\delta, a_\delta]\big\} = [p^n, a]$.

Let $\gamma > \delta$ be arbitrary and consider the γ-coordinate of a point x of the left-hand-side. Since $x \in [c, a]$ and $[c, a] = \varprojlim \big\{[c_\delta, a_\delta]; f^\gamma_\delta\big\}$ by (III) we have $x_\gamma \in [c_\gamma, a_\gamma]$. Since $x_\delta = f^\gamma_\delta(x_\gamma)$ we see x_γ is an element of $\big\{y \in [c_\gamma, a_\gamma] : f^\gamma_\delta(y) \in [p^n_\delta, a_\delta]\big\}$. Property (c) at earlier stages says this set equals $[p^n_\gamma, a_\gamma]$. Thus $x_\gamma \in [p^n_\gamma, a_\gamma]$. By (II) we know the set $\varprojlim \big\{[p^n_\delta, a_\delta]; f^\gamma_\delta\big\}$ is well defined and equals $[p^n, a]$. Since $x_\gamma \in [p^n_\gamma, a_\gamma]$ for all $\gamma > \delta$ we conclude $\big\{x \in [c, a] : x_\delta \in [p^n_\delta, a_\delta]\big\} \subset [p^n, a]$.

For the other inclusion let $x \in [p^n, a]$ be arbitrary. By (II) each $x_\delta \in [p^n_\delta, a_\delta]$. By Property (6) at stage δ we have $[p^n_\delta, a_\delta] \subset [c_\delta, a_\delta]$. Hence $x_\delta \in [c_\delta, a_\delta]$ and by (III) we have $x \in [c, a]$. We conclude $[p^n, a] \subset \big\{x \in [c, a] : x_\delta \in [p^n_\delta, a_\delta]\big\}$. □

We are ready to prove the main theorem.

Theorem 1. There exists a Bellamy continuum with a strong non-cut point.

Proof. By Theorem 1 of [1] the limit $X = \varprojlim\{X(\alpha); f^\alpha_\beta : \beta, \alpha < \omega_1\}$ is indecomposable with at most two composants. The *trivial composant* $E \subset X$ is the set $\bigcup\{X(\alpha) : \alpha < \omega_1\}$ of eventually constant ω_1-sequences. The *nontrivial composant* $X - E$ is equal to $\varprojlim\{C(\alpha); f^\alpha_\beta : \beta, \alpha < \omega_1\}$. The points $a = (a_\beta)$ and $c = (c_\beta)$ witness how $X - E$ is nonempty. Therefore X has exactly two composants.

Theorem 15 applied to $\{X(\alpha); f^\alpha_\beta : \beta, \alpha < \omega_1\}$ says there is a tail from $a = (a_\beta)$ to $c = (c_\beta)$. Lemma 4.4.2 says c does not weakly cut $X - E$.

Choose any two points $x \in E$ and $y \in X - E$ both distinct from c. Let $\widetilde{X}$ be obtained by treating $\{x, y\}$ as a single point. It follows $\widetilde{X}$ is an indecomposable continuum with exactly one composant and $c \in \widetilde{X}$ is not a weak cut point. $\square$

Finally we prove the lemmas cited throughout.

Lemma 4.4.4. Suppose $\{Y(\beta); f_\gamma^\beta : \gamma, \beta \in \Omega\}$ is an inverse system whose limit Y has points $p = (p_\beta)$ and $q = (q_\beta)$ and $a = (a_\beta)$. Suppose the set $\big\{\beta \in \Omega : [p_\beta, a_\beta, q_\beta]\big\}$ is cofinal. Then $[p, a, q]$.

Proof. First replace Ω with $\big\{\beta \in \Omega : a_\beta \in [p_\beta, a_\beta, q_\beta]\big\}$ hence assume each $[p_\beta, a_\beta, q_\beta]$. Now suppose $L \subset Y$ is a subcontinuum with $\{p, q\} \subset L$. Then each $\{p_\beta, q_\beta\} \subset \pi_\beta(L)$ and so $a_\beta \in \pi_\beta(L)$ since $\pi_\beta(L)$ is a subcontinuum. Now recall $L = \varprojlim\{\pi_\beta(L); f_\gamma^\beta : \gamma, \beta \in \Omega\}$. Since each $a_\beta \in \pi_\beta(L)$ we have $a \in L$. Since $L \subset Y$ is arbitrary we conclude $[p, a, q]$. $\square$

Corollary 4.4.5. Suppose $\{Y(\beta); f_\gamma^\beta : \gamma, \beta \in \Omega\}$ is an inverse system whose limit Y has points $p = (p_\beta)$ and $q = (q_\beta)$. Suppose $\big\{\beta \in \Omega : Y(\beta) = [p_\beta, q_\beta]\big\}$ is cofinal. Then $Y = [p, q]$.

Lemma 4.4.6. Suppose $\{X(\beta); f_\gamma^\beta : \gamma, \beta \in \Omega\}$ is an inverse system with points $a = (a_\beta)$ and $b = (b_\beta)$. Each $\pi_\gamma\big([a, b]\big) = \overline{\bigcup\big\{f_\gamma^\beta\big([a_\beta, b_\beta]\big) : \beta > \gamma\big\}}$.

Proof. Write $J_\gamma = \bigcup\big\{f_\gamma^\beta\big([a_\beta, b_\beta]\big) : \beta > \gamma\big\}$. We know $[a, b]$ has the form $\varprojlim\{I_\beta; f_\gamma^\beta : \gamma, \beta \in \Omega\}$ for some subcontinua $I_\beta \subset X(\beta)$. Since each I_β contains $\{a_\beta, b_\beta\}$ it contains $[a_\beta, b_\beta]$. By commutativity each $I_\gamma = f_\gamma^\beta(I_\beta)$ contains $f_\gamma^\beta\big([a_\beta, b_\beta]\big)$ for $\beta > \gamma$. hence I_γ contains J_γ and by closure contains $\overline{J_\gamma}$.

By commutativity each f_γ^β maps J_β onto J_γ. By continuity f_γ^β maps $\overline{J_\beta}$ onto $\overline{J_\gamma}$. Hence $J = \varprojlim\{\overline{J_\beta}; f_\gamma^\beta : \gamma, \beta \in \Omega\}$ is a well defined subcontinuum containing $\{a, b\}$. We have shown it is minimal with respect to containing $\{a, b\}$. We conclude $J = [a, b]$ and each $I_\beta = \overline{J_\beta}$ as required. $\square$

4.5 The Trivial Composant

We have already determined the weak cut structure of the nontrivial composant. Namely $c \in X - E$ is the only point to not weakly cut its composant.

This section examines the trivial composant $E \subset X$. We show E is weakly cut by its every point. Hence the continuum $\widetilde{X}$ from Theorem 1 has exactly one strong non-cut point.

To that end recall E is the set of eventually constant ω_1-sequences. The first claim is that the subcontinua of E share the property of being *eventually constant*.

Claim 20. Suppose the subcontinuum $L \subset E$ meets $X(\beta)$ and $X - X(\beta+1)$ for some $\beta < \omega_1$. Then $X(\beta+1) \subset L$.

Proof. By assumption L meets $X(\alpha) - X(\beta+1)$ for some $\alpha > \beta+1$. By hereditary unicoherence $X(\alpha) \cap L$ is a subcontinuum. Observe

$$f^{\alpha}_{\beta+1}\big(X(\alpha) \cap L\big) = f^{\alpha}_{\beta+1}\big((X(\alpha) \cap L) \cap X(\beta+1)\big) \cup f^{\alpha}_{\beta+1}\big((X(\alpha) \cap L) - X(\beta+1)\big).$$

Since the map is a retraction the first summand equals $L \cap X(\beta+1)$ which meets $X(\beta)$ by assumption and hence meets $D(\beta+1)$. By (b) the second summand is contained in $C(\beta+1)$.

Thus the subcontinuum $f^{\alpha}_{\beta+1}(X(\alpha) \cap L) \subset X(\beta+1)$ meets the distinct composants $C(\beta+1)$ and $D(\beta+1)$ hence equals $X(\beta+1)$. Since the second summand is contained in $C(\beta+1)$ the first summand $L \cap X(\beta+1)$ must contain $D(\beta+1)$. Since $L \cap X(\beta+1)$ is closed and $D(\beta+1)$ dense we have $L \cap X(\beta+1) = X(\beta+1)$ and so $X(\beta+1) \subset L$. $\square$

One consequence of Claim 20 is any subcontinuum that meets all $X(\alpha)$ must contain all $X(\alpha)$ and hence equal X. The next claim follows.

Claim 21. Each subcontinuum of E is contained in some $X(\alpha)$.

Claim 22. Each point of E weakly cuts its composant.

Proof. Let $b \in E$ be arbitrary. Then $b \in X(\beta)$ for some $\beta < \omega_1$. Choose any $a \in X(\beta) - b$ and $c \in X(\beta+2) - X(\beta+1)$. By Claim 20 each subcontinuum that includes a and c must contain $X(\beta+1)$. Hence the subcontinuum contains $X(\beta)$ and includes b. We conclude $[a, b, c]$. □

Theorem 2. There exists a Bellamy continuum with exactly one strong non-cut point.

Proof. Let X be the continuum from Theorem 1 and $X \to \widetilde{X}$ the quotient map that treats $\{x, y\}$ as the single point $z \in \widetilde{X}$. We have already shown $\widetilde{c} \in \widetilde{X}$ is a strong non-cut point. Now suppose $b \in \widetilde{X} - \widetilde{c}$. For $b = z$ we have $[r, b, s]$ for each $r \in \widetilde{\kappa(x)} - z$ and $s \in \widetilde{\kappa(y)} - z$.

Now assume $b \neq z$. Then $b = \widetilde{d}$ for some unique $d \in X - \{x, y\}$. The composant $\kappa(d)$ is one of E or $X - E$. In the first case Claim 22 says $[r, d, s]$ for some pair $r, s \in \kappa(d)$. In the second case Lemma 4.4.2 says the same.

Boundary bumping implies each continuum component of $X - d$ contains a nondegenerate subcontinuum hence has infinitely many points. That means we can reselect r and s outside $\{x, y\}$ if necessary. Hence $\widetilde{r}, \widetilde{s} \neq z$. We claim $[\widetilde{r}, b, \widetilde{s}]$ thus b weakly cuts $\widetilde{X}$.

Suppose to reach a contradiction $\{\widetilde{r}, \widetilde{s}\} \subset L \subset \widetilde{X} - b$ for some subcontinuum $L \subset \widetilde{X}$. Clearly $z \in L$ as otherwise $L = \widetilde{P}$ for some subcontinuum $P \subset X - \{x, y\}$. Then P contradicts how $[r, d, s]$.

Let C_r and C_s be the continuum components of $\widetilde{r}$ and $\widetilde{s}$ in $L - z$ respectively. Then $C_r = \widetilde{D}_r$ and $C_s = \widetilde{D}_s$ for semicontinua $D_r, D_d \subset X - \{x, y\}$. It follows $D_r, D_s \subset \kappa(d) - \{x, y\}$. Boundary bumping says $z \in \overline{C}_r$ and $z \in \overline{C}_s$. The definition of the quotient topology implies $\overline{D}_r$ and $\overline{D}_s$ meet $\{x, y\}$.

Without loss of generality $\kappa(d) = \kappa(x) = E$. Recall D_r and D_s are mapped into L which is proper. Continuity of the quotient says $\overline{D}_r$ and $\overline{D}_s$ are mapped into L hence proper. We cannot have $y \in \overline{D}_r$ as then the proper subcontinuum $\overline{D}_r$ meets both composants of X. Likewise $y \notin \overline{D}_s$. We conclude both $x \in \overline{D}_r$ and $x \in \overline{D}_s$.

Hence $\overline{D}_r \cup \overline{D}_s$ is a subcontinuum of X. Since X is indecomposable each summand is nowhere dense, and the same follows for the union. Thus the subcontinuum $\overline{D}_r \cup \overline{D}_s$ contradicts how $[r, d, s]$. We conclude no such subcontinuum $L \subset \widetilde{X}$ exists and therefore $[\widetilde{r}, b, \widetilde{s}]$. $\square$

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Chapter 5

Separable Indecomposable Continuum with Exactly One Composant

Abstract

Indecomposable continua with one composant are *large* in the sense of being non-metrisable. We adapt the method of Smith [18] to construct an example which is *small* in the sense of being separable.

5.1 Introduction

To date all known indecomposable continua with exactly one composant, henceforth called Bellamy continua, fall into two classes. Continua of the first class are Stone-Čech remainders of certain locally-compact spaces – and indeed these remainders need additional set-theoretic assumptions to be Bellamy [3, 5, 8].

Continua of the second class are those obtained by Bellamy and Smith by carefully constructing an ω_1-chain of metric continua and retractions so the limit has exactly two composants. We then select two points, one from each composant, and identify them to get exactly one composant [4, 17, 18, 19].

It is well-known that each Bellamy continuum is *large* in the sense of being non-metrisable [14]. This paper is about Bellamy continua that are *small* in the sense of being separable. The first class mentioned above provides no examples. Indeed Corollary 5.6 of [20] says the Stone-Čech remainder of any well-behaved locally-compact Hausdorff space is $\aleph_1$-cellular hence non-separable.

Section 3 contains preliminary results about separability of certain inverse limits of metric continua. The results are applied in Section 4 where we produce a modification of the inverse-system of Smith [18]. The modification itself is minor and only needed to make the limit separable – it is inessential to showing the limit has exactly two composants. We obtain a separable Bellamy continuum by *spot-welding* as usual.

Section 5 shows both composants are non-metrisable. One composant is separable and the other is non-separable. Hence our example is separable but not hereditarily separable. The problem is open whether there exists an hereditarily separable Bellamy continuum.

Section 6 extends familiar results about Bellamy continua by showing each hereditarily unicoherent metric continuum is a retract of a separable Bellamy continuum. The problem is open whether the same holds for each separable continuum. Section 7 shows our modification is necessary – the original limit of Smith fails to be separable.

5.2 Terminology and Notation

Throughout X is a continuum. That means a nondegenerate compact and connected Hausdorff space. For background on metric continua see [11] and [14]. The results cited here have analagous proofs for non-metric continua.

We call X *separable* to mean it has a dense countable subset, *hereditarily separable* to mean every subset is separable, and *dense-hereditarily separable* to mean every dense subset is separable. Metric continua are separable but the converse fails in general. For each cardinal α we say X is α*-cellular* to mean it admits a family of α-many pairwise disjoint open subsets. Clearly each $\aleph_1$-cellular continuum is non-separable.

For a subset $S \subset X$ denote by S° and $\overline{S}$ the interior and closure of S respectively. By *boundary bumping* we mean the principle that, for each closed $E \subset X$, each component C of E meets $\partial E = \overline{E} \cap \overline{X - E}$. For the non-metric proof see §47, III Theorem 2 of [11]. One corollary of boundary bumping is that the point $p \in X$ is in the closure of each continuum component of $X - p$.

Throughout all maps between continua are assumed to be continuous. We call $f : Y \to X$ a *retraction* to mean X is a subspace of the continuum Y and the restriction of f to X is the identity. The partition $\mathcal{P}$ of X into closed subsets is called *upper semicontinuous* to mean the following: For each $P \in \mathcal{P}$ and open $U \subset X$ containing P there is open $V \subset U$ with $P \subset V$ and V a union of elements of $\mathcal{P}$. Upper semicontinuity of the partition is equivalent to the quotient space $X/\mathcal{P}$ being a continuum.

For $b \in X$ we omit the curly braces and write $X - b$ instead of $X - \{b\}$ without confusion. For $a, b \in X$ we say X is *irreducible* about $\{a, b\}$ to mean no proper subcontinuum of X contains $\{a, b\}$. For $a, b \in X$ we write $[a, b]$ for the intersection of all subcontinua that contain $\{a, b\}$. Note $[a, b]$ is not in general connected as the interval notation suggests. Clearly $h\big([a, b]\big) = [h(a), h(b)]$ for each $a, b \in X$ and homeomorphism $h : X \to Y$.

We say X is *indecomposable* to mean it is not the union of two proper subcontinua. Equivalently each proper subcontinuum is nowhere dense. The *composant* $\kappa(x)$ of the point $x \in X$ is the union of all proper subcontinua that have x as an element. Indecomposable metric continua are partitioned into $\mathfrak{c}$ many pairwise disjoint composants [13]. In case $\kappa(x) \neq \kappa(y)$ then X is irreducible about $\{x, y\}$. There exist indecomposable non-metric continua with exactly one composant [4] henceforth called *Bellamy continua*.

We say X is *hereditarily unicoherent* to mean the intersection of any two subcontinua of X is empty or connected. Equivalently each interval $[a, b]$ is a continuum. It then follows $[a, b]$ is the unique subcontinuum of X irreducible about $\{a, b\}$.

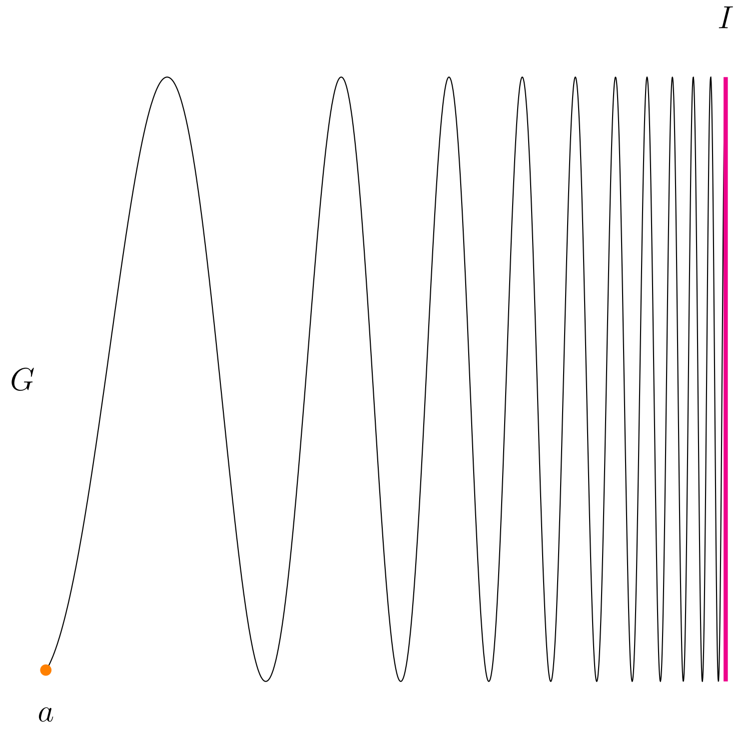

Figure 5.1: The $\sin(1/x)$ continuum

Throughout the $\sin(1/x)$ continuum is the metric continuum defined as the union of the graph $G = \big\{\big(x, \sin(1/x)\big) : -\pi/2 \leq x < 0\big\}$ and the arc $I = \{0\} \times [-1, 1]$. The composant $\kappa(a)$ of the endpoint $a = (-\pi/2, -1)$ is equal to G

and its every nondegenerate subcontinuum is equal to the closure of its interior.

The proper subcontinuum $R \subset X$ is called a *rung* to mean each other subcontinuum $K \subset X$ is either disjoint from, contained in, or contains R. For example the limiting arc I is a rung of the $\sin(1/x)$ continuum. By a *ladder* on X we mean a nested collection of rungs of X with dense union. Continua that admit ladders are rare. For example the $\sin(1/x)$ continuum admits no ladders. Note what we call rungs are sometimes called *terminal subcontinua*, but that term also has several unrelated meanings across continuum theory [6].

Throughout $\omega = \{0, 1, 2, \ldots\}$ is the first infinite ordinal and ω_1 the first uncountable ordinal. Each proper initial segment of ω_1 is countable and each countable subset has an upper bound. Each countable ordinal has a cofinal subset order-isomorphic to ω. For the ordered set Ω we say $\Psi \subset \Omega$ is *cofinal* to mean it has no upper bound in Ω.

The poset Ω is called *directed* to mean for each $\gamma, \beta \in \Omega$ there is $\alpha \in \Omega$ with $\gamma, \beta \leq \alpha$. Note most authors require $\gamma, \beta < \alpha$. The stronger condition prohibits Ω having a top element. In this paper it is convenient to allow a directed set to have a top element.

An *inverse system* over the directed set Ω consists of the following data: (1) a family of topological spaces $T(\alpha)$ for each $\alpha \in \Omega$ and (2) a family of continuous maps $f^\alpha_\beta : T(\alpha) \to T(\beta)$ for each $\beta \leq \alpha$ such that (3) we have $f^\beta_\gamma \circ f^\alpha_\beta = f^\alpha_\gamma$ whenever $\gamma \leq \beta \leq \alpha$. The property (3) is called *commutativity of the diagram*. The *inverse limit* T of the system is the space

$$\varprojlim \{T(\alpha); f^\alpha_\beta : \alpha, \beta \in \Omega\} = \Big\{ (x_\alpha) \in \prod_{\alpha \in \Omega} T(\alpha) : f^\alpha_\beta(x_\alpha) = x_\beta \text{ for all } \beta \leq \alpha \Big\}.$$

The functions f^α_β are called the *bonding maps*. Write $\pi_\beta : T \to T(\beta)$ for the restriction of the projection $\prod_\alpha T(\alpha) \to T(\beta)$. If each bonding map is surjective so is each π_β and we call the inverse system (limit) *surjective*. In case Ω has top element ∞ the inverse limit is a copy of $T(\infty)$. The inverse limit of a system of continua is a continuum.

5.3 The Successor Stage

We use transfinite recursion to construct the eponymous indecomposable continuum as the limit of a system $\{X(\alpha); f^{\alpha}_{\beta} : \alpha, \beta < \omega_1\}$ of metric continua and retractions. This section shows how to construct each $X(\beta + 1)$ from $X(\beta)$. The following section deals with limit ordinals.

To begin let $X(0)$ be the $\sin(1/x)$ continuum. Write a_0 for the endpoint and select a sequence q^n_0 in $X(0)$ with x-coordinates strictly increasing to 0 from below. Observe q^n_0 satisfies the following definition of being a *thick half-tail*.

Definition 5.3.1. For X a continuum we define a *half-tail* at $a \in X$ as a sequence $q^n \subset X$ with the properties:

(1) $[a, q^1] \subsetneq [a, q^2] \subsetneq \ldots$

(2) $\bigcup \big\{[a, q^n] : n \in \mathbb{N}\big\} = \kappa(a)$

(3) For each $x \in X$ and $n \in \mathbb{N}$ either $[a, q^n] \subset [a, x]$ or $[a, x] \subset [a, q^n]$.

Moreover the half-tail q^n is called *thick* to mean each $[a, q^n]$ is the closure of its interior.

Next use induction to find a family $\mathcal{D}_0 = \{D^1_0, D^2_0, \ldots\}$ of pairwise disjoint subsets of $\kappa(a_0)$ with each $D^n_0 \subset [a_0, q^n_0]$ dense. The fact that q^n_0 is thick implies that $\mathcal{D}_0$ is a *tailing family* as defined below.

Definition 5.3.2. Suppose the continuum X has a half-tail q^n at $a \in X$. By a *tailing family* on X we mean a pairwise disjoint collection $\mathcal{D} = \{D^1, D^2, \ldots\}$ of countable subsets of X with each $D^n \subset [a, q^n]$ and $D^n \cap [a, q^m]$ dense in $[a, q^m]$ for each $m \leq n$.

The notions of a half-tail and tailing family are pivotal to our example. Indeed as part of the construction we will at stage $\alpha < \omega_1$ choose a half-tail q^n_α at $a_\alpha \in$

$X(\alpha)$ and a tailing family $\mathcal{D}_\alpha$ on $X(\alpha)$ so both objectss behave nicely with respect to the bonding maps. This is made precise below.

In the next definition and throughout when we write for example $a_\beta \mapsto a_\gamma$ the map in question is understood to be the bonding map f^β_γ. Similarily for subsets $B \subset X(\beta)$ and $C \subset X(\gamma)$ we write $B \to C$ to mean $f^\beta_\gamma(B) = C$.

Definition 5.3.3. Suppose the continua $X(\beta)$ and $X(\gamma)$ have half-tails q^n_β and q^n_γ at $a_\beta \in X(\beta)$ and $a_\gamma \in X(\gamma)$ and tailing families $\mathcal{D}_\beta$ and $\mathcal{D}_\gamma$ respectively. The map $f : X(\beta) \to X(\gamma)$ is called *coherent* to mean $a_\beta \mapsto a_\gamma$ and $[a_\beta, q^n_\beta] \to [a_\gamma, r^n_\gamma]$ and f induces bijections $D^n_\beta \to D^n_\gamma$ for each $n \in \mathbb{N}$. The system $\{X(\beta); f^\beta_\gamma : \gamma, \beta < \alpha\}$ is called coherent to mean each bonding map is coherent.

At stage $\alpha < \omega_1$ we have already constructed the coherent inverse system $\{X(\beta); f^\beta_\gamma : \beta, \gamma < \alpha\}$ of hereditarily unicoherent metric continua and retractions. We assume the following objects have been specified for each $\beta < \alpha$:

(i) Half-tails q^n_β at $a_\beta \in X(\beta)$

(ii) Tailing families $\mathcal{D}_\beta = \{D^1_\beta, D^2_\beta, \ldots\}$ on $X(\beta)$

We also assume for each $\gamma, \delta < \alpha$ the two conditions hold:

(a) $\bigcup\{X(\delta) : \delta < \gamma\} \subset X(\gamma) - \kappa(a_\gamma)$

(b) $\bigcup \left\{ f^\gamma_\delta\big(X(\gamma) - X(\delta)\big) : \alpha > \gamma > \delta \right\} = \kappa(a_\delta)$

Conditions (a) and (b) come straight from [4] and will ensure the limit has exactly two composants. Coherence will ultimately be used to construct a tailing family on the inverse limit. Once we have shown the limit is a Bellamy continuum, the next lemma gives our main result.

Lemma 5.3.4. Suppose X admits a half-tail q^n and tailing family $\mathcal{D} = \{D^1, D^2, \ldots\}$. Then X is separable.

Proof. Clearly $\bigcup \mathcal{D} = D^1 \cup D^2 \cup \ldots$ has the cardinality of $\mathbb{N} \times \mathbb{N}$ which is well known to be countable. Now suppose $U \subset X$ is open. Since $\kappa(a)$ is dense it meets U. Since $\kappa(a) = \bigcup_n [a, q^n]$ some $[a, q^n]$ meets U. Since $[a, q^n] \cap U$ is open in $[a, q^n]$ it contains an element of D^n by the definition of a tailing family. We conclude $\bigcup \mathcal{D}$ is dense in X. □

We are now ready to begin the successor step. Suppose $\alpha = \beta + 1$ is a successor ordinal. We will construct the hereditarily unicoherent continuum $X(\beta + 1)$ and retraction $f_\beta^{\beta+1} : X(\beta + 1) \to X(\beta)$. Then we can define the bonding maps $f_\gamma^{\beta+1} = f_\beta^{\beta+1} \circ f_\gamma^\beta$. We will specify the objects (i) and (ii) when β is replaced by $\beta + 1$. Finally we will check the enlarged system is coherent and Conditions (a) and (b) hold for all $\gamma, \delta \le \beta + 1$.

To begin the construction of $X(\beta + 1)$ consider the following subset N of $X(\beta) \times [0, 1]$.

$$N = \Big(\bigcup \big\{[a_\beta, q_\beta^n] \times \{1/(2n-1), 1/2n\} : n \in \mathbb{N}\big\}\Big) \cup \big(X(\beta) \times \{0\}\big).$$

Define the points $b_0, b_1, b_2, \ldots \in N$.

$$\begin{aligned} b_{4n} &= \big(a_\beta, 1/(2n+1)\big) & \quad b_{4n+1} &= \big(q_\beta^n, 1/(2n+1)\big) \\ b_{4n+2} &= \big(q_\beta^n, 1/(2n+2)\big) & \quad b_{4n+3} &= \big(a_\beta, 1/(2n+2)\big) \end{aligned}$$

To obtain $X(\beta+1)$ make for each $n \in \mathbb{N}$ the identification $b_{4n+1} \sim b_{4n+2}$. Call that point $c_{2n+1} \in X(\beta + 1)$. Then make the identification $b_{4n+3} \sim b_{4(n+1)}$. Call that point $c_{2(n+1)} \in X(\beta + 1)$.

Write $J(2n)$ for the quotient space of each $[a_\beta, q_\beta^n] \times \{1/(2n+1)\}$ and $J(2n + 1)$ for the quotient space of $[a_\beta, q_\beta^n] \times \{1/2n\}$. Observe each $J(n)$ is irreducible from c_n to c_{n+1}.

Clearly the quotient space of $\bigcup\{J(n) : n \in \mathbb{N}\}$ is connected and its closure is the union with $X(\beta) \times \{0\}$. Therefore $X(\beta+1)$ is connected. Identify $X(\beta)$ with the subspace $X(\beta) \times \{0\}$. The projection $X(\beta) \times [0,1] \to X(\beta)$ respects the identifications and therefore induces a retraction $f_\beta^{\beta+1} : X(\beta+1) \to X(\beta)$.

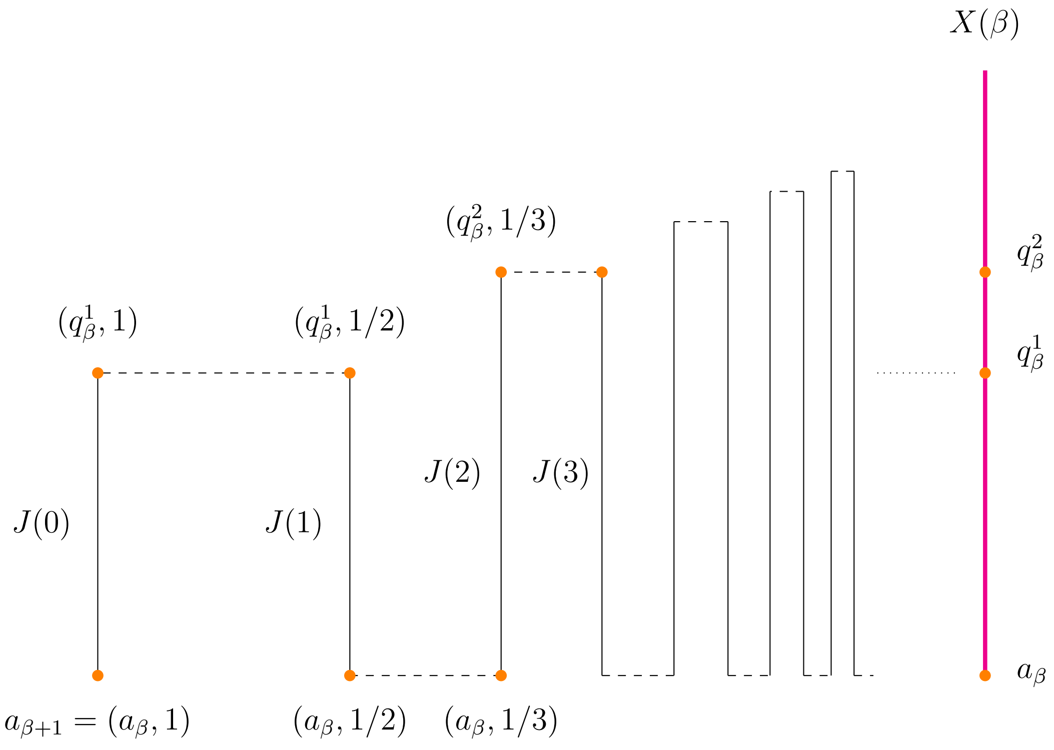

Figure 5.2: Schematic for $X(\beta+1)$. Dashed lines indicate identifications. The bonding map $f_\beta^{\beta+1}$ projects to the right.

Claim 1. $X(\beta+1)$ is compact metric.

Proof. Let $\mathcal{P}$ be the partition induced on N by the identifications. Each partition element is either a singleton or doubleton hence closed. Let $P \in \mathcal{P}$ be contained in the open $U \subset N$. We claim some open neighborhood W of P is a union of partition elements. Then $V = U \cap W$ witnesses how $\mathcal{P}$ is upper semicontinuous and $X(\beta+1)$ is compact metric by [14] Lemma 3.2.

For P a singleton of some $J(2n)$ take $W = J(2n) - \{b_{4n}, b_{4n+1}\}$. For P a singleton of some $J(2n+1)$ take $W = J(2n+1) - \{b_{4n+2}, b_{4n+3}\}$. For P a singleton of $X(\beta)$ observe $U \cap X(\beta)$ is an open neighborhood of P in $X(\beta)$. Since $F = f_\beta^{\beta+1}$ respects $\mathcal{P}$ the open set $W = F^{-1}\big(U \cap X(\beta)\big)$ is a union of partition elements.

For P a doubleton without loss of generality some $n \in \mathbb{N}$ has $P = \big\{b_{4n+1}, b_{4n+2}\big\}$. Since $J(2n) - b_{4n}$ and $J(2n+1) - b_{4n+3}$ are open in N so is the union $W = \big(J(2n) - b_{4n}\big) \cup \big(J(2n+1) - b_{4n+3}\big)$. The case for $P = \big\{b_{4n+3}, b_{4(n+1)}\big\}$ is similar. $\square$

Claim 2. The composant $\kappa(a_{\beta+1}) = X(\beta+1) - X(\beta)$. Hence Condition (a) holds for all $\gamma, \delta \leq \beta + 1$.

Proof. Let $n \in \mathbb{N}$ be arbitrary and $U \subset J(n) - \{c_n, c_{n+1}\}$ open. Since $J(n)$ is irreducible from c_n to c_{n+1} boundary bumping implies $J(n) - U = A \cup B$ is the disjoint union of two nonempty clopen sets that include c_n and c_{n+1} respectively. It follows

$$X(\beta+1) - U = \big(J(1) \cup \ldots J(n-1) \cup A\big) \cup \big(B \cup J(n+1) \cup \ldots \cup X(\beta)\big)$$

is a disjoint union of two clopen sets that include $a_{\beta+1}$ and contain $X(\beta)$ respectively.

Now suppose the subcontinuum K connects $a_{\beta+1}$ to $X(\beta)$. Observe for each $n \in \mathbb{N}$ the set $O = J(0) \cup J(1) \cup \ldots \cup J(n-1) - c_n$ is clopen in $X(\beta+1) - c_n$ and has $a_{\beta+1} \in O$. We conclude all $c_n \in K$.

For K proper it excludes some open $U \subset J(n) - \{c_n, c_{n+1}\}$. Then the two clopen sets from the first paragraph contradict how K is connected. We conclude $\kappa(a_{\beta+1}) \subset X(\beta+1) - X(\beta)$. The other inclusion is witnessed by the subcontinua $J(1) \cup J(2) \cup \ldots \cup J(n)$. $\square$

Recall each $J(n)$ is hereditarily unicoherent. For $n \leq m$ it is straightforward to prove by induction each subcontinuum $I(n,m) = J(n) \cup \ldots \cup J(m)$ is hereditarily unicoherent.

Claim 3. Each subcontinuum of $\kappa(a_{\beta+1})$ is contained in some $I(n,m)$.

Proof. Suppose $K \subset \kappa(a_{\beta+1})$ is a proper subcontinuum. Without loss of generality assume $a_{\beta+1} \in K$. We claim K meets only finitely many $J(n)$. For otherwise there is a sequence $n(1), n(2), \ldots$ with $n(i) \to \infty$ and elements $x_i \in K \cap J(n(i))$.

Consider the sequence $f_\beta^{\beta+1}(x_1), f_\beta^{\beta+1}(x_2), \ldots$ in $X(\beta)$. Since $X(\beta)$ is compact metric $f_\beta^{\beta+1}(x_i)$ has a subsequence tending to some $x \in X(\beta)$. It follows x_i has a subsequence tending to $(x,0)$. Since K is closed it includes $(x,0) \in X(\beta)$. But then Claim 2 says $K = X(\beta+1)$ and so $K \not\subset \kappa(a_{\beta+1})$ contrary to assumption. $\square$

Claim 4. $X(\beta+1)$ is hereditarily unicoherent.

Proof. Suppose K and L are proper subcontinua. For K and L contained in $\kappa(a_{\beta+1})$ Claim 3 says $K \cup L \subset I(n,m)$ for some $n \leq m$. Since $I(n,m)$ is hereditarily unicoherent $K \cap L$ is empty or connected. For $K, L \subset X(\beta)$ then $K \cap L$ is empty or connected since $X(\beta)$ is hereditarily unicoherent.

Now suppose K meets $\kappa(a_{\beta+1})$ and $X(\beta+1) - \kappa(a_{\beta+1}) = X(\beta)$. We claim $K = K' \cup J(n+1) \cup \ldots \cup X(\beta)$ for some $n \in \mathbb{N}$ and subcontinuum $K' \subset J(n)$ with $c_{n+1} \in K'$. To that end let $n \in \mathbb{N}$ be minimal with $K \cap J(n) \neq \varnothing$. Then $K \cap J(n-1) = \varnothing$ and so $c_n \notin K$. Observe the subcontinuum $I(0,n) \cup K$ connects $a_{\beta+1}$ to $X(\beta)$. Hence $I(0,n) \cup K = X(\beta+1)$ by Claim 2 and K contains $J(n+1) \cup \ldots \cup X(\beta)$. Since $O = J(0) \cup J(1) \cup \ldots \cup J(n) - c_{n+1}$ is clopen in $X(\beta+1) - c_{n+1}$ and disjoint from $X(\beta)$ we must have $c_{n+1} \in K$.

Suppose $K \cap J(n) = A \cup B$ is the disjoint union of two nonempty closed sets with $c_{n+1} \in A$. Since $K \cap J(n)$ is closed in $X(\beta+1)$ so are A and B. Observe

$C = J(n+1) \cup J(n+2) \cup \ldots \cup X(\beta)$ is closed and $C \cap J(n) = \{c_{n+1}\}$. Thus $K = \big((C \cap K) \cup A\big) \cup B$ is a disjoint union of nonempty closed sets. We conclude $K \cap J(n)$ is connected.

Thus $K = K' \cup J(n+1) \cup \ldots \cup X(\beta)$ and $L = L' \cup J(m+1) \cup \ldots \cup X(\beta)$ for some $m, n \in \mathbb{N}$ and subcontinua $K' \subset J(n)$ and $L' \subset J(m)$. For $m \leq n$ we have $K \cap L = L$ is connected. For $n \leq m$ we have $K \cap L = K$ is connected. For $n = m$ we have $K \cap L = (L' \cap K') \cup J(m+1) \cup \ldots \cup X(\beta)$ which is a subcontinuum by hereditary unicoherence of $J(n)$. □

Claim 5. Condition (b) holds for all $\gamma, \delta \leq \beta + 1$.

Proof. By induction each $\bigcup \big\{ f_\delta^\gamma \big(X(\gamma) - X(\delta) \big) : \beta + 1 > \gamma > \delta \big\} = \kappa(a_\delta)$. Thus we can factor the set $\bigcup \big\{ f_\delta^\gamma \big(X(\gamma) - X(\delta) \big) : \beta + 1 \geq \gamma > \delta \big\}$ as the union $f_\delta^{\beta+1}\big(X(\beta+1) - X(\delta)\big) \cup \kappa(a_\delta)$. So it is enough to show the second factor is contained in $\kappa(a_\delta)$. To that end write

$$f_\delta^{\beta+1}\big(X(\beta+1) - X(\delta)\big) = f_\delta^{\beta+1}\big(X(\beta+1) - X(\beta)\big) \cup f_\delta^{\beta+1}\big(X(\beta) - X(\delta)\big)$$

By definition $f_\delta^{\beta+1} = f_\delta^\beta \circ f_\beta^{\beta+1}$. Since $f_\beta^{\beta+1}$ is a retraction the second term equals $f_\delta^\beta \big(X(\beta) - X(\delta) \big)$ which is contained in $\kappa(a_\delta)$ by Condition (b) for earlier stages.

Claim 2 says $X(\beta+1) - X(\beta) = \kappa(a_{\beta+1})$. Since $q_{\beta+1}^n$ is a half-tail we have $\kappa(a_{\beta+1}) = \bigcup_n [a_{\beta+1}, q_{\beta+1}^n]$ by Property (2). Hence the first term can be written $f_\delta^{\beta+1}\big(\kappa(a_{\beta+1})\big) = f_\delta^{\beta+1}\big(\bigcup_n [a_{\beta+1}, q_{\beta+1}^n]\big) = \bigcup_n f_\delta^{\beta+1}\big([a_{\beta+1}, q_{\beta+1}^n]\big)$ which equals $\bigcup_n [a_\delta, q_\delta^n] = \kappa(a_\delta)$ by coherence of the bonding maps and Property (2) at stage δ respectively. □

Claim 6. The sequence $q_{\beta+1}^n = c_{2n-1}$ is a half-tail at $a_{\beta+1}$.

Proof. It is straightforward to verify $J(0) \cup J(1) \cup \ldots \cup J(2n-2)$ is a subcontinuum irreducible from $a_{\beta+1}$ to c_{2n-1}. By hereditary unicoherence that subcontinuum is $[a_{\beta+1}, c_{2n-1}]$. Since each $c_{2n+1} \notin [a_{\beta+1}, c_{2n-1}]$ we have $q^{n+1}_{\beta+1} \notin [a_{\beta+1}, q^n_{\beta+1}]$ which is Property (1). To prove Property (2) observe $\bigcup_n [a_{\beta+1}, q^n_{\beta+1}] = J(1) \cup J(2) \cup \ldots = \kappa(a_{\beta+1})$ by Claim 2.

To prove Property (3) suppose $x \in X(\beta+1) - [a_{\beta+1}, q^n_{\beta+1}]$. Observe $[a_{\beta+1}, c_{2n-1}] - c_{2n-1}$ is clopen in $X(\beta+1) - c_{2n-1}$. Thus the continuum $[a_{\beta+1}, x]$ includes the point c_{2n-1}. By Zorn's lemma the continuum $[a_{\beta+1}, x]$ contains a subcontinuum irreducible from $a_{\beta+1}$ to c_{2n-1}. Since $[a_{\beta+1}, c_{2n-1}]$ is the only such subcontinuum we have $[a_{\beta+1}, c_{2n-1}] \subset [a_{\beta+1}, x]$ and so $[a_{\beta+1}, q^n_{\beta+1}] \subset [a_{\beta+1}, x]$. □

For the successor continua in our system we need to use how the half-tail is thick. This holds only for successor stages. We will later see the limit continua are indecomposable. Hence their every subcontinuum has void interior and they cannnot admit a thick half-tail.

Claim 7. The half-tail $q^n_{\beta+1}$ is thick.

Proof. We claim each $J(n)^\circ$ contains the dense open subset $J(n) - \{c_n, c_{n+1}\}$. Hence $[a_{\beta+1}, q^n_{\beta+1}]^\circ$ contains the dense subset $J(1)^\circ \cup J(2)^\circ \cup \ldots \cup J(2n-2)^\circ$. Let $x \in J(n) - \{c_n, c_{n+1}\}$ be arbitrary. Recall $J(n)$ is a copy of some $[a_\beta, q^m_\beta]$ with x corresponding to some $y \in [a_\beta, q^m_\beta]$ and each of c_n, c_{n+1} corresponding to exactly one of a_β or q^m_β.

Choose open $U \subset X(\beta)$ with $y \in U \subset X(\beta) - \{a_\beta, q^m_\beta\}$ and positive $\varepsilon < \min\left\{\left|\frac{1}{m} - \frac{1}{m+1}\right|, \left|\frac{1}{m} - \frac{1}{m-1}\right|\right\}$. Observe $U \times \left(\frac{1}{m} - \varepsilon, \frac{1}{m} + \varepsilon\right)$ is an open subset of N is disjoint from $\{b_0, b_1, b_2, \ldots\}$. Hence it corresponds to an open neighborhood of x in $X(\beta+1)$. The choice of ε ensures it is contained in

$J(n) - \{c_n, c_{n+1}\}$. $\square$

Claim 8. There exists a tailing family $\mathcal{D}_{\beta+1} = \{D^1_{\beta+1}, D^2_{\beta+1}, \ldots\}$ on $X(\beta+1)$ that makes $f^{\beta+1}_\beta$ coherent.

Proof. We first construct the sets $D^M_{\beta+1}$. For now let $M \in \mathbb{N}$ be fixed. Since $X(\beta+1)$ is metric there is a countable basis $U_1, U_2 \ldots$ for $[a_{\beta+1}, q^M_{\beta+1}]$. By the first paragraph U_1 meets some $J(n) \subset [b, r^M]$ that maps homeomorphically onto $[a, q^m]$ for some $m \leq M$. Since $U_1 \cap J(n)$ is open in $J(n)$ the image $f^{\beta+1}_\beta(U_1) \cap [a_\beta, q^n_\beta]$ is open in $[a_\beta, q^n_\beta]$. Since $m \leq M$ the definition of a tailing family says $D^M_\beta \cap [a_\beta, q^m_\beta]$ is dense in $[a_\beta, q^m_\beta]$. Thus $f^{\beta+1}_\beta(U_1) \cap [a_\beta, q^m_\beta]$ contains infinitely many elements of $D^M_\beta \cap [a_\beta, q^m_\beta]$. Choose one such $d(1) \in D^M_\beta \cap [a_\beta, q^n_\beta]$ and select $c(1) \in U_1$ with $c(1) \mapsto d(1)$.

Proceed by induction. At stage r we have chosen distinct $c(i) \in U_i$ for $i = 1, 2, \ldots, r-1$ and $d(i) = f\big((c(i)\big)$ are distinct. Just like before $f^{\beta+1}_\beta(U_r)$ includes infinitely many elements of D^M_β. Select some $d(r) \in f(U_r) \cap D^M_\beta$ with $d(1), d(2), \ldots, d(r)$ distinct. Then select $c(r) \in U_r$ with $c(r) \mapsto d(r)$.

By construction we get a countable dense subset $E^M_{\beta+1} = \{c(1), c(2), \ldots\}$ of distinct elements of $[a_{\beta+1}, q^M_{\beta+1}]$. For each $d \in D^M - \{d(1), d(2), \ldots\}$ use surjectivity to select $c \in [a_{\beta+1}, q^M_{\beta+1}]$ with $c \mapsto d$. Adjoin all such d to $E^M_{\beta+1}$ to get the set $D^M_{\beta+1}$. By construction $f^{\beta+1}_\beta$ induces a bijection $D^M_{\beta+1} \to D^M_\beta$.

Now let M vary over $\mathbb{N}$. We get a family $\mathcal{D}_{\beta+1} = \{D^1_{\beta+1}, D^2_{\beta+1}, \ldots\}$ of countable subsets of $X(\beta+1)$ with each $f^{\beta+1}_\beta : D^M_{\beta+1} \to D^M_\beta$ a bijection and $D^M_{\beta+1} \subset [a_{\beta+1}, q^M_{\beta+1}]$ dense. Since the elements of $\mathcal{D}_\beta$ are pairwise disjoint so are the elements of $\mathcal{D}_{\beta+1}$. Claim 7 shows each $[a_{\beta+1}, q^m_{\beta+1}]$ is the closure of its interior. From this it follows $D^M_{\beta+1} \cap [a_{\beta+1}, q^m_{\beta+1}]$ is dense in $[a_{\beta+1}, q^m_{\beta+1}]$ whenever $m \leq M$. We conclude $\mathcal{D}_{\beta+1}$ is a tailing family.

To show coherence first recall by construction $J(0)$ is a copy of $[a_\beta, q^1_\beta]$ and projects under $f^{\beta+1}_\beta$ onto that copy. Since $a_{\beta+1} \in J(0)$ corresponds to the point $a_\beta \in [a_\beta, q^1_\beta]$ we have $a_{\beta+1} \mapsto a_\beta$. Recall we define $q^n_{\beta+1} = c_{2n-1}$ and by definition $c_{2n-1} \sim \big(q^n_\beta, 1/(2n-1)\big)$. Since $f^{\beta+1}_\beta$ is induced by the projection $X(\beta) \times [0,1] \to X(\beta)$ we have $q^n_{\beta+1} \mapsto q^n_\beta$. Finally recall $[a_{\beta+1}, c_{2n-1}] = J(0) \cup J(1) \cup \ldots \cup J(2n-2)$. By cooherence this set maps onto $[a_\beta, q^1_\beta] \cup [a_\beta, q^2_\beta] \cup \ldots \cup [a_\beta, q^n_\beta]$. By Property (1) for β the image equals $[a_\beta, q^n_\beta]$ as required. $\square$

5.4 The Limit Stage

This section deals with the limit stage of our construction. Henceforth assume $\alpha \leq \omega_1$ is a limit ordinal and $\{X(\beta); f_\gamma^\beta : \beta, \gamma < \alpha\}$ a coherent system of hereditarily unicoherent metric continua. For all $\beta, \gamma, \delta < \alpha$ we assume the objects (i) and (ii) from Section 5.4 have been specified and Conditions (a) and (b) hold.

We define $X(\alpha) = \varprojlim\{X(\beta); f_\gamma^\beta\}$ and each f_β^α as the projection from the inverse limit onto its factors. For each $\gamma < \alpha$ we identify $X(\gamma)$ with the subset $\big\{x \in X(\alpha) : x_\beta = x_\gamma \text{ for all } \beta > \gamma\big\}$ of $X(\alpha)$.

That $X(\alpha)$ is hereditarily unicoherent follows from a straightforward modification of [7] Corollary 1. To see $X(\alpha)$ is metric we observe by definition α is a countable ordinal. The product $\prod_{\beta<\alpha} X(\beta)$ of countably many metric spaces is itself a metric space. The inverse limit $X(\alpha)$ is by definition a subset of that product and therefore a metric space

It remains to show the enlarged system is coherent; to check Conditions (a) and (b) hold for the enlarged system; and to specify the data (i) and (ii) for $X(\alpha)$. By coherence at earlier stages each $a_\beta \mapsto a_\gamma$ and $q_\beta^n \mapsto q_\gamma^n$. Hence there are well defined points $a_\alpha = (a_\beta)_{\beta<\alpha}$ and $q_\alpha^n = (q_\beta^n)_{\beta<\alpha}$ in the limit $X(\alpha)$ with $a_\alpha \mapsto a_\beta$ and $q_\alpha^n \mapsto q_\beta^n$. For ease of notation write a and q^n instead of a_α and q_α^n respectively.

For the proof that q^n is a half-tail we refer to [2] where we introduce the more complicated notion of a *tail* and study inverse limits of tails. The proof for half-tails follows from a close reading of the proofs of [2] Claims 12 and 13 and Lemma 4.4.4. Hence we have the next claim.

Claim 9. The sequence q^n is a half-tail at a. Moreover for each $n \in \mathbb{N}$ we have $[a, q^n] = \varprojlim \big\{[a_\beta, q_\beta^n]; f_\gamma^\beta\big\}$.

Recall the proper subcontinuum $R \subset X$ is called a *rung* to mean each other subcontinuum $K \subset X$ is either disjoint from, contained in, or contains R. By a *ladder* on X we mean a nested collection of rungs of X with dense union. The proof that $X(\alpha)$ is indecomposable will follow from the existence of a ladder.

Lemma 5.4.1. Suppose X admits a ladder. Then X is indecomposable.

Proof. We first show rungs have void interior. Suppose otherwise the rung $R \subset X$ has $R^\circ \neq \varnothing$. Let $b \in X - R$ be arbitrary and C the component of b in $X - R$. Then b witnesses how $\overline{C} \not\subset R$. At the same time $C \subset X - R$ which implies $\overline{C} \subset X - R^\circ$ which in turn implies $R \not\subset \overline{C}$. Since R is a rung the subcontinua $\overline{C}$ and R must be disjoint. But this contradicts boundary bumping which says says $\overline{C}$ meets R. We conclude each $R^\circ = \varnothing$.

Now suppose $\mathcal{L}$ is a ladder on X and the proper subcontinuum $L \subset X$ has nonvoid interior. Since $\bigcup \mathcal{L}$ is dense some rung $R \in \mathcal{L}$ meets L° and $X - L$ hence contains L. Since R has void interior so does L. We conclude each subcontinuum of X has void interior. This is equivalent to X being indecomposable. $\square$

Since our simplest example of a rung is the limiting arc of the $\sin(1/x)$ continuum, and each $X(\beta + 1)$ looks like a $\sin(1/x)$ continuum limiting to $X(\beta)$, the next claim should be unsurprising.

Claim 10. Each $X(\gamma)$ is a rung of $X(\alpha)$.

Proof. The proof uses transfinite induction. Suppose for some $\widetilde{\alpha} \leq \alpha$ and all $\gamma \leq \beta < \widetilde{\alpha}$ that $X(\gamma) \subset X(\beta)$ is a rung. Now suppose the continuum $K \subset X(\widetilde{\alpha})$ meets $X(\gamma)$ and $X(\widetilde{\alpha}) - X(\gamma)$.

For $\widetilde{\alpha} = \widetilde{\beta} + 1$ a successor ordinal there are two possibilities. First that $K \subset X(\widetilde{\beta})$. In that case $K \subset X(\widetilde{\beta})$ meets $X(\gamma)$ and $X(\widetilde{\beta}) - X(\gamma)$ and we have $X(\gamma) \subset K$ since $X(\gamma) \subset X(\widetilde{\beta})$ is a rung by induction. Second that K meets $X(\widetilde{\beta} + 1) - X(\widetilde{\beta})$. It follows from Claim 2 that $X(\widetilde{\beta}) \subset X(\widetilde{\beta}+1)$ is a rung. Hence $X(\widetilde{\beta}) \subset K$ and $X(\gamma) \subset K$ by Property (a) at stage $\widetilde{\beta}$.

For $\widetilde{\alpha}$ a limit ordinal there are again two possibilities. First that $K \subset X(\beta)$ for some $\beta < \widetilde{\alpha}$. In that case $X(\gamma) \subset X(\beta)$ is a rung by induction hence $X(\gamma) \subset K$ as required. Second that K is contained in no $X(\beta)$. In that case recall $K = \varprojlim\{f_\delta^{\widetilde{\alpha}}(K); f_\delta^\beta : \beta, \delta < \widetilde{\alpha}\}$.

If there was $\beta < \widetilde{\alpha}$ with $f_\delta^{\widetilde{\alpha}}(K) \subset X(\beta)$ for all $\delta > \beta$ we would have $K \subset X(\beta)$ contrary to assumption. Thus for each $\beta < \widetilde{\alpha}$ there is $\delta > \beta$ with $f_\delta^{\widetilde{\alpha}}(K) \not\subset X(\beta)$. By induction $X(\beta) \subset X(\delta)$ is a rung. Hence the subcontinuum $f_\delta^{\widetilde{\alpha}}(K)$ of $X(\delta)$ contains $X(\beta)$ and its subset $X(\gamma)$. By commutativity $f_{\delta'}^{\widetilde{\alpha}}(K) = f_{\delta'}^{\delta} \circ f_\delta^{\widetilde{\alpha}}(K)$ also contains $X(\gamma)$ whenever $\gamma \leq \delta' \leq \delta$. In particular whenever $\gamma \leq \delta' \leq \beta$. Since $\beta < \widetilde{\alpha}$ is arbitrary we get $X(\gamma) \subset f_{\delta'}^{\widetilde{\alpha}}(K)$ for all $\gamma \leq \delta' < \widetilde{\alpha}$. It follows $X(\gamma) \subset K$ as required. □

Claim 11. $X(\alpha)$ is indecomposable.

Proof. Recall we define $X(\gamma) = \big\{(x_\beta) \in X(\alpha) : x_\beta = x_\gamma \ \forall \beta > \gamma\big\}$. By Lemma 5.4.1 it is enough to show $\{X(\beta) : \beta < \alpha\}$ is a ladder in $X(\alpha)$. To that end let $\gamma < \alpha$ be arbitrary and $U \subset X(\gamma)$ open. We must show $\pi_\gamma^{-1}(U)$ meets some $X(\beta)$. Let $x_\gamma \in U$ be arbitrary and consider the following $(x_\beta) \in X(\alpha)$. For $\beta \geq \gamma$ define $x_\beta = x_\gamma$. We have $f_\gamma^\beta(x_\beta) = x_\gamma$ since the bonding map is a retraction. For $\beta \leq \gamma$ define $x_\beta = f_\beta^\gamma(x_\gamma)$. It follows (x_β) is a well defined element of $X(\alpha)$. By definition $(x_\alpha) \in X(\gamma) \cap \pi_\gamma^{-1}(U)$. □

Claim 12. Condition (a) holds for all $\gamma, \delta \leq \alpha$.

Proof. By induction we only need to prove the case $\gamma = \alpha$. We must show $\kappa(a_\alpha)$ is disjoint from each $X(\beta)$. Since q^n is a half-tail at a each $x \in \kappa(a)$ is an element of some $[a, q^n] = \varprojlim[a_\beta, q_\beta^n]$. That means $x_{\beta+1} \in [a_{\beta+1}, q_{\beta+1}^n]$ for each $\beta < \alpha$. Since $q_{\beta+1}^n$ is a half-tail at $a_{\beta+1}$ we have $[a_{\beta+1}, q_{\beta+1}^n] \subset \kappa(a_{\beta+1})$ which is disjoint from $X(\beta)$ by Condition (a) for $\beta + 1$. We conclude $x_{\beta+1} \in X(\beta + 1) - X(\beta)$.

Since $x_\beta \in X(\beta)$ we have $x_{\beta+1} \neq x_\beta$ hence $x \notin X(\beta)$ by definition of the embedding $X(\beta) \to X(\alpha)$. $\square$

Claim 13. Condition (b) holds for all $\gamma, \delta \leq \alpha$.

Proof. By induction and commutativity it is enough to show $f^\alpha_\delta\big(X(\alpha) - X(\delta)\big) = \kappa(a_\delta)$ for each $\delta \leq \alpha$. Claim 12 says each $X(\beta)$ is disjoint from $\kappa(a_\alpha)$. Hence $X(\alpha) - X(\beta)$ contains $\kappa(a_\alpha)$ which maps onto $\kappa(a_\beta)$ by coherence. $\square$

Claim 14. There exists a tailing family $\mathcal{D}_\alpha = \{D^1_\alpha, D^2_\alpha, \ldots\}$ on $X(\alpha)$ that makes f^α_β coherent for each $\beta < \alpha$.

Proof. It is clear that $a \mapsto a_\beta$ and $q^n \mapsto q^n_\beta$. Claim 9 says each $[a, q^n] = \varprojlim \big\{[a_\beta, q^n_\beta]; f^\beta_\gamma\big\}$ hence $[a, q^n] \mapsto [a_\beta, q^n_\beta]$.

By induction each f^β_γ is coherent hence induces bijections $D^n_\beta \to D^n_\gamma$. That means the inverse limit $D^n_\alpha = \varprojlim \big\{D^n_\beta; f^\beta_\gamma\big\}$ is a well defined countable subset of $X(\alpha)$. Lemma 2.5.9 of [9] says the restrictions $f^\alpha_\beta : D^n_\alpha \to D^n_\beta$ are bijective. Claim 9 says each $D^n_\alpha \subset [a, q^n]$.

To see $D^1_\alpha, D^2_\alpha, \ldots$ are pairwise disjoint take $x \in D^n_\alpha$ and $y \in D^m_\alpha$ for some $m \neq n$. By definition $f^\alpha_0(x) \in D^n_0$ and $f^\alpha_0(y) \in D^m_0$. Since $\mathcal{D}_0$ is a tailing family D^m_0 and D^n_0 are disjoint and the result follows.

To show $D_\alpha = \{D^1_\alpha, D^2_\alpha, \ldots\}$ is a tailing family it remains to prove each $D^n_\alpha \cap [a, q^m]$ is dense in $[a, q^m]$ whenever $m \leq n$. To that end Claim 7 says the half-tail q^n_β is thick whenever $\beta < \alpha$ is a successor ordinal. Since α is a limit ordinal the set $\Gamma = \{\beta < \alpha : \beta \text{ is a successor ordinal}\}$ is cofinal in α. Hence we can write $X(\alpha) = \varprojlim\{X(\beta); f^\beta_\gamma : \beta, \gamma \in \Gamma\}$ where each $X(\beta)$ comes with a distinguished thick half-tail.

Recall $\pi_\beta = f^\alpha_\beta$ and the open subsets of $[a, q^m] = \varprojlim [a_\beta, q^m_\beta]$ have the form $\pi^{-1}_\beta(U)$ for open $U \subset [a_\beta, q^m_\beta]$. By thickness $[a_\beta, q^m_\beta]^\circ$ is dense in $[a_\beta, q^m_\beta]$. Hence

$V = U \cap [a_\beta, q_\beta^m]^\circ$ is a nonempty open subset of $X(\beta)$. Since $[a_\beta, q_\beta^m] \subset [a_\beta, q_\beta^n]$ we see V is open in $[a_\beta, q_\beta^n]$ as well. Since D_β^n is dense in $[a_\beta, q_\beta^n]$ there is $d \in V \cap D_\beta^n$. Since $D_\beta^n = \pi_\beta(D_\alpha^n)$ we have $\pi_\beta(x) = d$ for some $x \in D_\alpha^n$. But then $\pi_\beta(x) \in V \subset U$ and so $x \in \pi_\beta^{-1}(U)$ as required. □

Finally we have the main example.

Theorem 1. There exists a separable Bellamy continuum.

Proof. By transfinite recursion we have a coherent system of metric continua $\{X(\alpha); f_\beta^\alpha : \beta, \alpha < \omega_1\}$. Let X be the inverse limit. Recall this section assumes $\alpha \leq \omega_1$ is a limit ordinal. For the special case $\alpha = \omega_1$ we have $X = X(\alpha)$ and Claim 14 says X admits a tailing family. Lemma 5.3.4 then says X is separable.

Once we have expressed X as the inverse limit of a system of indecomposable metric continua, Conditions (a) and (b) and Theorem 1 of [4] will say X is indecomposable with at most two composants. The *trivial composant* $E \subset X$ is the set $\bigcup\{X(\alpha) : \alpha < \omega_1\}$ of eventually constant ω_1-sequences. The *nontrivial composant* $X - E$ is equal to the limit $\varprojlim\{\kappa(a_\alpha); f_\beta^\alpha : \beta, \alpha < \omega_1\}$. In this case the trivial composant contains the point $(a_\alpha)_{\alpha<\omega_1}$ hence is nonempty by construction.

Claim 11 says $X(\alpha)$ is indecomposable whenever $\alpha < \omega_1$ is a limit ordinal. We claim $\Gamma = \{\alpha < \omega_1 : \alpha \text{ is a limit ordinal}\}$ is cofinal in ω_1. Thus we can write $X = \varprojlim\{X(\alpha); f_\beta^\alpha : \alpha, \beta \in \Gamma\}$ as the inverse limit of a system of indecomposable metric continua.

To that end let $\beta < \omega_1$ be arbitrary and observe $\beta \times \omega$ is well-ordered under $(\delta, m) \leq (\gamma, n) \iff \big(m < n \text{ or } m = n \text{ and } \delta \leq \gamma\big)$. Hence $\beta \times \omega$ is isomorphic to some ordinal $\widetilde{\beta}$. It is clear $\beta \times \omega$ has no top element hence $\widetilde{\beta}$ is a limit ordinal. Since the initial segment $\beta \times \{1\}$ is a copy of β we have $\beta \leq \widetilde{\beta}$ and since $\beta \times \omega$ is countable we have $\widetilde{\beta} < \alpha$.

We conclude X has exactly two composants. Let the Bellamy continuum $\widetilde{X}$ be obtained by choosing any $x \in E$ and $y \in X - E$ and identifying $x \sim y$. Since $\widetilde{X}$ is the image of X under the quotient map it is separable. $\square$

5.5 Separability and Metrisability

We make some observations on the global properties of the inverse limit X from Section 5.4. The first is the tailing family $\mathcal{D} = \{D^1, D^2, \ldots\}$ on X has each D^n contained in the nontrivial composant $\varprojlim\{\kappa(a_\alpha); f^\alpha_\beta : \beta, \alpha < \omega_1\}$. Thus we have the following.

Claim 15. The nontrivial composant of X is separable.

Thus the nontrivial composant can be considered *small*. On the other hand Lemma 5.5.1 shows the trivial composant is *large*.

Lemma 5.5.1. Suppose the topological space T is the union of an ω_1-chain of proper closed subsets. Then T is non-separable.

Proof. Let $\mathcal{B} = \{B(\alpha) : \alpha < \omega_1\}$ be a chain of proper closed subsets of T. Suppose $D = \{d_1, d_2, \ldots\} \subset T$ is dense. For each $n \in \mathbb{N}$ there exists $\alpha(n) < \omega_1$ with $d_n \in B(\alpha(n))$. The countable subset $\{\alpha(n) : n \in \mathbb{N}\}$ has an upper bound $\alpha < \omega_1$. Since $\mathcal{B}$ is a chain we have $\{d_1, d_2, \ldots\} \subset B(\alpha)$. Since $B(\alpha)$ is closed and proper D is not dense. Hence T is non-separable. $\square$

The trivial composant of X is the union of the ω_1-chain $\{X(\alpha) : \alpha < \omega_1\}$ of proper subcontinua. Thus Lemma 5.5.1 gives the next two claims.

Claim 16. The trivial composant of X is non-separable.

Claim 17. The continuum X is separable but not hereditarily separable and not dense-hereditarily separable.

This suggests two open problems.

Question 1. Is the nontrivial composant of X hereditarily separable or dense-hereditarily separable?

Question 2. Does there exist an hereditarily separable or dense-hereditarily separable Bellamy continuum?

The Introduction mentions all known Bellamy continua are Stone-Čech remainders, or arise from inverse-limits constructions similar to Section 5.4. For a positive answer to Question 2 we imagine an entirely new class of examples would be needed to obviate Lemma 5.5.1.

Our next result is that neither composant of X is metrisable. Lemma 5.5.2 is similar to Lemma 5.5.1 and applies to the trivial composant.

Lemma 5.5.2. No metric space is the union of an ω_1-chain of compact proper subsets.

Proof. First recall [15] Theorem IV.5 (F) says compactness and sequential compactness are equivalent for metric spaces. Now suppose the metric space M is the union of the chain $\mathcal{K} = \{K(\alpha) : \alpha < \omega_1\}$ of compact proper subsets. We claim M is compact.

Suppose $x_1, x_2, \ldots$ is an arbitrary sequence in M. For each $n \in \mathbb{N}$ there exists $\alpha(n) < \omega_1$ with $x_n \in K(\alpha(n))$. The countable subset $\{\alpha(n) : n \in \mathbb{N}\}$ has an upper bound $\alpha < \omega_1$. Since $\mathcal{K}$ is a chain we have $\{x_1, x_2, \ldots\} \subset K(\alpha)$. Since $K(\alpha)$ is compact metric there is a point $x \in K(\alpha)$ and a subsequence (y_n) of (x_n) with $y_n \to x$. We conclude M is sequentially compact hence compact.

Since M is compact metric it admits a countable base $U_1, U_2, \ldots$ of open sets. Since $M = \bigcup \mathcal{K}$ each $U_n \in \mathbb{N}$ meets $K(\beta(n))$ for some $\beta(n) < \omega_1$. The countable subset $\{\beta(n) : n \in \mathbb{N}\}$ has an upper bound $\beta < \omega_1$. Since $K(\beta)$ meets all U_n it is dense. Since $K(\beta)$ is compact and M metric $K(\beta)$ is closed. Thus $K(\beta) = M$ contradicting the assumption that each $K(\beta)$ is proper. $\square$

The trivial composant of X is the union of the ω_1-chain $\bigcup\{X(\alpha) : \alpha < \omega_1\}$ of proper subcontinua. Thus Lemma 5.5.2 gives the next claim.

Claim 18. The trivial composant of X is non-metrisable.

Next we prove the same for the nontrivial composant.

Claim 19. The nontrivial composant of X is non-metrisable.

Proof. Clearly the nontrivial composant $X-E$ is connected. Claim 15 says $X-E$ is separable. Lemma 3 of [12] says $X-E$ is strongly indecomposable as defined in [12] Section 2. For a contradiction suppose $X-E$ is metrisable.

Let the $X \to \widetilde{X}$ be the quotient map from Theorem 1. Since $X-E$ is homeomorphic to its image we know $\widetilde{X}$ is a compactification of $X-E$. Then [12] Theorem 8 says $\widetilde{X}$ is irreducible between some pair of points $\{a, b\}$. But that means a and b have different composants contradicting how $\widetilde{X}$ is a Bellamy continuum. $\square$

To close the section we remark that metrisability cannot be droped from the hypothesis of Lemma 5.5.2. Example 5.8 of [1] is the closed unit ball in the Hilbert space $\ell^2(\omega_1)$ of square-summable functions $\omega_1 \to \mathbb{R}$ under the weak topology. In fact $\ell^2(\omega_1)$ is even a continuum and the elements of the ω_1-chain are nowhere dense subcontinua.

5.6 Embedding Properties

Bellamy [4] has shown each metric continuum is a retract of a Bellamy continuum. We use a similar technique to show each hereditarily unicoherent metric continuum is a retract of a separable Bellamy continuum.

For our example we took $X(0)$ the $\sin(1/x)$ continuum. There is no obstruction to using instead any hereditarily unicoherent metric continuum Y. In case Y admits thick half-tail q_0^n at the point $a_0 \in Y$ we can simply take $Y = X(0)$ as the bottom element of the system $\{X(\alpha); f_\beta^\alpha : \alpha, \beta < \omega_1\}$ from Section 5.3.

Writing X for the limit we see the projection $\pi_0 : X \to X(0)$ is a retraction. Therefore each $\pi_0(x)$ is a point of the trivial composant. Let the separable Bellamy continuum $\widetilde{X}$ be obtained by choosing some x in the nontrivial composant and identifying $x \sim \pi_0(x)$. Clearly π_0 induces a retraction $\widetilde{X} \to Y$ from a separable Bellamy continuum.

In case Y has no such tail, we must first build an hereditarily unicoherent metric continuum $X(0)$ with a thick half-tail q_0^n at the point $a_0 \in X(0)$ and a retraction $X(0) \to Y$. We then take $X(0)$ as the bottom element of the inverse system and compose the retractions $\widetilde{X} \to X(0)$ and $X(0) \to Y$ to get the desired retraction onto $\widetilde{X} \to Y$. It remains to construct such a continuum $X(0)$.

Lemma 5.6.1. Suppose the metric continuum Y is hereditarily unicoherent. There exists a retraction $X(0) \to Y$ from an hereditarily unicoherent metric continuum $X(0)$ with a thick half-tail q_0^n at the point $a_0 \in X(0)$.

Proof. Choose a countable dense subset $D = \{d(1), d(2), \ldots\}$ of Y. For the sequence $s = (1, 1, 2, 2, 1, 1, 2, 2, 3, 3, \ldots)$ define each $p_n = d(s(n))$. Define the closed subset $N \subset Y \times [0, 1]$.

$$N = \Big(\bigcup_{n \in \mathbb{N}} [p_1, p_n] \times \{1/n\} \Big) \cup \big(Y \times \{0\} \big).$$

To obtain $X(0)$ from N first make for each odd $n \in \mathbb{N}$ the identification $\big(p_n, 1/n\big) \sim \big(p_{n+1}, 1/(n+1)\big)$. Then make for each even $n \in \mathbb{N}$ the identification $\big(p_1, 1/n\big) \sim \big(p_1, 1/(n+1)\big)$.

The picture for $X(0)$ is similar to Figure 5.2. Then $X(0)$ is a metric space as a subset of the product $Y \times [0,1]$ of metric spaces. It is straightforward to show the first summand of N is connected with closure $X(0)$. Since $Y \times [0,1]$ is compact so is $X(0)$. Identify Y with the subset $Y \times \{0\}$ of $X(0)$.

A similar argument to Section 5.3 shows $X(0)$ is hereditarily unicoherent and $q_0^n = (p_n, 1/n)$ is a thick half-tail at $a_0 = (p_1, 1)$ Clearly the projection $N \to Y$ onto the first coordinate respects the partition. Hence it induces a retraction $X(0) \to Y$. □

The theorem follows.

Theorem 2. Each hereditarily unicoherent metric continuum is a retract of a separable Bellamy continuum.

There are several directions one might generalise Theorem 2. The first is to drop the reference to hereditary unicoherence.

Question 3. Is each metric continuum a retract of a separable Bellamy continuum?

The methods in Sections 5.3 and 5.4 rely heavily on hereditary unicoherence and do not generalise. One special case that seems approachable is when Y is arcwise connected.

If we take the proof of Lemma 5.6.1 and replace each $[p_1, p_n]$ with some arc from p_1 to p_n the resulting space $X(0)$ is a *spiral* over Y. That means the composant $\kappa(a_0)$ of $Y - X(0)$ of a_0 is an open ray. So while $X(0)$ is not itself hereditarily unicoherent, we see any two subcontinua of $\kappa(a_0)$ have connected intersection. Since the maps f_0^α map $X(\alpha) - X(0)$ into $\kappa(a_0)$ it seems likely one could adapt our construction while paying very close attention to where hereditary unicoherence is used, and thus get a retractions from separably Bellamy continua onto arcwise connected continua. In particular [14] Theorem 8.23 says this includes all locally connected continua.

One might also try to replace the hypothesis of Y being metrisable with merely being separable.

Question 4. Is each separable hereditarily unicoherent continuum a retract of a separable Bellamy continuum?

Secions 5.3 and 5.4 do not generalise to answer Question 4. This is because in the non-metric realm we might encounter the following type of subset.

Definition 5.6.2. The dense subset D of the topological space T is called *resolvable* to mean it is the disjoint union of two dense subsets. Otherwise we call the subset *irresolvable*.

Irresolvable sets are pathological objects that never occur in metric continua.

Lemma 5.6.3. Every dense subset of a metric continuum is resolvable.

Proof. Suppose X is a metric continuum. We first show each nonvoid open subset $U \subset X$ is uncountable. Corollary 5.5 of [14] says U contains a proper subcontinuum K. Then Urysohn's lemma (see [16] Theorem III.2) says K surjects onto $[0,1]$ hence is uncountable. We conclude U is uncountable.

Now suppose X is metric and $D \subset X$ dense. Since finite subsets are closed we see D is infinite. Let $U_1, U_2, \ldots$ be a countable basis for X. Since each U_i is infinite we can use induction to select distinct elements $a_1, b_1, a_2, b_2, \ldots \in X$ with each $a_i, b_i \in U_i$. By construction $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ are dense. Since $D(2) = D - A$ contains B it is also dense. For $D(1) = A$ we get a disjoint union $D = D(1) \cup D(2)$ into disjoint dense subsets. □

On the other hand Theorem 4.1 of [10] shows the construction of a countable irresolvable subset of the non-metric separable continuum $[0,1]^{\mathfrak{c}}$. The following lemma will be useful.

Lemma 5.6.4. Suppose the subsets D and F of the continuum X differ by only finitely many elements. Then neither or both of D and F are resolvable.

Proof. We claim D is resolvable if and only if $A = D \cup F$ is resolvable. First suppose $D = D(1) \cup D(2)$ is a disjoint union of dense subsets. For $A(1) = D(1)$ and $A(2) = D(2) \cup (A - D)$ the equality $A = A(1) \cup A(2)$ witnesses how A is resolvable.

Now suppose $A = A(1) \cup A(2)$ is a disjoint union of dense subsets. Clearly $D(1) = A(1) \cap D$ and $D(2) = A(2) \cap D$ are disjoint with union D. By assumption there are finite subsets $C(1), C(2) \subset X$ with $D(1) = A(1) - C(1)$ and $D(2) = A(2) - C(2)$. Since nonvoid open subsets of X are infinite we see $C(1)^\circ = C(2)^\circ = \varnothing$. Thus $\overline{D(1)} = \overline{A(1) - C(1)} = \overline{A(1)} - C(1)^\circ = \overline{A(1)} = X$.

Hence $D(1)$ is dense and likewise for $D(2)$. We conclude D is resolvable. The same proof shows F is resolvable if and only if $A = F \cup D$ is resolvable. We conclude D is resolvable if and only if F is resolvable. □

Henceforth suppose the Hausdorff continuum $X(\beta+1)$ has a thick half-tail $q^n_{\beta+1}$ at $a_{\beta+1} \in X(\beta+1)$ and $\mathcal{D}_{\beta+1} = \{D^1_{\beta+1}, D^2_{\beta+1}, \ldots\}$ is a tailing family on $X(\beta+1)$. Let the Hausdorff continuum $X(\beta+2)$ and thick half-tail $q^n_{\beta+2}$ at $a_{\beta+2} \in X(\beta+2)$ and bonding map $f^{\beta+2}_{\beta+1} : X(\beta+2) \to X(\beta+1)$ be as constructed in Section 5.3. The next claim shows the obstruction to choosing an appropriate $\mathcal{D}_{\beta+2}$ on $X(\beta+2)$.

Claim 20. Suppose $X(\beta+2)$ admits a tailing family that makes $f^{\beta+2}_{\beta+1}$ coherent. Then $[a_{\beta+1}, q^1_{\beta+1}] \cap D^2_{\beta+1}$ is resolvable as a subset of $[a_{\beta+1}, q^1_{\beta+1}]$.

Proof. Suppose the tailing family $\mathcal{D}_{\beta+2} = \{D^1_{\beta+2}, D^2_{\beta+2}, \ldots\}$ makes $f^{\beta+2}_{\beta+1}$ densely coherent. Define the subset $F = D^2_{\beta+2} - \{c_1, c_2\}$ of $X(\beta+2)$. Recall the subcontinua $J(0), J(1), J(2) \subset X(\beta+2)$ have dense interior and

$$J(0) \cap J(1) = \{c_1\} \qquad J(1) \cap J(2) = \{c_2\} \qquad J(0) \cap J(2) = \varnothing.$$

Therefore $F \cap J(0)$, $F \cap J(1)$ and $F \cap J(2)$ are pairwise disjoint and dense in $J(0)$, $J(1)$ and $J(2)$ respectively.

Recall the bonding map $f^{\beta+2}_{\beta+1}$ induces projections $J(0) \to [a_{\beta+1}, q^1_{\beta+1}]$ and $J(1) \to [a_{\beta+1}, q^1_{\beta+1}]$ and $J(2) \to [a_{\beta+1}, q^2_{\beta+1}]$. Therefore $A = f^{\beta+2}_{\beta+1}\big(F \cap J(0)\big)$ and $B = f^{\beta+2}_{\beta+1}\big(F \cap J(1)\big)$ are dense in $[a_{\beta+1}, q^1_{\beta+1}]$ and $f^{\beta+2}_{\beta+1}\big(F \cap J(2)\big)$ is dense in $[a_{\beta+1}, q^2_{\beta+1}]$. Since $[a_{\beta+1}, q^1_{\beta+1}] \subset [a_{\beta+1}, q^2_{\beta+1}]$ has dense interior we see $C = [a_{\beta+1}, q^1_{\beta+1}] \cap f^{\beta+2}_{\beta+1}\big(F \cap J(2)\big)$ is also dense in $[a_{\beta+1}, q^1_{\beta+1}]$.

By assumption $f^{\beta+2}_{\beta+1}$ induces a bijection $D^2_{\beta+2} \to D^2_{\beta+1}$ and therefore A, B and C are pairwise disjoint. Since F is contained in $J(0) \cup J(1) \cup J(2)$ we see $A \cup B \cup C = [a_{\beta+1}, q^1_{\beta+1}] \cap f^{\beta+2}_{\beta+1}(F)$. We conclude $[a_{\beta+1}, q^1_{\beta+1}] \cap f^{\beta+2}_{\beta+1}(F)$ is a resolvable subset of $[a_{\beta+1}, q^1_{\beta+1}]$.

By definition F differs from $D^2_{\beta+2}$ by only finitely many elements. Thus $f^{\beta+2}_{\beta+1}(F)$ differs from $f^{\beta+2}_{\beta+1}\big(D^2_{\beta+2}\big) = D^2_{\beta+1}$ by only finitely many elements. The same holds taking intersections with $[a_{\beta+1}, q^1_{\beta+1}]$. The result then follows from Lemma 5.6.4. □

Claim 20 says there is no way in general to define a suitable tailing family on $X(\beta + 2)$. For suppose $X(\beta + 1)$ and $[a_{\beta+1}, q^1_{\beta+1}]$ are non-metric. There is no guarantee a given dense subset of $[a_{\beta+1}, q^1_{\beta+1}]$ is resolvable. Thus a positive answer to Question 3 would go beyond the methods here.

We close with the remark that irresolvable sets can be generated by trying to extend the system $\{X(\alpha); f^\alpha_\beta : \alpha, \beta < \omega_1\}$ of continua from Sections 5.3 and 5.4 beyond ω_1.

Suppose $X = X(\omega_1)$ and $\mathcal{D}_{\omega_1}$ are constructed and $X(\omega_1 + 1)$ is as defined in

Section 5.3. For brevity write $a = a_{\omega_1}$ and $q^n = q^n_{\omega_1}$ and $\mathcal{D} = \mathcal{D}_{\omega_1}$. Suppose all countable dense subsets of each $[a, q^n]$ are resolvable. We give only the construction of $D^2_{\omega_1+1} \subset [a_{\omega_1+1}, q^2_{\omega_1+1}] = J(0) \cup J(1) \cup J(2)$ as the construction of $D^n_{\omega_1+1}$ is analogous.

First we split $[a, q^1] \cap D^2 = A \cup B \cup C$ into three pairwise disjoint dense subsets. Recall $J(0)$ and $J(1)$ are copies of $[a, q^1]$ and $J(2)$ a copy of $[a, q^2]$. Let $A' \subset J(0)$ and $B' \subset J(1)$ be the dense sets corresponding to A and B and $C' \subset J(2)$ the dense set corresponding to $D^2 - (A \cup B)$. Define $D^2_{\omega_1+1} = A' \cup B' \cup C'$.

In case all relevant countable dense subsets of $[a_{\omega_1+1}, q^n_{\omega_1+1}]$ are resolvable we define $X(\omega_1 + 2)$ and $\mathcal{D}_{\omega_1+2}$ similarly. Proceeding under the same assumption we continue to extend the system. Theorem 2 says we need no further assumptions to extend to limit ordinals. We claim we cannot extend to $\{X(\alpha); f^\alpha_\beta : \alpha, \beta < \Omega\}$ for any $|\Omega| > 2^{\mathfrak{c}}$.

For then Theorem 2 says the limit $X(\Omega)$ is separable. But at the same time Corollary 1.12 of [20] says every separable compactum is the image of $\beta\mathbb{N}$ hence has at most $|\beta\mathbb{N}| = 2^{\mathfrak{c}}$ points. But since the chain of subcontinua $X(\alpha) \subset X(\Omega)$ is strictly increasing $X(\Omega)$ has cardinality at least $|\Omega| > 2^{\mathfrak{c}}$ which is a contradiction.

We conclude the process terminates at $\{X(\alpha); f^\alpha_\beta : \alpha, \beta < \eta\}$ for some ordinal η with $|\eta| \leq 2^{\mathfrak{c}}$. Thus some relevant countable dense subset D of some $[a_\eta, q^n_\eta]$ is irresolvable.

Closer inspection of the proposed construction of each $D^n_{\eta+1}$ reveals some such D is built out of D^N_η for some $n \leq N$ by successively splitting sets into two dense disjoint halves or taking intersections with $[a_\eta, q^m_\eta]$ for some $n \leq m \leq N$.

5.7 Smith's Limit is Not Separable

The fundamental difference between our construction and that of Smith [18] is the use of identifications. We obtain $X(\beta+1)$ from the set

$$N = \Big(\bigcup_{n\in\mathbb{N}} [p_1, p_n] \times \{1/n\} \Big) \cup \big(Y \times \{0\} \big).$$

by making for each $n \in \mathbb{N}$ the identifications $\big(q^n_\beta, 1/(2n-1)\big) \sim \big(q^n_\beta, 1/2n\big)$ and $\big(a_\beta, 1/2n\big) \sim \big(a_\beta, 1/(2n+1)\big)$. Smith instead obtains $X(\beta+1)$ from N by attaching horizontal arcs from $\big(q^n_\beta, 1/(2n-1)\big)$ to $\big(q^n_\beta, 1/2n\big)$ and from $\big(a_\beta, 1/2n\big)$ to $\big(a_\beta, 1/(2n+1)\big)$ as in Figure 5.3.

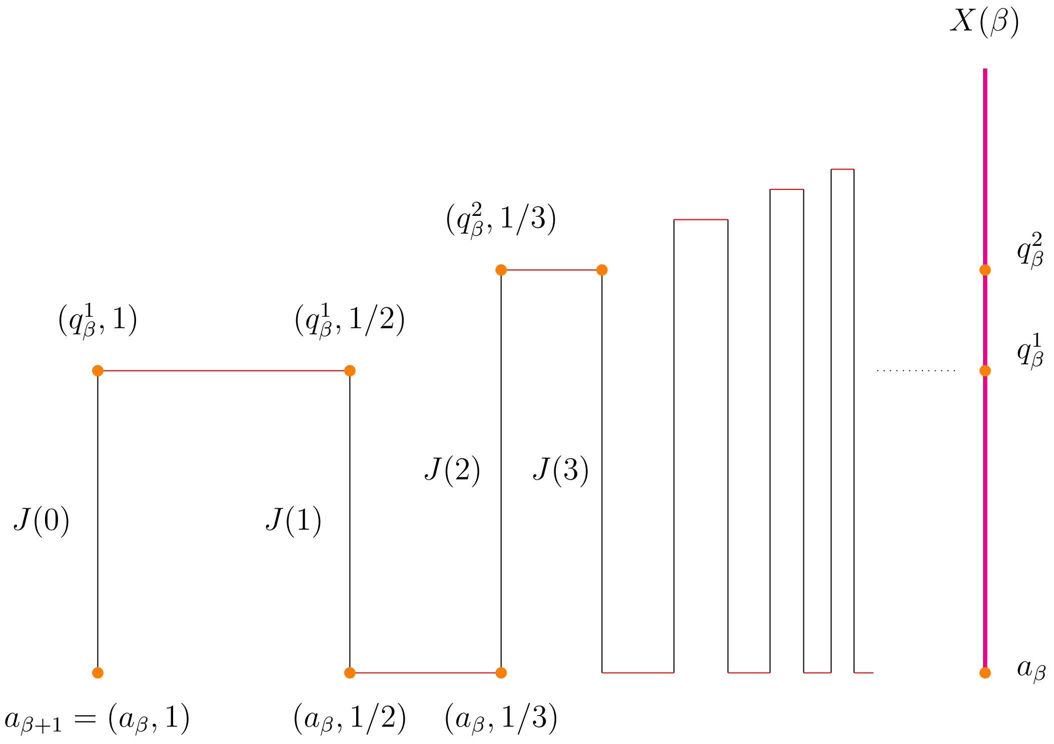

Figure 5.3: Schematic of the new $X(\beta+1)$. The arcs $I_{\beta+1}(n)$ are shown in red.

Suppose as in Section 5.4 we define each limit $X(\alpha)$ as the inverse limit of all previous $X(\beta)$. However, unlike before, we define each successor $X(\beta+1)$ as the union of the three subspaces of $X(\beta) \times [0,1]$.

$$\Big(\bigcup\Big\{[a_\beta, q_\beta^n] \times \{1/(2n-1), 1/2n\} : n \in \mathbb{N}\Big\}\Big) \cup \big(X(\beta) \times \{0\}\big)$$
$$\bigcup\Big\{\{q_\beta^n\} \times [1/2n, 1/(2n-1)] : n \in \mathbb{N}\Big\}$$
$$\bigcup\Big\{\{a_\beta\} \times [1/(2n+1), 1/2n] : n \in \mathbb{N}\Big\}$$

For each $n \in \mathbb{N}$ define the arcs $I_{\beta+1}(2n-1) = \{q_\beta^n\} \times [1/2n, 1/(2n-1)]$ and $I_{\beta+1}(2n) = \{a_\beta\} \times [1/(2n+1), 1/2n]$.

Claim 21. The new limit X is $\aleph_1$-cellular hence non-separable.

Proof. Let each $V_{\alpha+1} \subset X(\alpha)$ be the interior of $I_{\alpha+1}(2)$. Observe $f_\alpha^{\alpha+1}(V_{\alpha+1}) = \{a_\alpha\}$ thus each $f_\beta^{\alpha+1}(V_{\alpha+1}) = \{a_\beta\}$ for $\beta < \alpha + 1$. We claim the open sets $U_{\alpha+1} = \pi_{\alpha+1}^{-1}(V_{\alpha+1})$ of X are pairwise disjoint. For suppose $x \in U_{\alpha+1}$. Then by definition $x_{\alpha+1} \in V_{\alpha+1}$ and by commutativity $x_{\beta+1} \in f_{\beta+1}^{\alpha+1}(V_{\alpha+1})$ which equals $\{a_{\beta+1}\}$. Thus $x_{\beta+1} = a_{\beta+1}$. But since $a_{\beta+1} \notin V_{\beta+1}$ we must have $x \notin U_{\beta+1}$ for each $\beta + 1 < \alpha + 1$.

Since ω_1 is a limit ordinal the map by $\beta \mapsto \beta+1$ from ω_1 to itself is well defined and injective. Thus $\{U_{\alpha+1} : \alpha < \omega_1\}$ has the same cardinality as ω_1 namely $\aleph_1$. We conclude X is $\aleph_1$-cellular. □

The proof for the example of Smith [18] is similar, with the interiors of the sets $a_1^\alpha \times [1/2, 1]$ (page 594) playing the role of $V_{\alpha+1}$ and the points 1_α (page 595) playing the role of a_α.

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Chapter 6

Indecomposable Continua Without Non-Block Points

Abstract

We generalise results about dense proper semicontinua of the Stone-Čech remainder $\mathbb{H}^*$ of the half-line [2] to a wider class of indecomposable continua due to Bellamy and Rubin called *wave remainders* that consistently have exactly one composant [7]. Thus all known indecomposable continua have either a dense proper semicontinuum at every point or at no points. We also show each point of a wave remainder weakly cuts its composant.

6.1 Introduction

The author is interested in betweenness properties of indecomposable continua with exactly one composant [2, 3] henceforth called *Bellamy continua*. In particular the existence of (*proper*) *coastal points*. That means points that are elements of some dense proper semicontinuum (that is properly contained in a composant).

There are two known classes of Bellamy continua. The first is due to Bellamy and Smith [8, 19, 20, 21]. These continua are obtained by *spot-welding* two points of an indecomposable continuum with exactly two composants. For this class the coastal question is easy to answer: Since the quotient space of each composant is a dense proper semicontinuum, each point of these examples is proper coastal.

Continua of the second class are Stone-Čech remainders of spaces called *waves*. These spaces resemble the half-line $\mathbb{H}$ enough that their Stone-Čech remainders are indecomposable continua with the same number of composants as $\mathbb{H}^*$. That composant number is known to be axiom sensitive. In particular wave remainders being Bellamy continua is equivalent to the set theoretic axiom Near Coherence of Filters (NCF).

The author has shown NCF is equivalent to $\mathbb{H}^*$ having no coastal points [2]. This paper shows the same for the Stone-Čech remainder of an arbitrary wave $\mathbb{W}$. Moreover we show each $x \in \mathbb{W}^*$ is proper coastal if and only if the near coherence class corresponding to $\kappa(x)$ has a Q-point. It follows a given point being proper non-block depends only on its composant.

The generalisation is unsurprising. The upshot is all known Bellamy continua are coastal either at every point or at no points. We see no reason this should hold in general. But it is clear new techniques would be required to construct a counterexample. The structure of such counterexamples (if they exist) is examined in the final section of [2].

We also show $\mathbb{W}^*$ shares the property with $\mathbb{H}^*$ that, regardless of the composant number, each point weakly cuts between two points of its composant. The fact is nontrivial as there exist Bellamy continua with strong non-cut points [3].

The paper is organised as follows: Section 6.3 identifies the global structure of waves. The wave $\mathbb{W}$ is the union of a countable succession of subcontinua $T(0.5) \cup T(1.5) \cup \ldots$. Each $T(p)$ meets only the $T(q)$ immediately before and after. Each intersection $T(p) \cap T(q)$ is connected and its removal disconnects the space into a *bounded* and *unbounded* component.

Section 6.4 defines standard subcontinua in analogy with $\mathbb{H}^*$ and shows each *definable* subcontinuum of $\mathbb{W}^*$ is the intersection of a family of standard subcontinua. Section 6.5 introduces *dominoes* as those standard subcontinua $\mathbb{K}_\mathcal{D} \subset \mathbb{W}^*$ with each $K_i = T(m.5) \cup \ldots \cup T(n.5)$ for some $m \leq n$ and shows each proper subcontinuum is contained in a domino. Section 6.6 shows the intersection of two dominoes $\mathbb{K}_\mathcal{D}$ and $\mathbb{L}_\mathcal{E}$ is constrained by the relationship between $\mathcal{D}$ and $\mathcal{E}$ in analogy with $\mathbb{H}^*$.

Section 6.7 uses results from Section 6.5 and 6.6 to relate coastal points of $\mathbb{W}^*$ and Q-points of ω^*. Section 6.8 uses results from Section 6.4 to show each point of $\mathbb{W}^*$ weakly cuts its composant.

6.2 Terminology and Notation

Throughout a *compactum* means a nondegenerate compact Hausdorff space and a *continuum* means a connected compactum. For $a \in Y$ and subsets $A_n \subset Y$ we write $A_n \to a$ to mean for each open neighborhood U of a we have $A_N, A_{N+1}, \ldots \subset U$ for some $N \in \mathbb{N}$. For A_n closed this is equivalent to convergence to $\{a\}$ in the Vietoris topology. For background on metric continua see [16] and [17]. The results cited here have analagous proofs for non-metric continua.

The continuum X is called *irreducible* about $P \subset X$ to mean no proper subcontinuum of X contains P. For $b \in X$ we omit the curly braces and write $X - b$ to mean $X - \{b\}$. The subspace $A \subset X$ is called a *semicontinuum* to mean for each $a, c \in A$ some subcontinuum $K \subset A$ has $\{a, c\} \subset K$. Each subspace $A \subset X$ is partitioned into maximal semicontinua called the *continuum components* of A.

For $a, b, c \in X$ we say b *cuts* (resp. *weakly cuts*) a *from* c and write $\langle a, b, c\rangle$ (resp. $[a, b, c]$) to mean $b \in C$ for each connected set (resp. subcontinuum) $C \subset X$ with $\{a, c\} \subset C$. Equivalently a and c have different (continuum) components in $X - b$. We say b weakly cuts the semicontinuum $A \subset X$ to mean $b \in A$ and $[a, b, c]$ for some $a, c \in A$. For X irreducible about $\{a, c\}$ we have $[a, b, c]$ for each $b \in X$.

By *boundary bumping* we mean the principle that, for each closed $E \subset X$, each component C of E meets $\partial E = \overline{E} \cap \overline{X - E}$. For the non-metric proof see §47, III Theorem 2 of [16]. One corollary of boundary bumping is that the point $p \in X$ is in the closure of each continuum component of $X - p$.

We call X *indecomposable* to mean it cannot be expressed as the union of two proper subcontinua. Equivalently each proper subcontinuum is nowhere dense. We call X hereditarily indecomposable to mean each of its subcontinua is indecomposable. The *composant* $\kappa(x)$ of the point $x \in X$ is the union of all proper subcontinua that have x as an element. Composants are semicontinua.

By a *spiral* over the continuum K we mean a continuum X containing a dense open ray A with $X - A$ homeomorphic to K. The familiar $\sin(1/x)$ continuum is an example of a spiral over $[0, 1]$. There exist spirals over arbitrary metric continua [1].

For the locally-compact noncompact metric space T write $\omega T = T \cup \{\infty\}$ for the one point compactification and βT for the *Stone-Čech compactification*. That means the set of closed ultrafilters on T with basis $\{\widetilde{U} : U \subset T \text{ open}\}$ for each $\widetilde{U} = \{\mathcal{A} \in \beta T : A \subset U \text{ for some } A \in \mathcal{A}\}$. We identify each $x \in T$ with the principal ultrafilter $\{A \subset T : A \text{ closed with } x \in A\}$. For each $A \subset T$ write A^* for the set $\widetilde{A} - A$. In particular $T^* = \beta T - T$ is called the *Stone-Čech remainder* of T. Throughout ω^* and $\mathbb{H}^*$ are the Stone-Čech remainders of the non negative integers $\omega = \{0, 1, 2, \ldots\}$ and non negative half-line $\mathbb{H} = [0, \infty)$. See [15] and [12] for background on $\mathbb{H}^*$ and ω^* respectively and [23] for background on the Stone-Čech compactification in general.

The connected locally-compact space $\mathbb{W}$ is called a *wave* to mean the one point compactification $\omega\mathbb{W}$ has a countable basis of connected neighborhoods at ∞ and is irreducible from some fixed $\mathbf{0} \in \mathbb{W}$ to $\infty \in \omega\mathbb{W}$. The remainder $\mathbb{W}^* = \beta\mathbb{W} - \mathbb{W}$ is an indecomposable continuum [7] with the same number of composants as $\mathbb{H}^*$.

Each continuous $f : X \to Y$ extends to a continuous $f : \beta X \to \beta Y$ called the *Stone-Čech lift* given by $f(\mathcal{D}) = \{A \subset Y : A \text{ closed and } f^{-1}(A) \in \mathcal{D}\}$. Equivalently $f(\mathcal{D})$ is the ultrafilter generated by the family $\left\{\overline{f(D)} : D \in \mathcal{D}\right\}$. For f defined only on a subset $X' \subset X$ we define $f(\mathcal{D})$ as the ultrafilter generated by the family $\left\{\overline{f(D)} : D \in \mathcal{D} \text{ and } D \subset X'\right\}$.

For elements $\mathcal{D}, \mathcal{E} \in \omega^*$ we write $\mathcal{E} \leq \mathcal{D}$ to mean $f(\mathcal{D}) = \mathcal{E}$ for some monotone finite-to-one $f : \omega \to \omega$ and $\mathcal{D} \sim \mathcal{E}$ to mean both $\mathcal{D} \leq \mathcal{E}$ and $\mathcal{E} \leq \mathcal{D}$. Two elements of ω^* are said to *nearly cohere* to mean they have a common lower bound. The number of near coherence classes is independent of ZFC.

Near Coherence of Filters (NCF) is the axiom that there is exactly one near coherence class. NCF is equivalent to each wave remainder having exactly one composant [7, 11]. In particular NCF implies there are no *Q-points* [10]. We call $\mathcal{E} \in \omega^*$ a Q-point to mean it is $\leq$-minimal. Equivalently each finite-to-one function $\omega \to \omega$ is injective when restricted to some element of $\mathcal{E}$.

6.3 Global Structure of Waves

The *local* structure of an arbitrary wave remainder $\mathbb{W}^*$ can be very different from $\mathbb{H}^*$. See Section 6.4 for examples. This section shows $\mathbb{W}$ shares enough *global* structure with $\mathbb{H}$ to generalise the arguments from [2] about dense proper semicontinua.

Throughout the letter $\mathbb{W}$ refers to a wave. Theorem 1 breaks $\mathbb{W}$ into a linear succession of *chunks* with each meeting only those directly before and after as described in Definition 6.3.1.

Definition 6.3.1. We call the family $\mathcal{T} = \{P(0), T(0.5), P(1), T(1.5) \ldots\}$ of subcontinua of $\mathbb{W}$ a *bicycle-chain* to mean the following:

(1) $\mathbf{0} \in P(0)$

(2) $\mathbb{W} = T(0.5) \cup T(1.5) \cup \ldots$

(3) $T(p) \to \infty$ as $p \to \infty$

(4) $T(p) \cap T(q) = P\big((p+q)/2\big)$ for $|p-q| = 1$ and $T(p) \cap T(q) = \varnothing$ otherwise.

(5) Each $T(n.5)$ is irreducible about $P(n) \cup P(n+1)$.

(6) Each $\omega\mathbb{W} - P(n+1)$ has exactly two components: One component contains $P(0) \cup P(1) \cup \ldots \cup P(n)$. The other component contains $P(n+2) \cup \ldots \cup \{\infty\}$.

For the simplest wave $\mathbb{W} = \mathbb{H}$ we can get a bicycle-chain by taking each $P(n) = \{n\}$ and $T(n.5) = [n, n+1]$. Bellamy [7] mentions the following sort of example.

Example 6.3.2. Let $X_0, X_1, X_2, \ldots$ be copies of the same indecomposable continuum with at least two distinct composants $A_i, B_i \subset X_i$ and with homeomorphic nondegenerate proper subcontinua $K_i \subset A_i$ and $L_i \subset B_i$. Obtain $\mathbb{W}$ from the topological disjoint union $\bigcup X_i$ by identifying each L_i with

K_{i+1}. Then $\omega\mathbb{W}$ is irreducible from each point $\mathbf{0} \in K_0$ to ∞. It is straightforward to check $\mathcal{T} = \{P(0), T(0.5), P(1), T(1.5)\ldots\}$ is a bicycle-chain for each $P(n) = X_{2n}$ and $T(n.5) = X_{2n} \cup X_{2n+1} \cup X_{2n+2}$.

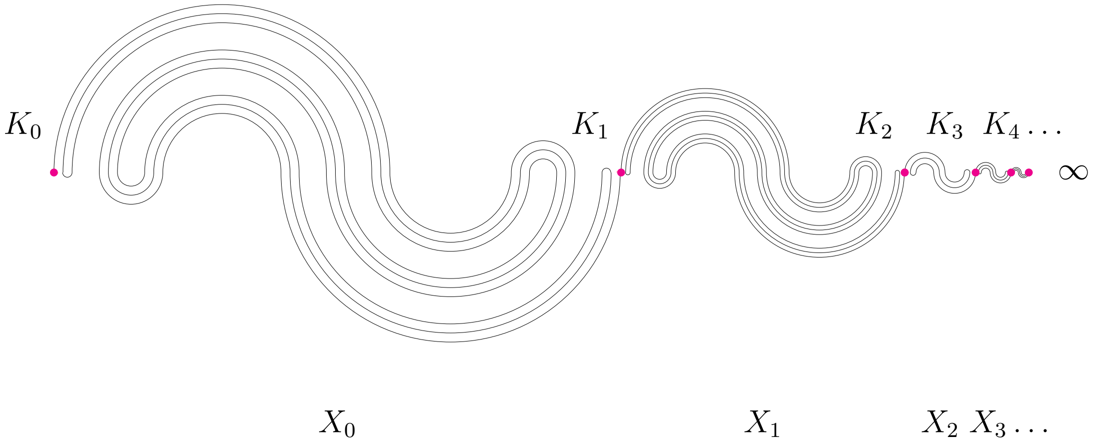

Figure 6.1: One particular instance of the wave $\mathbb{W}$ from Example 6.3.2

We prove the existence of bicycle-chains in several parts. First recall the definition and Lemma 6 from Bellamy [7]. Theorem 1 is really nothing more than an extension of that lemma.

Definition 6.3.3. (Bellamy Definition page 205) We call $\{p_0, p_1, \ldots\} \subset \mathbb{W}$ a *cofinal sequence of cut points* to mean $p_0 = \mathbf{0}$, all p_n are distinct, each p_n cuts p_{n-1} from ∞, and $p_n \to \infty$.

Lemma 6.3.4. (Bellamy Lemma 6) There exists a wave $\widehat{\mathbb{W}}$ with a cofinal sequence of cut points and monotone surjection $M : \omega\mathbb{W} \to \omega\widehat{\mathbb{W}}$ with $\mathbf{0} \mapsto \widehat{\mathbf{0}}$, $\infty \mapsto \widehat{\infty}$ and $\{\infty\}$ a fibre of M.

Recall some results about continua irreducible between two points. For X irredicible about $\{a, c\}$ define $S = \big\{p \in X - \{a, c\} : \langle a, p, c\rangle\big\}$. Chapter IV.3 of [17] shows each $X - p$ has exactly two components L_p of a and R_p of c. Moreover S is linearly ordered by $p < q \iff L_p \subset L_q \iff R_q \subset R_p \iff p \in L_q \iff q \in R_p$.

Lemma 6.3.5. There are pairwise disjoint subcontinua $P(n) \subset \mathbb{W}$ with $\mathbf{0} \in P(0)$ and $P(n) \to \infty$ and each $\omega\mathbb{W} - P(n+1)$ has exactly two components: One component contains $P(0) \cup P(1) \cup \ldots \cup P(n)$ and the other contains $P(n+2) \cup \ldots \cup \{\infty\}$.

Proof. Take $M : \omega\mathbb{W} \to \omega\widehat{\mathbb{W}}$ and $\{p_0, p_1, \ldots\} \subset \widehat{\mathbb{W}}$ from Definition 6.3.3 and Lemma 6.3.4. For each $n > 0$ write L_n and R_n respectively for the components of $\widehat{\mathbf{0}}$ and $\widehat{\infty}$ in $\widehat{\mathbb{W}} - p_n$. Since p_n cuts p_{n-1} from ∞ we have $p_{n-1} \notin R_n \implies p_{n-1} \in L_n \implies p_{n-1} < p_n$. Since n is arbitrary we get $p_m < p_n \iff m < n$. It follows each $\{p_0, p_1, \ldots p_{n-1}\} \subset L_n$ and $\{p_{n+1}, \ldots, \widehat{\infty}\} \subset R_n$.

Since M is monotone [13] Theorem 6.1.28 says $P(n+1) = M^{-1}(p_{n+1})$ are subcontinua. Since L_{n+1} and R_{n+1} are open and connected the same theorem says $M^{-1}(L_{n+1})$ and $M^{-1}(R_{n+1})$ are connected. Clearly they are open, nonempty and pairwise disjoint, and satisfy $M^{-1}(L_{n+1}) \cup M^{-1}(R_{n+1}) = \omega\mathbb{W} - P(n+1)$. We conclude they are the components of $\omega\mathbb{W} - P(n+1)$. Taking preimages we see that $\{P(0), P(1), \ldots P(n)\} \subset M^{-1}(L_{n+1})$ and $\{P(n+2), \ldots, \infty\} \subset M^{-1}(R_{n+1})$ as required.

To see $P(n) \to \infty$ suppose U is open at ∞. Then $\infty \notin \omega\mathbb{W} - U$. Since $\{\infty\}$ is a fibre $\widehat{\infty} \notin M(\omega\mathbb{W} - U)$. By compactness $M(\omega\mathbb{W} - U)$ is closed hence $\widehat{\omega}\mathbb{W} - M(\omega\mathbb{W} - U)$ open at $\widehat{\infty}$. By definition $p_n \to \widehat{\infty}$ so $\widehat{\omega}\mathbb{W} - M(\omega\mathbb{W} - U)$ contains some tail $\{p_N, p_{N+1}, \ldots\}$. It follows

$$\{P(N), P(N+1), \ldots\} \subset M^{-1}\big(\widehat{\omega}\mathbb{W} - M(\omega\mathbb{W} - U)\big) = \omega\mathbb{W} - M^{-1}\big(M(\omega\mathbb{W} - U)\big).$$

Since $M^{-1}\big(M(\omega\mathbb{W} - U)\big)$ contains $\omega\mathbb{W} - U$ the right-hand-side is contained in $\omega\mathbb{W} - (\omega\mathbb{W} - U) = U$. Hence $\{P(N), P(N+1), \ldots\} \subset U$ as required. $\square$

Henceforth fix some family $\{P(0), P(1), \ldots\}$ of subcontinua of $\mathbb{W}$ satisfying Lemma 6.3.5. For $n > 0$ we write L_n and R_n for the components of $\mathbf{0}$ and ∞ in $\omega\mathbb{W} - P(n)$.

Lemma 6.3.6. Suppose $\{p_0, p_1, \dots\} \subset \mathbb{W}$ is a cofinal sequence of cutpoints. For each R_n the component of $\mathbf{0}$ in $\mathbb{W} - p_n$ we have $R_n \to \infty$.

Proof. Suppose U is open at ∞. Then $C \subset U$ for some connected neighborhood C of ∞. Since $P(n) \to \infty$ some tail $\{P(N), P(N+1), \dots\}$ is contained in C. To see $R_{N+1}, R_{N+2} \dots \subset C$ suppose $m > N$. By boundary bumping $L_m \cup P(m)$ is a subcontinuum. Since $P(m) \subset C$ we know $L_m \cup C$ is a subcontinuum joining $\mathbf{0}$ to ∞. By irreducibility $L_m \cup C = \omega\mathbb{W}$. Therefore C contains $\omega\mathbb{W} - L_m$ which contains R_m as required. □

Theorem 1. $\mathbb{W}$ admits a bicycle-chain.

Proof. We claim the sets $P(n)$ and $T(n.5) = \omega\mathbb{W} - (L_n \cup R_{n+1})$ form a bicycle-chain. By boundary bumping $A(n) = L_n \cup P(n)$ and $B(n) = P(n+1) \cup R_{n+1}$ are subcontinua. Since $P(n) \subset L_{n+1} \subset \omega\mathbb{W} - R_{n+1}$ and $P(n+1) \subset R_n \subset \omega\mathbb{W} - L_n$ we know $A(n)$ and $B(n)$ are disjoint. Since $\omega\mathbb{W}$ is connected $C(n) = \omega\mathbb{W} - A(n) \cup B(n)$ is nonempty. Theorem 11.8 of [17] says $C(n)$ is connected.

We claim $C(n) \cap C(m) = \varnothing$ for all $n \neq m$. For $n < m$ Lemma 6.3.5 says $P(n) \subset L_m$. Since the continuum $L_m \cup B(n)$ joins $\mathbf{0}$ to ∞ we have $L_m \cup B(n) = \omega\mathbb{W}$ hence $\omega\mathbb{W} - B(n) \subset L_m$. In particular $C(n) = \omega\mathbb{W} - A(n) \cup B(n) \subset \omega\mathbb{W} - B(n) \subset L_m$ and so $C(n) \subset A(m)$. We conclude $C(n)$ is disjoint from $\omega\mathbb{W} - A(m) \cup B(m) = C(m)$.

By connectedness $\overline{C(n)}$ meets $A(n)$ and $B(n)$. Since L_n and R_{n+1} are open and disjoint from $C(n)$ the boundary is contained in $P(n) \cup P(n+1)$. We conclude $T(n.5) = P(n) \cup \overline{C(n)} \cup P(n+1) = P(n) \cup C(n) \cup P(n+1)$ is a continuum. Observe $P(n), P(n+1), L_n, R_{n+1}$ are pairwise disjoint. Hence $T(n.5) = \omega\mathbb{W} - (L_n \cup R_{n+1})$.

By construction each $T(i) \cap T(j)$ is nonempty only for $\{i, j\} = \{n, n+1\}$ and then equals $P(n+1)$. This proves (4). To prove (3) observe $T(n.5)$ is contained in R_{n-2}. Lemma 6.3.6 says $R_{n-2} \to \infty$ which implies $T(n.5) \to \infty$.

Lemma 6.3.5 proves (6). To prove (5) suppose $P(n) \cup P(n+1) \subset T$ for the subcontinuum $T \subset \omega\mathbb{W}$. Then $\big(L_n \cup P(n)\big) \cup T \cup \big(P(n+1) \cup R_{n+1}\big) = L_n \cup T \cup R_{n+1}$ is a subcontinuum that joins **0** to ∞. By irreducibility we have $L_n \cup T \cup R_{n+1} = \omega\mathbb{W}$ and so $\omega\mathbb{W} - (L_n \cup R_{n+1}) \subset T$ and so $T(n.5) \subset T$ by definition.

To prove (2) let $x \in \mathbb{W}$ be arbitrary. Since $R_n \to \infty$ there $N \in \mathbb{N}$ with $x \notin R_{N+1}$. Without loss of generality N is minimal with this property. For $x \in P(N)$ we have $x \in T(N.5)$ by definition. Otherwise $x \in R_N$ and so $x \notin L_N$ which implies $x \in \omega\mathbb{W} - (L_N \cup R_{N+1})$ and in turn $x \in T(N.5)$. $\square$

Henceforth fix some bicycle-chain $\mathcal{T} = \{P(0), T(0.5), P(1), T(1.5)\ldots\}$ on $\mathbb{W}$. For each $m, n \in \mathbb{N}$ define the subcontinuum $T(m, n+1) = T(m.5) \cup \ldots \cup T(n.5)$. The following observation will be useful for simplifying proofs.

Proposition 6.3.7. Suppose $0 = N_0 < N_1 < \ldots$ is an increasing sequence of integers. For each $n \in \mathbb{N}$ define $\widetilde{P}(n) = P(N_n)$ and $\widetilde{T}(n.5) = T(N_n, N_{n+1})$. Then $\widetilde{\mathcal{T}} = \{\widetilde{P}(0), \widetilde{T}(0.5), \widetilde{P}(1), \widetilde{T}(1.5)\ldots\}$ is a bicycle-chain.

6.4 Standard Subcontinua: Regular Hulls

We define standard subcontinua in analogy with $\mathbb{H}^*$. These objects have been studied under the guise of abstract ultracoproducts of continua [4]. Thus we often use results and terminology on ultracoproducts for Examples 6.4.7 – 6.4.10 and throughout.

Definition 6.4.1. Suppose $\mathbb{A} = \{A_1, A_2 \ldots\}$ are subcompacta of $\mathbb{W}$ with $A_n \to \infty$. Suppose the ultrafilter $\mathcal{D} \in \omega^*$ has some set $D \in \mathcal{D}$ that makes $A_D = \bigcup\{A_n : n \in D\}$ a disjoint union. Define the *standard subcompactum* $\mathbb{A}_\mathcal{D} = \bigcap\{A_D^* : D \in \mathcal{D}\}$ of $\mathbb{W}^*$. In case $\mathbb{A}_\mathcal{D}$ is connected we call it a *standard subcontinuum*. For each $A_n = \{a_n\}$ a singleton we write $a_\mathcal{D}$ instead of $\mathbb{A}_\mathcal{D}$.

We justify the terminology.

Lemma 6.4.2. Each $\mathbb{A}_\mathcal{D} \subset \mathbb{W}^*$ is closed, nonempty and proper.

Proof. Each $D \in \mathcal{D}$ has the form $D = \{d(1), d(2), \ldots\}$ for some $d(n) \to \infty$. From $A_n \to \infty$ it is easy to see $A_{d(n)} \to \infty$. It follows $A_D = \bigcup_n A_{d(n)}$ is noncompact and $A_D \cup \{\infty\}$ closed in $\omega\mathbb{W}$. Hence A_D is closed in $\mathbb{W}$. By definition of the topology on $\mathbb{W}^*$ the set $A_D^* \subset \mathbb{W}^*$ is closed. Since A_D^* is a copy of the Stone-Čech remainder of A_D it is nonempty. By compactness of $\mathbb{W}^*$ the intersection $\mathbb{A}_\mathcal{D}$ of the family $\{A_D^* : D \in \mathcal{D}\}$ of closed subsets is nonempty.

To see $\mathbb{A}_\mathcal{D}$ is proper observe the ultrafilter $\mathcal{D}$ includes exactly one of $E(1) = \{e(1), e(3), \ldots\}$ or $E(2) = \{e(2), e(4), \ldots\}$. Without loss of generality $E(1) \in \mathcal{D}$. Let $\mathcal{E}$ be an ultrafilter with $E(2) \in \mathcal{E}$. Then $A_{E(1)}$ and $A_{E(2)}$ are disjoint hence have disjoint remainders. Since $\mathbb{A}_\mathcal{D} \subset A_{E(1)}^*$ and $\mathbb{A}_\mathcal{E} \subset A_{E(2)}^*$ are both nonempty they are both proper. □

Relations between standard subcompacta are complicated by how they are *coordinate independent* objects specified using *coordinates*: For any given $\mathbb{A}$

and $\mathcal{D}$ there are distinct $\mathbb{B}$ and $\mathcal{E}$ with $\mathbb{A}_\mathcal{D} = \mathbb{B}_\mathcal{E}$. For the simplest example let $\sigma : \omega \to \omega$ be a permutation and each $B_n = A_{\sigma(n)}$ and $\mathcal{E} = \sigma(\mathcal{D})$. We recall several facts that account for this extra freedom of definition. The first appears as [5] Lemma 4.6.

Proposition 6.4.3. The standard subcompactum $\mathbb{A}_\mathcal{D}$ is connected exactly if $\{n \in \omega : A_n \text{ is connected}\} \in \mathcal{D}$.

The next follows from [14] Proposition 11 and [22] Theorem 7.

Proposition 6.4.4. The standard subcontinuum $\mathbb{K}_\mathcal{D}$ is (hereditarily) indecomposable exactly if $\{n \in \omega : K_n \text{ is (hereditarily) indecomposable}\} \in \mathcal{D}$.

The next are generalisations of well known facts about $\mathbb{H}^*$ [15].

Proposition 6.4.5. For standard subcontinua $\mathbb{K}_\mathcal{D} \subset \mathbb{L}_\mathcal{E}$ if and only if there are sets $D \in \mathcal{D}$ and $E \in \mathcal{E}$ and a finite-to-one map $f : D \to E$ with $f(\mathcal{D}) = \mathcal{E}$ and $K_n \subset L_{f(n)}$ for each $n \in D$. In particular $\{n \in \omega : K_n \subset L_n\} \in \mathcal{D} \implies \mathbb{K}_\mathcal{D} \subset \mathbb{L}_\mathcal{D}$.

Proposition 6.4.6. For standard subcontinua $\mathbb{K}_\mathcal{D} = \mathbb{L}_\mathcal{E}$ if and only if there are sets $D \in \mathcal{D}$ and $E \in \mathcal{E}$ and a permutation $\sigma : D \to E$ with $\sigma(\mathcal{D}) = \mathcal{E}$ and $K_n = L_{\sigma(n)}$ for each $n \in D$. In particular $\{n \in \omega : K_n = L_n\} \in \mathcal{D} \implies \mathbb{K}_\mathcal{D} = \mathbb{L}_\mathcal{D}$.

Standard subcontinua witness how the *local* structure of $\mathbb{W}^*$ and $\mathbb{H}^*$ can diverge. For Examples 6.4.7 – 6.4.10 the wave $\mathbb{W}$ is obtained by arranging end-to-end countably many spirals over the continua $K_1, K_2, \dots$.

Example 6.4.7. Let each K_n be a copy of the pseudo-arc. Proposition 6.4.4 says $\mathbb{K}_\mathcal{D}$ is an hereditarily indecomposable subcontinuum of $\mathbb{W}^*$. Compare [15] Corollary 5.11 which says $\mathbb{H}^*$ contains no hereditarily indecomposable subcontinua.

Example 6.4.8. Let each $K_n = A_n \cup B_n$ be the union of two copies of the Knaster Buckethandle with $A_n \cap B_n = \{x_n\}$ a singleton. Proposition 6.4.4 says $\mathbb{A}_\mathcal{D}$ and

$\mathbb{B}_\mathcal{D}$ are indecomposable subcontinua of $\mathbb{W}^*$ with $\mathbb{A}_\mathcal{D} \cap \mathbb{B}_\mathcal{D} = \{x_\mathcal{D}\}$. Compare [15] Theorem 5.9 which says any two indecomposable subcontinua of $\mathbb{H}^*$ are disjoint or nested.

Example 6.4.9. Let each K_n be a circle with $A_n, B_n \subset K_n$ arcs such that $A_n \cup B_n = K_n$ and $A_n \cap B_n = \{x_n, y_n\}$ a doubleton. Then $\mathbb{A}_\mathcal{D} \cap \mathbb{B}_\mathcal{D} = \{x_\mathcal{D}, y_\mathcal{D}\}$ is disconnected. Compare [15] Theorem 5.6 which says the intersection of any two subcontinua of $\mathbb{H}^*$ is empty or connected. Compare also Theorem 5.8 which says each decomposable subcontinuum of $\mathbb{H}^*$ has a dense set of cut points. It can be shown by joining points using *ultra arcs* that the *ultra circle* $\mathbb{K}_\mathcal{D}$ has no cut points.

Example 6.4.10. Let each K_n be a triod with *arms* $A_n, B_n, D_n \subset K_n$ and corresponding endpoints $a_n, b_n, d_n \in K_n$. Proposition 3.11 of [6] says each $\mathbb{K}_\mathcal{D}$ is irreducible about $\{a_\mathcal{D}, b_\mathcal{D}, d_\mathcal{D}\}$ and Theorem 3.14 says $\mathbb{K}_\mathcal{D}$ is not irreducible between any two points. Compare [15] Theorem 5.8 which says each decomposable continuum $K \subset \mathbb{H}^*$ is an interval of some standard subcontinuum. Thus by Theorem 2.11 we see K is irreducible between two points.

Theorem 2 is an analogue of [15] Theorem 5.1 which assumes $\mathbb{W} = \mathbb{H}$ and does not demand that K contain a standard subcontinuum. In fact every subcontinuum of $\mathbb{H}^*$ does so by [15] Corollary 5.11.

Theorem 2 is used in Section 6.8 to prove each point of $\mathbb{W}^*$ weakly cuts its composant.

Theorem 2. Suppose the proper subcontinuum $K \subset \mathbb{W}^*$ contains a standard subcontinuum. For each $x \in \mathbb{W}^* - K$ we have $K \subset L \subset \mathbb{W}^* - x$ for some standard subcontinuum $L \subset \mathbb{W}^*$.

The first step is to find a *small enough* neighborhood U^* of x so $\mathbb{W} - U$ breaks into manageable pieces.

Notation 6.4.11. For closed $Y \subset \mathbb{W}$ we write $Y = \bigoplus_{i=1}^{\infty} X_i$ to mean Y is the disjoint union of the compacta X_i and each X_i is clopen in Y and $X_i \to \infty$. In that case we say Y is the *free union* of the compacta X_i.

The proof of Theorem 2 has two stages. First we look at an easy special case. Then we reduce the general case to the special case.

Lemma 6.4.12. Suppose $U_n \subset T(n.5) - P(n) \cup P(n+1)$ are nonempty open sets. For $U = U_1 \cup U_2 \cup \ldots$ There are compacta X_i with $\mathbb{W} - U = \bigoplus_{i=1}^{\infty} X_i$.

Proof. Since each $T(n.5)$ is irreducible about $P(n) \cup P(n+1)$ we have a disjoint union $T(n.5) - U_n = A(n) \cup B(n+1)$ of clopen subsets with $P(n) \subset A(n)$ and $P(n+1) \subset B(n+1)$. Clearly $B(n+1)$ meets $A(n+1)$ and no other $A(n)$ and $B(m)$ intersect. For each $X_n = A(n) \cup B(n)$ we see $\mathbb{W} - U = \bigcup\{X_n : n \in \omega\}$ is a disjoint union of closed sets. Recall each $P(n) \subset A(n) \subset X_n$.

To show X_n is open in $\mathbb{W} - U$ first observe it is disjoint from the subcontinua $L_{n-1} \cup P(n-1)$ and $P(n+1) \cup R_{n+1}$ and their union which is closed. Hence it is enough to show X_n is open in $T = T(n-1, n+1) - U$. But T is covered by the disjoint closed sets X_{n-1}, X_n, and X_{n+1}. So $T - X_n = T \cap (X_{n-1} \cup X_n)$ is closed in T. Hence X_n is open in T.

Observe each closed set X_n is contained in the subcontinuum $T(n-1, n+1)$ and therefore X_n is compact. Since $X_n \subset T(n-1, n+1) \subset R_{n-2}$ Lemma 6.3.6 says $X_n \to \infty$. We conclude $\mathbb{W} - U = \bigoplus_{i=1}^{\infty} X_i$. □

Lemma 6.4.13. Suppose $V^* \subset \mathbb{W}^*$ is open and $x \in V^*$. There is open $U \subset V$ with $x \in U^*$ and compacta X_i with $\mathbb{W} - U = \bigoplus_{i=1}^{\infty} X_i$.

Proof. Since V^* is nonempty $\overline{V}$ meets $\mathbb{W}^*$. Hence V has noncompact closure in $\mathbb{W}$ and is contained in no continuum $T(0, n)$. That means V meets infinitely many $T(n_1), T(n_2), \ldots$ for some $n_1 < n_2 < \ldots$.

For $N = 0,1,2,3$ define each $T_i^N = T(n_{4i+N})$ Since $T(i) \to \infty$ each $T_i^N \to \infty$. Thus $B^N = \bigcup\{T(n_{4i+N}) : i \in \omega\}$ is closed. Since B^N are disjoint x is an element of at most one $\overline{B^N}$. Without loss of generality $x \notin \overline{B^0}$.

Thus $U = V - B^0$ is open with $x \in U^*$. Define the subcontinua $\widetilde{P}(m) = P(n_{4m})$ and $\widetilde{T}(m.5) = T(n_{4m}, n_{4m+4})$. Apply Lemma 6.4.12 to the bicycle-chain $\widetilde{\mathcal{T}}$ to see $\mathbb{W} - U$ is the required free union. $\square$

Next we use the free union $\mathbb{W} - U = \bigoplus_{i=1}^{\infty} X_i$ from Lemma 6.4.13 to find the continuum L from Theorem 2 as a component of $(\mathbb{W} - U)^*$. Just like before we start with a special case then show it is equivalent to the general case.

Lemma 6.4.14. Suppose $U \subset \mathbb{W}$ is open and $\mathbb{W} - U = \bigoplus_{i=1}^{\infty} X_i$. Suppose each subcontinuum $K_n \subset X_n$ and the subcontinuum $K \subset \mathbb{W}^* - U^*$ contains $\mathbb{K}_{\mathcal{D}}$. There is a standard subcontinuum $\mathbb{L}_{\mathcal{D}}$ with $K \subset \mathbb{L}_{\mathcal{D}} \subset \mathbb{W}^* - U^*$.

Proof. Let $F : (\mathbb{W} - U) \to \omega$ be the continuous surjection taking each x to the unique $i \in \omega$ with $x \in X_i$. Then F induces a surjection $G : (\mathbb{W} - U)^* \to \omega^*$. Since $K \subset \mathbb{W}^* - U^* = (\mathbb{W} - U)^*$ is connected there is a unique $\mathcal{D} \in \omega^*$ with $G(K) = \{\mathcal{D}\}$. Thus K is a subcontinuum of the ultracoproduct $G^{-1}(\mathcal{D}) = \mathbb{X}_{\mathcal{D}}$.

Since $\mathbb{K}_{\mathcal{D}} \subset K$ we know K is contained in the component of $\mathbb{K}_{\mathcal{D}}$ in $\mathbb{X}_{\mathcal{D}}$. Theorem 3.1 of [6] says that component is $\mathbb{L}_{\mathcal{D}}$ for each L_n the component of K_n in X_n. We conclude $K \subset \mathbb{L}_{\mathcal{D}} \subset \mathbb{W}^* - U^*$ as required. $\square$

To address the more general case we use the following proposition. The proof follows from induction and §47, II Theorem 2 of [16] which says the quasicomponents of a compactum coincide with the components.

Proposition 6.4.15. Suppose X is a compactum and $K_1, K_2, \ldots, K_n \subset X$ subcontinua. There are compacta $Y_1, \ldots, Y_m \subset X$ with $X = \bigoplus_{i=1}^{m} Y_i$ a free union and components $A_i \subset Y_i$ with $K_1 \cup \ldots \cup K_n \subset A_1 \cup \ldots \cup A_m$.

We might also have to discard some summands of $\bigoplus_{i=1}^{\infty} X_i$. The next fact follows from the definition.

Proposition 6.4.16. Suppose $X_1, X_2, \ldots \subset \mathbb{W}$ are compacta and $\bigoplus_{i=1}^{\infty} X_i$ is a well-defined free union. Now suppose each compactum $Y_1, Y_2, \ldots$ is equal to some X_i. Then $\bigcup_{i=1}^{\infty} Y_i$ is closed in $\mathbb{W}$ and $\bigoplus_{i=1}^{\infty} Y_i$ is a well defined free union.

Lemma 6.4.17. Suppose $U \subset \mathbb{W}$ is open and $\mathbb{W} - U = \bigoplus_{n=1}^{\infty} X_n$. Suppose the subcontinuum $K \subset \mathbb{W}^* - U^*$ contains a standard subcontinuum. There is a standard subcontinuum $\mathbb{L}_{\mathcal{E}}$ with $K \subset \mathbb{L}_{\mathcal{E}} \subset \mathbb{W}^* - U^*$.

Proof. By assumption there is a standard subcontinuum $\mathbb{K}_{\mathcal{D}} \subset K$. Proposition 6.4.6 says we can redefine K_n over some $\mathcal{D}$-small set hence assume all $K_n \subset \mathbb{W} - U$. Then each continuum K_n is contained in exactly one of the clopen sets X_N of $\mathbb{W} - U$.

Let $N \in \omega$ be fixed. Since X_N is compact and $K_n \to \infty$ there are only finitely many $K_{i(1)} K_{i(2)}, \ldots, K_{i(N)}$ contained in X_N. Let $\widetilde{Y}_1^N, \widetilde{Y}_2^N, \ldots, \widetilde{Y}_{m(N)}^N \subset X_N$ and $\widetilde{A}_i^N \subset \widetilde{Y}_i^N$ be the compacta and components from Proposition 6.4.15.

Now let N vary over ω. We can write $\mathbb{W} - U = \bigoplus_{n=1}^{\infty} Y_n$ where each compactum Y_n equals some $\widetilde{Y}_i^N$. Define each A_n as the unique $\widetilde{A}_i^N$ with $\widetilde{Y}_i^N = Y_n$. Clearly each A_n is a component of Y_n and each K_i is contained in exactly one A_n. Let $\{Z_i : i \in \omega\}$ be the subcollection of all Y_j that contain some K_n. Proposition 6.4.16 says $\bigoplus_{n=1}^{\infty} Z_i$ is a welli defined free union and equals $\mathbb{W} - V$ for some open $V \subset \mathbb{W}$. Clearly $U \subset V \implies \mathbb{W}^* - V^* \subset \mathbb{W}^* - U^*$. Relabel the components A_i to have each $A_i \subset Z_i$.

Define the finite to one map $f : \omega \to \omega$ where $f(n)$ is the unique i with $K_n \subset A_i$. Proposition 6.4.5 says $\mathbb{K}_{\mathcal{D}} \subset \mathbb{A}_{f(\mathcal{D})}$ which implies $A = K \cup \mathbb{A}_{f(\mathcal{D})}$ is a continuum. Now apply Lemma 6.4.14 with U, X_n, K_n, K, and $\mathcal{D}$ replaced by V, Z_n, A_n, A and $f(\mathcal{D})$ respectively. We conclude $A \subset \mathbb{L}_{f(\mathcal{D})} \subset \mathbb{W}^* - V^*$

for some standard subcontinuum $\mathbb{L}_{f(\mathcal{D})}$. In particular $K \subset \mathbb{L}_{\mathcal{E}} \subset \mathbb{W}^* - U^*$ for $\mathcal{E} = f(\mathcal{D})$. $\square$

Finally we put it all together.

Theorem 2. Suppose the proper subcontinuum $K \subset \mathbb{W}^*$ contains a standard subcontinuum. For each $x \in \mathbb{W}^* - K$ we have $K \subset L \subset \mathbb{W}^* - x$ for some standard subcontinuum $L \subset \mathbb{W}^*$.

Proof. Since $x \notin K$ there is a basic open $V^* \subset \mathbb{W}^*$ with $x \in V^* \subset \mathbb{W}^* - K$. Lemma 6.4.13 says some open $U \subset V$ has $x \in U^*$ and $\mathbb{W}^* - U^* = \bigoplus_{n=1}^{\infty} X_n$. Since K contains a standard subcontinuum Lemma 6.4.17 says some standard subcontinuum $L = \mathbb{L}_{\mathcal{E}}$ has $K \subset \mathbb{L}_{\mathcal{E}} \subset \mathbb{W}^* - U^*$ and so $K \subset \mathbb{L}_{\mathcal{E}} \subset \mathbb{W}^* - x$. $\square$

6.5 Standard Subcontinua: Dominoes

The previous section gave approximations from above of certain well behaved subcontinua of $\mathbb{W}^*$. This section shows even badly behaved subcontinua of $\mathbb{W}^*$ are contained in a particularly simple type of standard subcontinuum.

Definition 6.5.1. We call the standard subcontinuum $\mathbb{K}_{\mathcal{D}} \subset \mathbb{W}^*$ a $\mathcal{T}$*-domino* to mean each $K_i = T(m_i, n_i)$ for some $m_i, n_i \in \mathbb{N}$. We call the standard subcontinuum $\mathbb{I}_{\mathcal{D}} \subset \mathbb{H}^*$ a *domino* to mean each I_i has endpoints in $\mathbb{N}$.

We first prove dominoes are sufficient to cover $\mathbb{H}^*$.

Lemma 6.5.2. Each proper subcontinuum of $\mathbb{H}^*$ is contained in a domino.

Proof. Let $K \subset \mathbb{H}^*$ be an arbitrary proper subcontinuum. Corollary 5.2 of [15] says K is contained in some nondegenerate standard subcontinuum $\mathbb{I}_{\mathcal{D}}$. Choose an increasing sequence $(x_m) \subset \mathbb{N}$ so each $[x_m, x_{m+1}]$ contains some I_n.

In case $E = \big\{n \in \omega : \text{no } x_m \in I_n\big\} \in \mathcal{D}$ define each $y_n = x_n$ and proceed to the next paragraph. Otherwise $E^c = \big\{n \in \omega : \text{some } x_m \in I_n\big\} \in \mathcal{D}$. Since $\mathbb{I}_{\mathcal{D}}$ is well defined some $D \in \mathcal{D}$ with $D \subset E^c$ has $\{I_d : d \in D\}$ pairwise disjoint. The ultrafilter $\mathcal{D}$ includes exactly one of $D(1) = \{d(1), d(3), \ldots\}$ or $D(2) = \{d(2), d(4), \ldots\}$. Without loss of generality $D(1) \in \mathcal{D}$. Then let $(y_m) \subset \mathbb{N}$ be the subsequence $\{x_m : x_m \in I_{d(2)} \cup I_{d(4)} \cup \ldots\}$.

In either case $\widetilde{E} = \big\{r \in \omega : \text{no } y_m \in I_r\big\} \in \mathcal{D}$. Then each I_r is contained in some unique $[y_m, y_{m+1}]$. Define the finite-to-one monotone $f : \widetilde{E} \to \omega$ by $I_r \subset [y_{f(r)}, y_{f(r)+1}]$ and extend it to ω. For $J_n = [y_{f(n)}, y_{f(n)+1}]$ we claim $\mathbb{J}_{f(\mathcal{D})}$ is a well defined standard subcontinuum hence a domino containing $\mathbb{I}_{\mathcal{D}}$.

To that end consider the partition of $\widetilde{E}$ into $E(1) = \widetilde{E} \cap f^{-1}\big(\{1, 3, \ldots\}\big)$ and $E(2) = \widetilde{E} \cap f^{-1}\big(\{2, 4, \ldots\}\big)$. Without loss of generality $E(1) \in \mathcal{D}$ hence $f\big(E(1)\big) \in f(\mathcal{D})$. Since y_n is increasing the family $\big\{[y_n, y_{n+1}] : n = 1, 3, \ldots\big\}$ is pairwise disjoint. Hence the subfamily $\big\{J_n : n \in f\big(E(1)\big)\big\}$ is pairwise disjoint. $\square$

Next we show how to *pull up* dominoes from $\mathbb{H}^*$ to $\mathbb{W}^*$ using a special type of map.

Definition 6.5.3. The continuous surjection $F : \mathbb{W} \to \mathbb{H}$ is called a $\mathcal{T}$*-reduction* to mean each $F\big(T(n,m)\big) = [n,m]$ and $F\big(P(m)\big) = \{m\}$.

Notation 6.5.4. We write $\widetilde{\mathcal{T}} \le \mathcal{T}$ to mean there exists an increasing sequence $0 = N_0 < N_1 < \ldots$ of integers such that $\widetilde{\mathcal{T}}$ is as defined in Proposition 6.3.7.

Lemma 6.5.5. Suppose $K \subset \mathbb{W}^*$ is a proper subcontinuum. For some $\widetilde{\mathcal{T}} \le \mathcal{T}$ there is a $\widetilde{\mathcal{T}}$-reduction $F : \mathbb{W} \to \mathbb{H}$ with $F(K) \subset \mathbb{H}^*$ contained in a domino.

Proof. Let $K \subset \mathbb{W}^*$ be a proper subcontinuum. Lemma 6.4.13 says $K \subset \mathbb{W}^* - U^*$ for some open $U \subset \mathbb{W}$ with $\mathbb{W} - U = \bigoplus_{i=1}^{\infty} X_i$. Mimicking the proof we can assume, without loss of generality, there are $0 = N_0 < N_1 < N_2 < \ldots$ with U disjoint from all $P(N_i)$. By passing to a subsequence we can assume each $N_i + 3 \le N_{i+1}$ and U meets each $T(N_i, N_{i+1})$.

Since $T(N_i, N_{i+1})$ is irreducible about $P(N_i) \cup P(N_{i+1})$ the set $T(N_i, N_{i+1}) - U = A_i \cup B_i$ is a disjoint union of closed sets A_i containing $P(N_i)$ and B_i containing $P(N_{i+1})$. Urysohn's lemma ([18] Theorem III.2) gives continuous sujections $g_i : A_i \to [N_i, N_i + 1]$ with $g_i\big(P(N_i)\big) = \{N_i\}$ and $h_i : B_i \to [N_{i+1} - 1, N_{i+1}]$ with $h_i\big(P(N_{i+1})\big) = \{N_{i+1}\}$. Write their combination as $f_i : A_i \cup B_i \to [N_i, N_{i+1}]$.

The Tietze extension theorem ([18] Theorem III.3) says there is a continuous extension $f_i : T(N_i, N_{i+1}) \to [N_i, N_{i+1}]$ which is surjective since the domain is connected. Let $H : \mathbb{W} \to \mathbb{H}$ be the combination of $f_1, f_2, \ldots$. We claim H is continuous. Since $\mathbb{W} - U = \bigoplus_{n=1}^{\infty} X_i$ is a disjoint union of clopen sets it is enough to show the restriction to each fixed X_i is continuous.

Since X_i is compact it is contained in some $T(0, N_{n+1})$. Proposition 2.1.13 of [13] says the combination $f : T(0, N_{n+1}) \to [0, N_{n+1}]$ of $f_1, f_2, \dots f_n$ is continuous. Since the restriction of f to X_i coincides with H we get continuity of H. We conclude all restrictions are continuous.

By construction we have

$$H(\mathbb{W} - U) = \bigcup_{i=0}^{\infty} \Big([N_i, N_i + 1] \cup [N_{i+1} - 1, N_{i+1}] \Big).$$

It follows $H(K) \subset (\mathbb{W} - U)^*$ is proper.

Observe $H\big(T(N_i, N_{i+1})\big) = [N_i, N_{i+1}]$ and $H\big(P(N_i)\big) = \{N_i\}$. Hence each $H\big(T(N_i, N_j)\big) = [N_i, N_j]$. Choose an homeomorphism $G : \mathbb{H} \to \mathbb{H}$ with each $N_i \mapsto i$. It follows $F = G \circ H$ is a $\widetilde{\mathcal{T}}$-reduction for each $\widetilde{T}(n.5) = T(N_n, N_{n+1})$ and $\widetilde{P}(n) = P(N_n)$. Since $H(K)$ is proper so is $F(K)$ hence is contained in a domino by Lemma 6.5.2. $\square$

Lemma 6.5.6. Suppose $F : \mathbb{W} \to \mathbb{H}$ is a $\mathcal{T}$-reduction and $K \subset \mathbb{W}^*$ a standard subcontinuum with $F(K)$ contained in a domino. Then K is contained in a $\mathcal{T}$-domino.

Proof. By assumption $F(K)$ is contained in some domino $\mathbb{I}_{\mathcal{D}}$. There are $m_i, n_i \in \mathbb{N}$ with each $I_i = [m_i, n_i]$. Define the closed subsets $A_i = F^{-1}(I_i)$ of $\mathbb{W}$.

Observe each A_i is contained in $T(m_i - 1, n_i + 1)$ hence is compact. Since $\mathbb{I}_{\mathcal{D}}$ is well defined there is $D \in \mathcal{D}$ with $\{I_n : n \in D\}$ pairwise disjoint. This implies $\{A_n : n \in D\}$ are pairwise disjoint and $\mathbb{A}_{\mathcal{D}}$ well defined. It follows from definition that $F^{-1}(\mathbb{I}_{\mathcal{D}}) = \mathbb{A}_{\mathcal{D}}$

Define each subcontinuum $K_i = T(m_i - 1, n_i + 1)$. Proposition 6.4.5 says $K \subset \mathbb{A}_{\mathcal{D}} \subset \mathbb{K}_{\mathcal{D}}$. It remains to show $\mathbb{K}_{\mathcal{D}}$ is a well defined standard subcontinuum hence a $\mathcal{T}$-domino containing K.

To that end write $D = \{d(1), d(2), \dots\}$ in increasing order. The ultrafilter $\mathcal{D}$ includes exactly one of $D(1) = \{d(1), d(3), \dots\}$ or $D(2) = \{d(2), d(4), \dots\}$.

Without loss of generality $D(1) \in \mathcal{D}$. To see $\{K_n : n \in D(1)\}$ are pairwise disjoint take any $d(m), d(n) \in D(1)$ with $d(m) < d(n)$ and observe $d(m) < d(m+1) < d(n)$.

Then $I_{d(m)} = [r_i, r_j]$, $I_{d(m+1)} = [r_k, r_l]$, and $I_{d(n)} = [r_p, r_q]$ for some $i, j, k, l, p, q \in \omega$ and $r_i < r_j < r_k < r_l < r_p < r_q$. It follows that $r_j + 1 \leq r_k < r_l \leq r_p - 1$. Hence $r_j + 1 < r_p - 1$ and $T(r_i - 1, r_j + 1)$ and $T(r_p - 1, r_q + 1)$ are disjoint. By definition the two sets are $K_{d(m)}$ and $K_{d(n)}$ respectively. $\square$

Theorem 3. Each proper subcontinuum of $\mathbb{W}^*$ is contained in a $\mathcal{T}$-domino.

Proof. Suppose $K \subset \mathbb{W}^*$ is a proper subcontinuum. Lemma 6.5.5 says there is $\widetilde{\mathcal{T}} \leq \mathcal{T}$ and a $\widetilde{\mathcal{T}}$-reduction $F : \mathbb{W} \to \mathbb{H}$ with $F(K)$ contained in a domino. Lemma 6.5.6 says K is contained in a $\widetilde{\mathcal{T}}$-domino $\mathbb{K}_{\mathcal{D}}$. Since each $K_n = \widetilde{T}(i, j) = T(N_i, N_j)$ for some increasing sequence $N_0 < N_1 < \ldots$ it follows $\mathbb{K}_{\mathcal{D}}$ is also a $\mathcal{T}$-domino. $\square$

6.6 How Dominoes Interact

We show dominoes interact like standard subcontinua of $\mathbb{H}^*$. First we construct a map that relates the two spaces.

Lemma 6.6.1. There is a $\mathcal{T}$-reduction $f : \mathbb{W} \to \mathbb{H}$

Proof. The Tietze extension theorem gives a family of continuous surjections $g_n : T(n.5) \to [n, n+1]$ with each $g_n\big(P(n)\big) = \{n\}$ and $g_n\big(P(n+1)\big) = \{n+1\}$. Proposition 2.1.13 of [13] says each family $\{g_1, g_2, \ldots, g_m\}$ has a continuous combination $f_m : T(0, m+1) \to \mathbb{H}$.

To see the combination of $\{f_1, f_2, \ldots\}$ is continuous first recall each component L_n of $\mathbf{0}$ in $\mathbb{W} - P(n)$ is open and contained in $T(0, n+1)$. Let $G_n : L_n \to \mathbb{H}$ be the restriction of f_n. Then observe each $x \in \mathbb{W}$ is an element of some $T(N.5) \subset L_{N+1}$ and therefore $\bigcup_n L_n = \mathbb{W}$. Proposition 2.1.11 of [13] says the combination of $\{G_1, G_2, \ldots\}$ is continuous. Clearly it coincides with the combination of $\{f_1, f_2, \ldots\}$. $\square$

Definition 6.6.2. We say the subcontinua $L, K \subset X$ *intersect properly* to mean the three sets $K \cap L$, $K - L$ and $L - K$ are nonempty.

Proposition 6.6.3 is straightforward to prove.

Proposition 6.6.3. The subcontinuum $T(m, n)$ of $\mathbb{W}$ contains (resp. intersects properly with) $T(a, b)$ if and only if the interval $[m, n]$ of $\mathbb{H}$ contains (resp. intersects properly with) $[a, b]$.

Theorem 4. For dominoes $\mathbb{K}_{\mathcal{D}}, \mathbb{L}_{\mathcal{E}} \subset \mathbb{W}^*$ exactly one of the four holds.

(1) $\mathbb{K}_{\mathcal{D}}$ and $\mathbb{L}_{\mathcal{E}}$ are disjoint

(2) $\mathbb{K}_{\mathcal{D}} \subset \mathbb{L}_{\mathcal{E}}$ and $\mathcal{E} \leq \mathcal{D}$

(3) $\mathbb{L}_{\mathcal{E}} \subset \mathbb{K}_{\mathcal{D}}$ and $\mathcal{D} \leq \mathcal{E}$

(4) $\mathbb{K}_\mathcal{D}$ and $\mathbb{L}_\mathcal{E}$ intersect properly and $\mathcal{D} \sim \mathcal{E}$.

Proof. By assumption each $K_n = T(a_n, b_n)$ and $L_n = T(c_n, d_n)$ for some $a_n, b_n, c_n, d_n \in \omega$. Lemma 6.6.1 says there exists a $\mathcal{T}$-reduction $f : \mathbb{W} \to \mathbb{H}$. For $I_n = [a_n, b_n]$ and $J_n = [c_n, d_n]$ the definition of the Stone-Čech lift $f : \beta\mathbb{W} \to \beta\mathbb{H}$ implies $f(\mathbb{K}_\mathcal{D}) = \mathbb{I}_\mathcal{D}$ and $f(\mathbb{L}_\mathcal{E}) = \mathbb{J}_\mathcal{E}$.

For $\mathbb{K}_\mathcal{D}$ and $\mathbb{L}_\mathcal{E}$ disjoint clearly $(2)-(4)$ fail. Now assume $\mathbb{K}_\mathcal{D}$ meets $\mathbb{L}_\mathcal{E}$. Then $f(\mathbb{K}_\mathcal{D}) = \mathbb{I}_\mathcal{D}$ meets $f(\mathbb{L}_\mathcal{E}) = \mathbb{J}_\mathcal{E}$. Let $(2') - (4')$ be the statements $(2) - (4)$ with $\mathbb{K}_\mathcal{D}$ and $\mathbb{L}_\mathcal{E}$ replaced by $\mathbb{I}_\mathcal{D}$ and $\mathbb{J}_\mathcal{E}$ respectively. Theorem 5.3 of [15] says exactly one of $(2') - (4')$ holds.

Suppose $(2')$ holds. Proposition 6.4.5 says there are $D \in \mathcal{D}$ and $E \in \mathcal{E}$ and a finite-to-one $g : D \to E$ with each $I_n \subset J_{g(n)}$. It follows $F(K_n) \subset F(L_{g(n)})$ and so $[a_n, b_n] \subset [c_{g(n)}, d_{g(n)}]$. Proposition 6.6.3 then says $K_n \subset L_{g(n)}$ for each $n \in D$. We conclude $\mathbb{K}_\mathcal{D} \subset \mathbb{L}_\mathcal{E}$ by Proposition 6.4.5 . The case for $(3')$ is similar.

Suppose $(4')$ holds. Theorem 5.3 of [15] implies there are $D \in \mathcal{D}$ and $E \in \mathcal{E}$ and a permutation $\sigma : D \to E$ such that I_n meets $J_{\sigma(n)}$ for each $n \in D$. The ultrafilter $\mathcal{D}$ cannot include the set $\{n \in D : I_n \subset J_{\sigma(n)}\}$ as then Proposition 6.4.5 says $\mathbb{I}_\mathcal{D} \subset \mathbb{J}_\mathcal{E}$. We conclude $\{n \in D : I_n$ properly intersects $J_{\sigma(n)}\} \in \mathcal{D}$ contrary to assumption. It follows $\{n \in D : K_n$ properly intersects $L_{\sigma(n)}\} \in \mathcal{D}$ which again by Proposition 6.4.5 implies $\mathbb{K}_\mathcal{D}$ and $\mathbb{L}_\mathcal{E}$ intersect properly. The function σ witnesses how $\mathcal{D} \sim \mathcal{E}$. □

Theorem 4 gives an explicit bijection between the composants of $\mathbb{H}^*$ and near coherence classes of ω^*. By mimicking the proof of [10] Theorem 4.1 with standard subcontinua of $\mathbb{H}^*$ replaced by dominoes of $\mathbb{W}^*$ we see each composant $E \subset \mathbb{W}^*$ corresponds to the near coherence class

$$\mathscr{E} = \big\{\mathcal{D} \in \omega^* : E \text{ contains some domino } \mathbb{K}_\mathcal{D}\big\}$$

and each near coherence class $\mathscr{E} \subset \omega^*$ corresponds to the composant

$$E = \bigcup \big\{\mathbb{K}_\mathcal{D} : \mathbb{K}_\mathcal{D} \text{ is a domino and } D \in \mathscr{E}\big\}.$$

6.7 Composants and Q-Points

We are ready to prove our first main theorem.

Theorem 5. Suppose the composant $E \subset \mathbb{W}^*$ corresponds to a near coherence class with no Q-point. Then each proper semicontinuum of E is nowhere dense.

Proof. Suppose S is a proper semicontinuum of E. Fix some $x \in S$ and write $S = \bigcup \mathcal{S}$ as a union of proper subcontinua with $x \in L$ for each $L \in \mathcal{S}$. For each $L \in \mathcal{S}$ Theorem 3 says there is $\mathbb{K}^L$ and $\mathcal{D}(L)$ such that the domino $\mathbb{K}^L_{\mathcal{D}(L)}$ contains L.

Choose $y \in E - S$ and let $K \subset E$ be a subcontinuum with $\{x, y\} \subset K$. Theorem 3 says K is contained in a domino $\mathbb{K}_{\mathcal{D}}$. The points x and y witness how $\mathbb{K}_{\mathcal{D}}$ properly intersects each $\mathbb{K}^L_{\mathcal{D}(L)}$. Theorem 4 says each $\mathcal{D}(L) \sim \mathcal{D}$.

Since $\mathcal{D}$ is not a Q-point some $\mathcal{E} \in \omega^*$ has $\mathcal{E} < \mathcal{D}$. That means $f(\mathcal{D}) = \mathcal{E}$ for some monotone finite-to-one function $f : \omega \to \omega$. Let each $a_n, b_n \in \omega$ respectively be the greatest and least elements with $\bigcup\{K_m : f(m) = n\} \subset T(a_n, b_n)$.

Define each $L_n = T(a_n, b_n)$. Since $\mathbb{K}_{\mathcal{D}}$ is well defined there is $D \in \mathcal{D}$ with $\{K_n : n \in D\}$ pairwise disjoint. It follows $\big\{L_n : n \in f(D)\big\}$ is pairwise disjoint. Hence $\mathbb{L}_{\mathcal{E}}$ is a well defined domino with $\mathbb{K}_{\mathcal{D}} \subset \mathbb{L}_{\mathcal{E}}$.

Since each $\mathcal{D}(L) \sim \mathcal{D}$ we have $\mathcal{E} < \mathcal{D}(L)$. Thus each $\mathbb{K}^L_{\mathcal{D}(L)} \subset \mathbb{L}_{\mathcal{E}}$ by Theorem 4 (2). We conclude S is contained in the subcontinuum $\mathbb{L}_{\mathcal{E}}$ which is nowhere dense since $\mathbb{W}^*$ is indecomposable. □

Next we deal with composants where the corresponding near coherence class has a Q-point. The next lemma will be useful. It might seem trivially true to a reader unfamiliar with Proposition 6.4.5.

Lemma 6.7.1. For the point $x_{\mathcal{D}}$ and standard subcontinuum $\mathbb{K}_{\mathcal{D}}$ we have $x_{\mathcal{D}} \in \mathbb{K}_{\mathcal{D}}$ if and only if $\{n \in \omega : x_n \in K_n\} \in \mathcal{D}$.

Proof. Suppose $x_{\mathcal{D}} \in \mathbb{K}_{\mathcal{D}}$. Proposition 6.4.5 gives sets $D_1, D_2 \in \mathcal{D}$ and a function $f : D_1 \to D_2$ with $f(\mathcal{D}) = \mathcal{D}$ and $x_n \in K_{f(n)}$ for each $n \in D_1$. Extend f to a function $F : \omega \to \omega$.

To see $F(\mathcal{D}) = \mathcal{D}$ observe $\{D \in \mathcal{D} : D \subset D_1\}$ is cofinal in $\mathcal{D}$. Hence $\{F(D) : D \in \mathcal{D} \text{ and } D \subset D_1\} = \{f(D) : D \in \mathcal{D} \text{ and } D \subset D_1\}$ is cofinal in $\{F(D) : D \in \mathcal{D}\}$. Recall the second set generates $f(\mathcal{D})$ while the third generates $F(\mathcal{D})$. We conclude $F(\mathcal{D}) = f(\mathcal{D})$.

By [12] Theorem 9.2 (a) we have $D_3 = \{n \in \omega : F(n) = n\} \in \mathcal{D}$. For each $n \in D_1 \cap D_3$ we have $x_n \in K_n$. Since the set $C = \{n \in \omega : x_n \in K_n\}$ contains $D_1 \cap D_3 \in \mathcal{D}$ we have $C \in \mathcal{D}$. The other direction of proof follows from Proposition 6.4.5. □

Next we introduce a way to slightly expand a given domino.

Notation 6.7.2. For dominoes $\mathbb{K}_{\mathcal{D}}$ and $\mathbb{L}_{\mathcal{D}}$ with each $K_n = T(a_n, b_n)$ and $L_n = T(c_n, d_n)$ we write $\mathbb{K}_{\mathcal{D}} \Subset \mathbb{L}_{\mathcal{D}}$ to mean each $c_n < a_n \leq b_n < d_n$.

Lemma 6.7.3. For each domino $\mathbb{K}_{\mathcal{D}}$ there is s domino $\mathbb{L}_{\mathcal{D}}$ with $\mathbb{K}_{\mathcal{D}} \Subset \mathbb{L}_{\mathcal{D}}$.

Proof. By assumption some $D \in \mathcal{D}$ has $\{T(a_n, b_n) : n \in D\}$ pairwise disjoint. Write $D = \{d(1), d(2), \ldots\}$ in increasing order. Then each $a_{d(n)} \leq b_{d(n)} < a_{d(n+1)} \leq b_{d(n+1)}$. For $N = 0, 1, 2$ the ultrafilter $\mathcal{D}$ includes exactly one of the three sets $D(N) = \{d(3n + N) : n \in \omega\}$. Without loss of generality $D(0) \in \mathcal{D}$.

Observe each

$$b_{d(3n)} < a_{d(3n+1)} < a_{d(3n+2)} < a_{d(3n+3)} \implies b_{d(3n)} + 1 < a_{d(3n+3)} - 1.$$

It follows the sets $\big\{T(a_n - 1, b_n + 1) : n \in D(0)\big\}$ are pairwise disjoint. For each $n \in D(0)$ define $L_n = T(a_n - 1, b_n + 1)$. For each $n \notin D(0)$ chose $L_n = T(c_n, d_n)$ for some $c_n < a_n \leq b_n < d_n$. It follows $\mathbb{L}_{\mathcal{D}}$ is well defined and $\mathbb{K}_{\mathcal{D}} \Subset \mathbb{L}_{\mathcal{D}}$. □

The next lemma constructs a dense proper semicontinuum. Theorem 6 uses these semicontinua to show each point is coastal.

Lemma 6.7.4. Suppose $\mathcal{D}$ is a Q-point and $\mathbb{K}_{\mathcal{D}}$ a domino with each $K_i = T(m_i, n_i)$ for $m_i < n_i$. Suppose each $x_i \in P(m_{\mathcal{D}})$ and $p_i \in P(n_{\mathcal{D}})$. Then the continuum component of $x_{\mathcal{D}}$ in $\mathbb{W}^* - p_{\mathcal{D}}$ is dense.

Proof. Suppose for a contradiction the continuum component C of $x_{\mathcal{D}}$ in $\mathbb{W}^* - p_{\mathcal{D}}$ is not dense. Since $\mathbb{W}^*$ is indecomposable $\overline{C}$ and $\mathbb{K}_{\mathcal{D}}$ hence $\overline{C} \cup \mathbb{K}_{\mathcal{D}}$ are nowhere dense. Theorem 3 says $\overline{C} \cup \mathbb{K}_{\mathcal{D}} \subset \widetilde{\mathbb{A}}_{\mathcal{F}}$ for some domino $\widetilde{\mathbb{A}}_{\mathcal{F}}$. Since $\mathbb{K}_{\mathcal{D}} \subset \widetilde{\mathbb{A}}_{\mathcal{F}}$ we have by Proposition 6.4.5 sets $D \in \mathcal{D}$ and $F \in \mathcal{F}$ and a finite-to-one $f : \omega \to \omega$ with $K_n \subset \widetilde{A}_{f(n)}$ for each $n \in D$.

Write $D = \{d(1), d(2), \ldots\}$ in increasing order. Since $\mathcal{D}$ is a Q-point f is injective over some $D' \subset D$. By discarding some elements and relabelling we can assume $D = D'$. Thus there are bijections $F : d(n) \mapsto n$ and $G : f(d(n)) \mapsto n$. Define each $q_n = p_{d(n)}$ and $y_n = x_{d(n)}$ and $A_n = \widetilde{A}_{f(d(n))}$. Proposition 6.4.6 says $q_{F(\mathcal{D})} = p_{\mathcal{D}}$ and $y_{F(\mathcal{D})} = x_{\mathcal{D}}$ and $\mathbb{A}_{G(\mathcal{F})} = \widetilde{\mathbb{A}}_{\mathcal{F}}$. Since each $y_n \in A_n$ we have $F(\mathcal{D}) = G(\mathcal{F})$. Henceforth write $\mathcal{E} = F(\mathcal{D}) = G(\mathcal{F})$ and each $M_n = m_{d(n)}$ and $N_n = n_{d(n)}$.

Lemma 6.7.3 says $\mathbb{A}_{\mathcal{E}} \Subset \mathbb{B}_{\mathcal{E}}$ for some domino $\mathbb{B}_{\mathcal{E}}$ with each $B_i = T(a_i, b_i)$. Consider the domino $\mathbb{L}_{\mathcal{E}}$ for each $L_i = T(a_i, M_i)$. Lemma 6.7.1 says $P(a_{\mathcal{E}}) \subset \mathbb{L}_{\mathcal{E}} - \mathbb{A}_{\mathcal{E}}$ and so $P(a_{\mathcal{E}}) \subset \mathbb{L}_{\mathcal{E}} - \overline{C} \cup \mathbb{K}_{\mathcal{D}}$ thus $\mathbb{L}_{\mathcal{E}}$ meets $\mathbb{W}^* - C$. Recall also $x_{\mathcal{D}} = y_{\mathcal{E}} \in P(M_{\mathcal{E}}) \subset \mathbb{L}_{\mathcal{E}}$ and so $x_{\mathcal{D}} \in \mathbb{L}_{\mathcal{E}}$. We claim $p_{\mathcal{D}} \notin \mathbb{L}_{\mathcal{E}}$ which contradicts the definition of C.

Recall each $q_n \in P(N_n)$. By assumption we have $M_n < N_n$ which implies $T(a_n, M_n) \cap P(N_n) = \varnothing$ and in turn $q_n \notin L_n$. Then Lemma 6.7.1 says $q_{\mathcal{E}} \notin \mathbb{L}_{\mathcal{E}}$. Since $p_{\mathcal{D}} = q_{\mathcal{E}}$ the result follows. □

Theorem 6. Suppose the composant $E \subset \mathbb{W}^*$ corresponds to a near coherence class with a Q-point. Then each point of E is a proper coastal point.

Proof. Let $x \in E$ be arbitrary. Theorem 3 says $x \in \mathbb{K}_{\mathcal{D}}$ for some domino $\mathbb{K}_{\mathcal{D}}$. Let

$\mathcal{E}$ be a Q-point in the near coherence class corresponding to E. Then $\mathcal{F} \leq \mathcal{E}$ and $\mathcal{F} \leq \mathcal{D}$ for some $\mathcal{F} \in \omega^*$. Section 5 of [9] shows how to construct a monotone finite-to-one function $f : \omega \to \omega$ with $f(\mathcal{D}) = f(\mathcal{E}) = \mathcal{F}$.

Let each a_n and b_n respectively be the greatest and least elements of ω with $\bigcup\{K_m : f(m) = n\} \subset T(a_n, b_n)$. Define each subcontinuum $L_n = T(a_n, b_n)$. Since $\mathbb{K}_\mathcal{D}$ is well defined there is $D \in \mathcal{D}$ with $\{K_n : n \in D\}$ pairwise disjoint. It follows $\big\{L_n : n \in f(D)\big\}$ are pairwise disjoint hence $\mathbb{L}_\mathcal{F}$ is a well defined domino that contains $\mathbb{K}_\mathcal{D}$.

Lemma 6.7.3 says $\mathbb{L}_\mathcal{F} \Subset \mathbb{A}_\mathcal{F}$ for some domino $\mathbb{A}_\mathcal{F}$ with $A_n = T(c_n, d_n)$. Choose points $p_n \in P(d_n)$ and $x_n \in P(c_n)$ and define each $B_n = T(c_n, d_n - 1)$. Since $\mathcal{F} \leq \mathcal{E}$ we know $\mathcal{F}$ is a Q-point. Clearly $x \in \mathbb{L}_\mathcal{F} \subset \mathbb{B}_\mathcal{F}$ and Lemma 6.7.1 says $\mathbb{B}_\mathcal{F} \subset \mathbb{W}^* - p_\mathcal{F}$. Thus x is an element of the continuum component of $x_\mathcal{F}$ in $\mathbb{W}^* - p_\mathcal{F}$. Finally Lemma 6.7.4 says that continuum component is a dense semicontinuum. $\square$

Theorems 5 and 6 together give the following.

Theorem 7. For a composant $E \subset \mathbb{W}^*$ and corresponding near coherence class $\mathscr{E} \subset \omega^*$ the following are equivalent.

(1) $\mathscr{E}$ has a Q-point.

(2) E properly contains a dense semicontinuum.

(3) Each $x \in E$ is an element of some dense proper semicontinuum of E.

Since NCF implies there are no Q-points we have the global result.

Theorem 8. NCF is equivalent to some (every) wave remainder containing no dense proper semicontinuum.

6.8 Weak Cut Points

It is easy to see for $\mathbb{H}^*$ having more than one composant each point is clearly a weak cut point. Lemma 3.7 of [2] is the stronger result that, regardless of the composant number, each point of $\mathbb{H}^*$ weakly cuts its composant. This section generalises the result to arbitrary waves.

Lemma 6.8.1. Suppose $a < m < n < b$ and the subcontinuum $K \subset \omega\mathbb{W}$ meets $P(a)$ and $P(b)$. Then $T(m,n) \subset K$

Proof. Lemma 6.3.5 says R_a contains $P(m)$ hence contains $T(m,n)$. Likewise L_b contains $P(n)$ and $T(m,n)$. We conclude $T(m,n)$ is disjoint from $L_a \cup R_b$. Boundary bumping says $L_a \cup P(a)$ and $P(b) \cup R_b$ are subcontinua. Then the subcontinuum $L_a \cup P(a) \cup K \cup P(b) \cup R_b$ joins **0** to ∞ hence equals $\omega\mathbb{W}$. Thus K contains the complement of $L_a \cup P(a) \cup P(b) \cup R_b$. Since $a < m < n < b$ we know $P(a)$ and $P(b)$ are disjoint from $T(m,n)$ and so K contains $T(m,n)$. □

Lemma 6.8.2. Suppose $K_i = T(m_i, n_i)$ and $L_i = T(a_i, b_i)$ and $\mathbb{K}_\mathcal{D} \Subset \mathbb{L}_\mathcal{D}$. Each standard subcontinuum $\mathbb{B}_\mathcal{E}$ with points $p_\mathcal{D} \in P(a_\mathcal{D})$ and $q_\mathcal{D} \in P(b_\mathcal{D})$ contains $\mathbb{K}_\mathcal{D}$.

Proof. Proposition 6.4.5 gives maps $g : D_1 \to E_1$ and $h : D_2 \to E_2$ that witness how $p_\mathcal{D} \in \mathbb{B}_\mathcal{E}$ and $q_\mathcal{D} \in \mathbb{B}_\mathcal{E}$ respectively. Since $\mathbb{L}_\mathcal{D}$ is well defined there is $D \in \mathcal{D}$ with the family $\{L_n : n \in D\}$ pairwise disjoint. Proposition 6.4.6 says we can reindex D over ω hence assume the families $\{L_n : n \in \omega\}$ and $\{K_n : n \in \omega\}$ are pairwise disjoint. This implies $a_1 < b_1 < a_2 < b_2 < \ldots$.

Lemma 6.7.1 says by passing to a further element of $\mathcal{D}$ and reindexing again we can ensure each $p_n \in P(a_n)$ and $q_n \in P(b_n)$. Define the set $Q = \{p_1, q_1, p_2, q_2, \ldots\}$. Lemma 6.8.1 says each $B_n \cap Q$ has four possible forms:

$$
\begin{aligned}
{}_pQ_q &= \{p_i, q_i, \ldots, p_j, q_j\} & {}_qQ_p &= \{q_i, p_{i+1}, \ldots, q_j, p_{j+1}\} \\
{}_qQ_q &= \{q_i, p_{i+1}, \ldots, p_j, q_j\} & {}_pQ_p &= \{p_i, q_i, \ldots, q_j, p_{j+1}\}
\end{aligned}
$$

For exactly one choice of $x, y \in \{p, q\}$ the ultrafilter $\mathcal{E}$ includes the set

$$ {}_xE_y = \{n \in \omega : B_n \text{ has the form } {}_xQ_y\} $$

First suppose $E = {}_pE_q \in \mathcal{E}$. Observe g and h coincide over the element $D = g^{-1}(E) = h^{-1}(E)$ of $\mathcal{D}$. Thus for all $n \in D$ we have $\{p_n, q_n\} \subset B_{g(n)}$ and so $K_n \subset B_{g(n)}$ by Lemma 6.8.1. From Proposition 6.4.5 we have $\mathbb{K}_\mathcal{D} \subset \mathbb{B}_\mathcal{E}$.

Next suppose $E = {}_qE_q \in \mathcal{E}$. Write $E = \{a(1), a(2), \ldots\}$ in increasing order. For each $n \in \omega$ there are $i(n), j(n) \in \omega$ with

$$ B_{a(n)} \cap Q = \{q_{i(n)}, p_{i(n)+1}, q_{i(n)+1}, \ldots, p_{j(n)}, q_{j(n)}\}. $$

Passing to a subset of E and relabelling we can assume without loss of generality that $B_{a(n)} \cap Q$ are pairwise disjoint. The set $g^{-1}(E)$ is of the form

$$ \bigcup_{n=1}^{\infty} \{i(n)+1, \ldots, j(n)\}. $$

Since $h(\mathcal{D}) = \mathcal{E}$ we have $h^{-1}(E) \in \mathcal{D}$. Observe $h^{-1}(E) = D_0 \cup D_1 \cup D_2$ is the disjoint union of the sets:

$$ D_0 = \bigcup_{n=1}^{\infty} \{i(n)+1, \ldots, j(n)\} $$
$$ D_1 = \{i(2m+1) : m \in \omega\} \qquad D_2 = \{i(2m) : m \in \omega\} $$

The ultrafilter $\mathcal{D}$ includes one exactly one of the sets D_0, D_1, D_2. Observe each $q_{i(2m)} \in B_{a(2m)}$ but $p_{i(2m)} \in B_{a(2m-1)}$. Thus $g(i(2m)) = a(2m)$ and $h(i(2m)) = a(2m-1)$ and the sets $g(D_2)$ and $h(D_2)$ are disjoint. Since both sets are elements of the ultrafilter $\mathcal{E}$ we conclude $D_2 \notin \mathcal{D}$. Likewise $D_1 \notin \mathcal{D}$ and therefore $D_0 \in \mathcal{D}$. By definition g and h coincide over D_0. Like before we conclude $\mathbb{K}_\mathcal{D} \subset \mathbb{B}_\mathcal{E}$.

The proof for ${}_pE_p$ is similar. The set D_0 stays the same but we take

$$ D_1 = \{j(2m+1)+1 : m \in \omega\} \qquad D_2 = \{j(2m)+1 : m \in \omega\}. $$

The proof for ${}_qE_p$ uses five sets with

$$D_1 = \{i(2m+1) : m \in \omega\} \quad D_2 = \{i(2m) : m \in \omega\}$$

$$D_3 = \{j(2m+1)+1 : m \in \omega\} \quad D_4 = \{j(2m)+1 : m \in \omega\}.$$

□

Theorem 9. Each point of $\mathbb{W}^*$ weakly cuts its composant.

Proof. Let $x \in \mathbb{W}^*$ be arbitrary. Choose dominoes $\mathbb{K}_\mathcal{D}$ and $\mathbb{L}_\mathcal{D}$ with $x \in \mathbb{K}_\mathcal{D} \Subset \mathbb{L}_\mathcal{D}$ and each $L_n = T(a_n, b_n)$. Choose points $p_n \in P(a_n)$ and $q_n \in P(b_n)$. Since $\mathbb{L}_\mathcal{D}$ is proper $p_\mathcal{D}, x, q_\mathcal{D}$ share a composant. We claim $[p_\mathcal{D}, x, q_\mathcal{D}]$.

For suppose the subcontinuum $K \subset \mathbb{W}^*$ has $\{p_\mathcal{D}, q_\mathcal{D}\} \subset K \subset \mathbb{W}^* - x$. Since K contains the standard subcontinuum $\{p_\mathcal{D}\}$ Theorem 2 says there is a standard subcontinuum L with $K \subset L \subset \mathbb{W}^* - x$. Since $\{p_\mathcal{D}, q_\mathcal{D}\} \subset L$ we have by Lemma 6.8.2 that $\mathbb{K}_\mathcal{D} \subset L$ and so $x \in L$ which is a contradiction. We conclude no such K exists. □

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Conclusion and Open Problems

Theorem 1 of Chapter 3 shows the three motivating questions from the Introduction are equivalent. In this section we summarise our progress on those questions. Then we discuss some open problems that arise from our research, and some possible approaches and obstructions. We freely borrow terminology and notation from all previous chapters.

Non-Coastal Continua in ZFC

The example of $\mathbb{H}^*$ being a continuum without coastal points from Chapter 2 relies on assuming Near Coherence of Filters. In the absence of additional axioms the problem remains open.

To date all known ZFC Bellamy continua arise from inverse limit constructions similar to Chapters 4 and 5. One possible approach to find a continuum without coastal points is to modify these constructions. The following fact is encouraging.

Lemma 1. The trivial composant E of the inverse limit X from Chapter 4 contains no dense proper subcontinuum of itself.

Proof. Suppose $E = \bigcup\{X(\alpha) : \alpha < \omega_1\}$ contains the dense semicontinuum S. Let $\gamma < \omega_1$ be the least value with $S \cap X(\gamma) \neq \varnothing$. Since each subcontinuum $X(\alpha)$ is closed and nowhere dense S meets $E - X(\alpha)$. Thus there are points $x \in X(\gamma)$ and $x(\alpha) \in E - X(\alpha)$ and continua $K(\alpha) \subset S$ with $\{x, x(\alpha)\} \subset K(\alpha)$.

Chapter 4 Claim 20 says $K(\alpha+1)$ contains $X(\alpha+1)$ for each $\alpha > \gamma$. Hence S contains $\bigcup\{X(\alpha+1) : \alpha > \gamma\}$. Since $X(\alpha)$ are nested the union equals E. We conclude E contains no dense proper subcontinuum of itself. $\square$

Condition (b) from Chapter 4 says each $f^\alpha_\beta\big(\kappa(c_\alpha)\big) = \kappa(c_\beta)$. This makes the nontrivial composant $X - E = \varprojlim\{\kappa(c_\alpha); f^\alpha_\beta\}$ nonempty. In fact the condition is needlessly strong. To get a limit with two composants, it is enough to assume $f_\gamma^\beta\big(C(\beta)\big) \subset C(\gamma)$. Under the weaker condition Bellamy [11] has observed there is no reason *a priori* to have $X - E$ nonempty.

If the system has $\bigcap \big\{ f_0^\alpha\big(C(\alpha)\big) : \alpha < \omega_1 \big\} = \varnothing$ then $X - E = \varnothing$ and the limit X equals the trivial composant. Hence Lemma 1 says X has no coastal points. However no examples are known where the intersection is empty.

In any case $\big\{ f_0^\alpha\big(C(\alpha)\big) : \alpha < \omega_1 \big\}$ is a decreasing ω_1-chain of dense semicontinua. Since each semicontinuum is dense, basic properties of ω_1 ensure no countable subset weakly cuts between two open sets of $C(0)$. Thus $X(0)$ must be more like the unit square than the one-dimensional examples from Chapters 4 and 5.

Even if we forget about inverse limits, it is not obvious there exists a metric continuum with a decreasing ω_1-chain of dense semicontinua with empty intersection. Thanks go to Włodzimierz Holsztyński [58] for the example.

Example 2. By choosing a countable open basis for $[0,1]$ and using transfinite induction we can define an injection $f : \omega_1 \to [0,1]$ with each open $U \subset [0,1]$ having uncountable intersection with $f(\omega_1)$. For each $\beta < \omega_1$ let $\Gamma(\beta) \subset [0,1]^2$ be the union of all *crosses* at the points $N(\beta) = \big\{ \big(f(\alpha), f(\alpha)\big) : \beta < \alpha < \omega_1 \big\}$.

$$\Gamma(\beta) = \big\{ (x,y) \in [0,1]^2 : x = f(\alpha) \text{ or } y = f(\alpha) \text{ for some } \beta < \alpha < \omega_1 \big\}$$

The choice of f ensures each $\Gamma(\beta)$ is dense. Moreover $\Gamma(\beta)$ is arc-connected since any two $\big(f(\alpha), f(\alpha)\big)$ are joined by the union of a vertical line segment and horizontal line segment. Since f is injective each given $\{x, y\}$ is eventually disjoint from $\big\{ f(\alpha) : \beta < \alpha < \omega_1 \big\}$ and thus $(x,y) \notin \Gamma(\beta)$. We conclude the intersection is empty.

Unfortunately the sets $\Gamma(\beta)$ above cannot be built into the sort of inverse system that interests us. For K.P Hart has observed – as examiner for the author's thesis – that $\Gamma(\beta)$ eventually fail to be σ-compact. For suppose we have an inverse system $\{X(\alpha); f^\alpha_\beta : \beta, \alpha < \omega_1\}$ of metric continua with $X(0) = [0,1]^2$. Proposition 11.14 of [43] says the composants of $X(\alpha)$ are σ-compact. Since σ-compactness is preserved by continuous images we cannot have composants $C(\alpha) \subset X(\alpha)$ with $f^\alpha_0\big(C(\alpha)\big) = \Gamma(\alpha)$ for all $\alpha < \omega_1$.

To see σ-compactness eventually fails observe that if $\Gamma(\beta)$ is σ-compact then so is the closed subset $\{(x,y) \in \Gamma(\beta) : y = 1\}$. But that subset is just a copy of $F(\beta) = \big\{f(\alpha) : \beta < \alpha < \omega_1\big\}$. Now let $\beta < \omega_1$ be high enough that $\mathbb{Q} \cap F(\beta) = \varnothing$. Then for any $\alpha > \beta$ the set $F(\alpha) \subset [0,1]$ is the complement of a dense countable subset. From here it is straightforward to show the only compact subsets of $F(\alpha)$ are singletons. Since $F(\alpha)$ is uncountable it cannot be σ-compact.

Non-Coastal Continua Under Different Axioms

The second possibility is that our motivating questions are independent of ZFC. One proof strategy is to try and construct a model where every continuum is coastal. However the required background in set-theoretic forcing is beyond the scope of this project.

It would be interesting to know whether any familiar axioms – for example the Continuum Hypothesis or Axiom of Constructibility – have a bearing on the questions. It is conceivable that $\mathfrak{c} = \omega_1$ might give us more control over Bellamy-style inverse limits and make it easier to engineer an empty nontrivial composant. For example it allows the following (very pretty) variation of Example 2 due to David Sumner Lipham [20].

Example 3. Let D be the closed unit disc with enumeration $\{p(\alpha) : \alpha < \omega_1\}$. Since each $\{p(\beta) : \beta < \alpha\}$ is countable $\Gamma(\alpha) = D - \{p(\beta) : \beta < \alpha\}$ is dense. Clearly the intersection of all $\Gamma(\alpha)$ is empty. To see $\Gamma(\alpha)$ is arc-connected let $x, y \in \Gamma(\alpha)$ be arbitrary. Choose a line segment $L \subset D$ that bisects D into two halves that include x and y respectively. For each $b \in L$ consider the arc $l(b) = |xb| \cup |yb|$. Observe for $b_1 \neq b_2$ the arcs $l(b_1)$ and $l(b_2)$ meet only at the endpoints. Since $\{p(\beta) : \beta < \alpha\}$ is countable and there are uncountably many $l(b)$ one of them is contained in $\Gamma(\alpha)$.

We remark that assuming CH seems against the spirit of Bellamy's original ZFC example. However it would be interesting if the assumption led to an empty nontrivial composant, as then we would have two examples of a non-coastal continuum under the two mutually exclusive axioms NCF and CH.

Partially Coastal Bellamy Continua

Chapter 3 takes a non-coastal continuum X and shows how to construct a partially coastal continuum. Simply identify any $x \in X$ with the endpoint $1 \in [0,1]$. The resulting decomposable continuum resembles a lollipop. It is coastal at every point except the image of $0 \in [0,1]$.

Partially coastal Bellamy continua prove more elusive.

Question 4. Does there exist a partially coastal Bellamy continuum?

Indeed all known Bellamy continua are coastal at no point or every point. [3] shows for the Stone-Čech remainder $\mathbb{H}^*$ both cases can occur depending on the set theory. [4] shows the same for Bellamy and Rubin's *wave remainders*.

In ZFC all known Bellamy continua are formed by taking an indecomposable continuum with exactly two composants E and $X - E$ and *spot-welding* a point from each composant. The quotient spaces of E and $X - E$ are dense proper semicontinua. Hence the resulting continuum is coastal at every point. Even if we could engineer $X - E = \varnothing$ it still follows from Lemma 1 that X is coastal at no point.

Partially coastal Bellamy continua – if they exist – are very strange animals. [3] shows that, despite having no thick subcontinua, their coastal points knit together into a semicontinuum with dense interior.

This is reminiscent of the dense open semicontinuum $\mathbb{H}$ of $\beta\mathbb{H}$. One approach that looks promising is to find a metric semicontinuum E with βE a Bellamy continuum and E exactly the set of coastal points.

Recent results of Lipham [36] show E cannot be separable – in particular it cannot occur as a subset of a metric continuum – and must have the following property.

Definition 5. The topological space E is called *strongly indecomposable* to mean for any two open $U, V \subset E$ we can write $E - U = A \cup B$ as a disjoint union of clopen sets both of which meet V.

Composants of indecomposable continua are an example of strongly indecomposable spaces.

Lemma 6. Suppose E is a separable metric semicontinuum. No compactification of E is a Bellamy continuum.

Proof. Suppose γE is an indecomposable compactification. Corollary 10 of [36] says βE is indecomposable. Corollary 5 then says E is strongly indecomposable. Theorem 8 then says γE is irreducible hence not a Bellamy continuum. □

Strongly indecomposable spaces are in a sense *badly behaved*. For E to also have dense interior in βE it must be *nice enough* to be locally-compact over a dense subset. Having both properties at once is a rather pathological situation. Our only example is $E = X - p$ for p the unique strong non-cut point of the Bellamy continuum X constructed in Chapter 4. The semicontinuum $X - p$ is locally-compact, noncompact, and strongly indecomposable by [36] Lemma 3. However our construction of such a continuum and point [5] is laborious and we know next to nothing about the Stone-Čech compactification of the point-complement.

The above example is non-metric. The problem is open to find a metric example.

Question 7. Does there exist a strongly indecomposable noncompact metric semicontinuum that is locally compact over a dense subset?

David Sumner Lipham [35] has pointed out no separable example exists, and moreover the same holds if we weaken the condition of being a semicontinuum to being a reducible space – meaning any two points are joined by a proper closed connected subset. In fact the proof can be pushed further to the following remarkable fact.

Theorem 8. Suppose the reducible Tychonoff space E is strongly indecomposable and has a compact neighborhood. Every indecomposable compactification of E is a Bellamy continuum. In particular βE is a Bellamy continuum.

Proof. We first show each closed connected proper subset of E is nowhere dense. For suppose such $C \subset E$ has $C^\circ \neq \varnothing$. In the notation of Definition 5 take $V = C^\circ$ and $U = E - C$. Then $C = E - U = A \cup B$ is the disjoint union of two nonempty clopen sets. This contradicts how C is connected.

Suppose X is an indecomposable compactification of E. Fix $x \in E$ and let $y \in X$ be arbitrary. We will find a proper subcontinuum $K \subset X$ with $\{x, y\} \subset K$. Hence each $y \in X$ shares a composant with the point $x \in X$.

By assumption some open $U \subset E$ has compact closure in E. It follows U is open in X. Since the composant of $y \in X$ is dense some proper subcontinuum $K_1 \subset X$ includes y and meets U. Since E is reducible some closed connected set $C \subset E$ includes x and meets U. Let the subcontinuum $K_2 \subset X$ be the closure of C in X.

Since X is indecomposable the subcontinua $K_1, K_2 \subset X$ are nowhere dense. Hence $K_1 \cup K_2$ is nowhere dense. In particular $K_1 \cup K_2$ is a proper subcontinuum that witnesses how x and y share a composant. We conclude X is a Bellamy continuum.

Corollary 5 of [36] says βE is an indecomposable continuum. Hence it is an indecomposable compactification of E and a Bellamy continuum by the above.

□

Corollary 9. Suppose X is a Bellamy continuum and $p \in X$ a strong non-cut point. Then $\beta(X - p)$ is a Bellamy continuum.

We give the proof there is no separable answer to Question 7.

Lemma 10. Suppose the noncompact metric space E is strongly indecomposable and locally-compact over a dense subset. Then E cannot be both separable and reducible.

Proof. For E separable [36] Theorem 8 says E has a metrizable indecomposable compactification X. For E reducible Theorem 8 says X is a Bellamy continuum. But this contradicts how indecomposable metric continua have exactly $\mathfrak{c}$ composants. $\square$

Another concrete example worth investigation is $E \subset \mathbb{H}^*$ a composant where the corresponding near coherence class has no Q-point. While NCF is required to ensure $E = \mathbb{H}^*$ such a composant always exists under ZFC [32].

Question 11. Suppose the composant $E \subset \mathbb{H}^*$ corresponds to a near coherence class with no Q-point. Determine the composant number and coastal properties of the indecomposable continuum βE.

Small Bellamy Continua

Chapter 5 constructs a Bellamy continuum that is *small* in the sense of being separable. Two stricter *smallness* conditions that our example fails are being hereditarily separable or dense-hereditarily separable. This suggests a more general class of problems.

Problem 12. Choose a suitable notion of *smallness*. Construct or prove there exists no small Bellamy continuum.

Each of the following *smallness* properties implies those below.

Definition 13. The space X is called *Fréchet-Urysohn* to mean the sequential closure of each subset equals the closure.

Definition 14. The space X is called *sequential* to mean each sequentially closed subset is closed.

Definition 15. The space X is said to have *countable tightness* to mean for each $A \subset X$ and $x \in \overline{A}$ there exists a countable $B \subset A$ with $x \in \overline{B}$.

One smallness property that is too strong is X being first-countable at some point. Then the standard argument [43] to prove metric continua have uncountably many composants can be repeated for X.

A similar problem is to look for non-metric indecomposable continua with at least one *small* composant.

Problem 16. Choose a suitable notion of *smallness*. Construct or prove there exists no non-metric indecomposable continuum with a small composant.

Chapter 5 gives a solution for the notion of separability. For metrisability the problem is open: It is unknown if there exists a non-metric indecomposable continuum X with a metrisable composant E. We observe by [36] Theorem 8 that for E both metrisable and separable X has infinitely many composants.

Strong Non-Cut Points of Bellamy Continua

Chapter 4 constructs a Bellamy Continuum with exactly one strong non-cut point. This suggests a broad class of problems about the possible cardinalities and topological properties of the set of strong non-cut points of a Bellamy continuum.

For example we see no obstruction to generalising the methods of Chapter 4 to replace the points $c_\alpha \in X(\alpha)$ with copies of the Hilbert cube and thus prove the next conjecture.

Conjecture 17. Each metric continuum M embeds as a set of strong non-cut points of a Bellamy continuum.

We propose the following strategy. First embed M in the Hilbert cube $[0,1]^{\mathbb{N}}$ which we embed in turn as the subspace $\{0\} \times [0,1]^{\mathbb{N}}$ of $H = [0,1] \times [0,1]^{\mathbb{N}}$. To define $X(0)$ start with the two halves $Q \cup R$ of the quinary buckethandle on the plane. Glue $c_0 = (0,1)$ to some fixed $c \in (0,1] \times [0,1]^{\mathbb{N}}$ and attach arcs $A(x)$ from each $(x,-1)$ to $(x,1)$ so the resulting continuum is indecomposable and $A(x) \to H$ as $x \to 0$.

Write $H(0)$ for the copy of H in $X(0)$ and $M(0)$ for the copy of M in $H(0)$. Select (q_0^n) to form the long end of the tail as before. The role of the short end will be played by a chain $R_0(n)$ of arcs from a_0 to some point $P_0^n \in (0,1] \times [0,1]^{\mathbb{N}} \subset H(0)$ whose union is dense in $H(0)$. Note this makes the arcs disjoint from $M(0)$.

To construct $X(1)$ again start with $Q \cup R$. Glue the point $a_0 \in X(0)$ to $(1,1) \in Q$. Instead of gluing $[[a_0, c_0]]$ to $(0,1)$ attach a Hilbert cube $H(1)$ at the point corresponding to c. Let each $R_1(n)$ be the union of the arc from a_1 to $(0,1)$ and the copy of $R_0(n)$ in $H(1)$. Then we have $H(1) \to H(0)$ and $M(1) \to M(0)$ and each $R_1(n) \to R_0(n)$.

For $2/5 \leq k < 1$ define $B(k)$ as in Chapter 4. For $0 < k \leq 1/5$ each $B(k)$ was previously two copies of some $[p_0^n, c_0]$ strung from $(k,-1)$ to $(k,-1)$ and joined at p_0^n. Instead define each $B(k)$ using $R_0(n)$ in place of $[p_0^n, c_0]$. It follows $B(k) \to H(1)$ as $k \to 0$ and $X(1)$ is indecomposable.

The construction of each successor $X(\beta+1)$ is similar. For limit ordinals we simply take the inverse limit. Since H is not hereditarily unicoherent we must take care to record at each stage $\alpha < \omega_1$ the exact extent to which $X(\alpha)$ is hereditarily unicoherent for use in later stages.

We predict the resulting inverse system has each $H(\alpha) \to H(\beta)$ and $M(\alpha) \to M(\beta)$ and $R_\alpha(n) \to R_\beta(n)$ for $\beta \leq \alpha$. Moreover each point of $M(\beta)$ is the image of a unique point of $X(\alpha)$. Namely the corresponding point of $M(\alpha)$. It should follow each point of $\varprojlim M(\alpha)$ does not weakly cut its composant.

One drawback of the construction is it gives at most a proper subcontinuum of strong non-cut points. Since the limit is indecomposable that subcontinuum is nowhere dense. On the other end of the spectrum we might imagine the following.

Conjecture 18. There exists a Bellamy continuum that is irreducible about the set of strong non-cut points.

One approach to Conjecture 18 is again start with the two halves $Q \cup R$ of the quinary buckethandle on the plane $\mathbb{R}^2 \times \{0\}$. For each $k(n) = 1 - 2/5^n$ attach the arc $\{k(n)\} \times [-1, 1] \times \{0\}$ and vertical segment $V_n = \{k(n)\} \times \{-1\} \times [0, 1/n]$. Attach arcs $A(x)$ from the remaining points $(x, -1, 0)$ to $(x, 1, 0)$ that tend to V_n and the resulting continuum $X(0)$ is indecomposable.

It follows $X(0)$ is irreducible about $\big\{(k(n), -1, 1/n) : n \in \mathbb{N}\big\}$ and no $p_n = \big(k(n), -1, 1/n\big)$ weakly cuts its composant. To ensure the same for the limit we should arrange the preimage of p_n to be a unique point that does not weakly cut its composant at each stage. However it is not obvious how to define each $X(\beta+1)$ to achieve this.

Question 19. Does there exist a Bellamy continuum where every point is a strong non-cut point?

Our construction for Conjecture 17 yields many possible variations. For example let F be a fan over the totally disconnected compact metric space D with vertex $c \in F$. We see no obstruction to replace the Hilbert cubes $H(\beta)$ with copies $F(\beta)$ of the fan and thus solve the following.

Conjecture 20. Each totally disconnected compact metric space D embeds as the set of strong non-cut points of a Bellamy continuum.

This time the continua that play the role of $[p_\alpha^n, a_\alpha]$ or $R_\alpha(n)$ take more work to construct. We first observe for $Z \subset F$ the set of strong non-cut points $F - Z$ is continuumwise connected (indeed this holds for any hereditarily unicoherent continuum) and dense.

Thus we can select a dense subset $\{b_0, b_1, b_2, \ldots\} \subset F - Z$ and follow the original method of Bellamy [11] to construct the continua $B_0(n)$ by gluing copies of $[b_0, b_1], [b_1, b_2], \ldots, [b_{n-1}, b_n]$ together end-to-end. Then define $S_0(n)$ as a copy of the arc from a_0 to $(0,1)$ with $(0,1)$ glued to the endpoint $b_0 \in B_0(n)$.

Observe $S_0(n)$ is not a subcontinuum of $X(0)$. However it can still be used in place of $[p_0^n, a_0]$ or $R_0(n)$ to define the subcontinua $B(k) \subset X(1)$ for $0 < k \leq 1/5$. The bonding map projects each copy of $[b_{n-1}, b_n]$ to the corresponding subset of $F(0)$.

We predict the resulting inverse system has each $F(\alpha) \to F(\beta)$ and $D(\alpha) \to D(\beta)$ for $\beta \leq \alpha$. Moreover each point of $D(\beta) \subset F(\beta)$ is the image of a unique point of $X(\alpha)$. Namely the corresponding point of $D(\alpha)$. It follows each point of $\varprojlim D(\alpha)$ does not weakly cut its composant. Similar arguments to Chapter 4 should prove the strong non-cut points are exhausted by $\varprojlim D(\alpha)$.

Note Conjecture 20 predicts $\varprojlim D(\alpha)$ is exactly the set of strong non-cut points. This is stronger than Conjecture 17 which only says the set of strong non-cut points contains $\varprojlim M(\alpha)$. We imagine the full proof would apply equally well to the points $x \in \varprojlim H(\alpha)$ with $x_0 \in H(0) - \bigcup_n R_0(n)$. Note such points are uniquely determined by the first entry.

Another conjecture that might be solved through a straightforward modification of the methods of Chapter 4 is the following.

Conjecture 21. For each metric continuum M there is a Bellamy continuum X and embedding $f : M \to X$ with $\big[a, b, c\big]_M \iff \big[f(a), f(b), f(c)\big]_X$ for all $a, b, c \in M$.

The strategy here is to take a spiral T over M with endpoint $c \in T$. Let $T(0)$ be a copy of T with $M(0) \subset T(0)$ corrsponding to $M \subset T$. Like before glue c to the point $(0, 1) \in Q \cup R$ and attach arcs $A(x)$ as appropriate.

Choose any long end $(q_0^n) \subset X(0) - T(0)$ and choose the short end $(p_0^n) \subset T(0)$ with $\bigcup \big\{[p_0^n, c_0] : n \in \mathbb{N}\big\}$ equal to the open ray $T(0) - M(0)$.

We can use $[p_0^n, a_0]$ to define the subcontinua $B(k) \subset X(1)$ for $0 < k \le 1/5$ and proceed by induction. We predict each $T(\alpha) \to T(\beta)$ and the restriction $f^\alpha_\beta : M(\alpha) \to M(\beta)$ is an homeomorphism and thus we have an embedding $M \to \varprojlim M(\alpha)$ with the required property.

Separable Bellamy Continua as Remainders

To date all known Bellamy continua arise either from inverse limits or as wave remainders. Chapter 5 constructs a separable Bellamy continuum using inverse limits. The problem is open whether we can do the same thing using wave remainders. That is to say assume NCF and constuct a separable Bellamy continuum that is an image of $\mathbb{H}^*$ under some *structure-preserving* map.

Problem 22 (NCF)**.** Choose a suitable notion of *structure-preserving* maps. Find or show there exists no separable Bellamy continuum X and structure-preserving $f : \mathbb{H}^* \to X$.

In fact the example $\widetilde{X}$ from Chapter 5 is already an image of $\mathbb{H}^*$. To see this first observe since $\widetilde{X}$ is non-metric the weight is at least $\aleph_1$. Since $\widetilde{X}$ is a quotient of an $\aleph_1$-chain of metric continua the weight is at most $\aleph_0 \times \aleph_1 = \aleph_1$. We conclude $\widetilde{X}$ has weight $\aleph_1$ hence by [23] is a continuous image of $\mathbb{H}^*$.

This result is unsatisfying for two reasons. First it holds for $\widetilde{X}$ replaced by any continuum of weight $\aleph_1$ or less so doesn't suggest $\mathbb{H}^*$ and $\widetilde{X}$ have very much in common. Second it is independent of NCF.

One notion of *structure-preserving* maps makes sense only when X is the remainder of some compactification $\gamma\mathbb{H}$. Then the identity map on $\mathbb{H}$ has Stone-Čech lift $F : \beta\mathbb{H} \to \gamma\mathbb{H}$ which restricts to a surjection $f : \mathbb{H}^* \to X$. Below we list three more *structure-preserving* notions from strongest to weakest.

Definition 23. The map $f : Y \to X$ is called *monotone* to mean $f^{-1}(K) \subset Y$ is a continuum for each continuum $K \subset X$.

Definition 24. The map $f : Y \to X$ is called *confluent* to mean each component of $f^{-1}(K)$ maps onto K for each continuum $K \subset X$.

Definition 25. The map $f : Y \to X$ is called *weakly confluent* to mean some component of $f^{-1}(K)$ maps onto K for each continuum $K \subset X$.

We can give a negative answer to Problem 22 for monotone maps, assuming the following result.

Conjecture 26. Suppose $\mathbb{I}^1_{\mathcal{F}} \subset \mathbb{I}^2_{\mathcal{F}} \subset \ldots$ is a chain of standard subcontinua. Some standard subcontinuum $\mathbb{I}_{\mathcal{F}}$ contains the union of the chain.

While acting as examiner for this thesis K.P Hart proposed that the continuum K_{10} from their paper [22] could be used to construct a counterexample to the above conjecture.

Suppose $f : \mathbb{H}^* \to X$ is monotone onto a separable indecomposable continuum. It follows from Zorn's lemma there is a subcontinuum $K \subset \mathbb{H}^*$ with the restriction $f : K \to X$ being a proper map. That means it maps proper subcontinua to proper subcontinua. Hence f induces a bijection between the composants of X and K. It is not hard to show K is indecomposable. For X a Bellamy continuum this raises the following question.

Question 27. Does $\mathbb{H}^*$ contain any Bellamy proper subcontinua?

We imagine the answer depends strongly on the choice of set theory. The simplest indecomposable subcontinua of $\mathbb{H}^*$ are the layers of standard subcontinua. Layers having one composant can be characterised in terms of ω^* and the ordering of shapes.

Lemma 28. For each standard subcontinuum $\mathbb{I}_{\mathcal{F}} \subset \mathbb{H}^*$ and layer $L \subset \mathbb{I}_{\mathcal{F}}$ the following are equivalent.

(1) L is a Bellamy continuum.

(2) For each standard subcontinua $\mathbb{J}_{\mathcal{D}}, \mathbb{K}_{\mathcal{E}} \subset L$ there are finite-to-one monotone maps $f, g : \omega \to \omega$ with $f(\mathcal{D}) = g(\mathcal{E}) > \mathcal{F}$.

This resembles a *local version* of Near Coherence of Filters. Perhaps an easier problem is the following.

Question 29. Let $\mathcal{F} \in \omega^*$ be fixed. Write $\mathcal{D} \sim \mathcal{E}$ to mean there exist $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ with $\mathbb{J}_\mathcal{D}$ and $\mathbb{K}_\mathcal{E}$ contained in the same layer of $\mathbb{I}_\mathcal{F}$. Does this define an equivalence relation?

Distal points

Distal points are a type of non-cut point more specialised than non-block points. The definition and terminology are due to Paul Bankston [8].

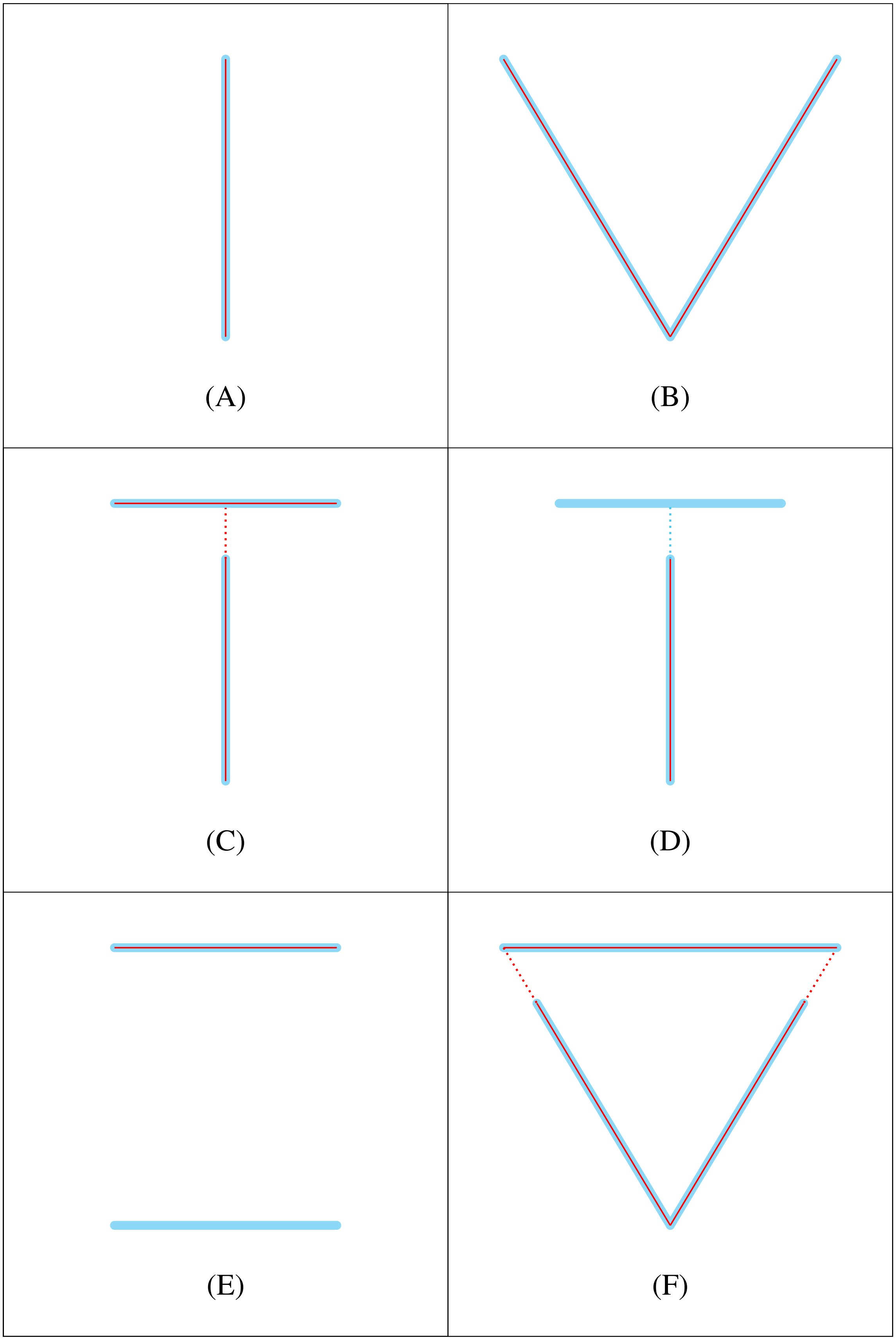
Definition 30. Each point a of the continuum X induces a preorder $\leq_a$ on X where $b \leq_a c \iff [a,b,c]$. The point c is called *a-distal* to mean it is $\leq_a$-maximal. We call $c \in X$ a *distal point* to mean it is *a-distal* for some $a \in X$.

The name suggests a-distal points are *as far as possible* from $a \in X$. This intuition does not always hold. For example the circle has each $\leq_a$ the trivial preorder. Thus no two elements are comparable and every point is distal.

Lemma 31. Distal points are non-block.

Proof. Suppose $c \in X$ is a-distal. By definition the continuum component C of a in $X - c$ is a semicontinuum with $c \notin C$. We claim C is dense and therefore c is non-block. For otherwise there exists a point $d \in X - \overline{C}$. It follows that a and d are in different continuum-components of $X - c$ and so c is an element of every subcontinuum that contains $\{a, d\}$. This shows $c \leq_a d$. Now recall $\overline{C}$ is a subcontinuum with $c \in \overline{C}$ and we chose d to have $d \notin \overline{C}$. Thus $\overline{C}$ witnesses how $d \not\leq_a c$. Taken together with $c \leq_a d$ this shows c is not maximal, contradicting the assumption that C is not dense. □

The next page shows $\leq_a$ for familiar choices of X and $a \in X$. The blue points are the elements of the preorder. For two points joined by a red line the point with higher y-coordinate is above the point with the lower y-coordinate. Two points joined by a horizontal red line are equivalent under the preorder.


(A)
(B)
(C)
(D)
(E)
(F)

(A) For $X = [0,1]$ and $a = 0$ the preorder is isomorphic to $[0,1]$ itself. The minimum and maximum elements are just 0 and 1 respectively.

(B) For $X = [0,1]$ and $a \in X$ an interior point the preorder has exactly two branches, both isomorphic to $[0,1]$. The minimum elements is a and the two maximal elements are 0 and 1.

(C) For X the sin$(1/x)$ continuum with endpoint $a \in X$ the preorder is isomorphic to the disjoint union of $[0,1)$ and $\mathfrak{c}$ many equivalent maximal elements. These correspond to the points of the limiting arc. The picture is the same for any compactification of a ray or the Knaster Buckethandle with a the unique endpoint.

(D) For X the figure 9 continuum with endpoint $a \in X$ the preorder is isomorphic to the disjoint union of $[0,1]$ and $\mathfrak{c}$ many incomparable maximal elements. These correspond to the points of the *head* that are not points of the *stick*.

(E) Let C be the circle and $R \subset [0,1] \times C$ a ray with remainder $\{0\} \times C$. Define $X = \big(\{0\} \times C\big) \cup \big\{x \in [0,1] \times C : d(x,R) \leq d(x,C)\big\}$ as the *ribbon tending to a circle*. For $a \in X$ in the ribbon every two points of the remainder are equivalent and no point is comparable with a point of the ribbon.

(F) For X the solenoid and $a \in X$ the preorder has two branches isomorphic to $[0,1)$ and $\mathfrak{c}$ many equivalent maximal elements. These correspond to the points of the composants other than $\kappa(a)$.

Distal points are always non-block but the converse can fail. For example let X be the harmonic fan with vertex r and endpoints $p_1, p_2, \ldots, p$. Let $|pr|$ be the unique arc from p to r. The union $S = |p_1 r| \cup |p_2 r| \cup \ldots$ is a dense proper semicontinuum and therefore each $x \in |pr| - r$ is a non-block point. To see x is never distal write $X = S \cup |rx| \cup |px|$ and observe that $x <_y p$ for all $y \in S \cup |rx|$ and $x <_y r$ for all $y \in |px|$.

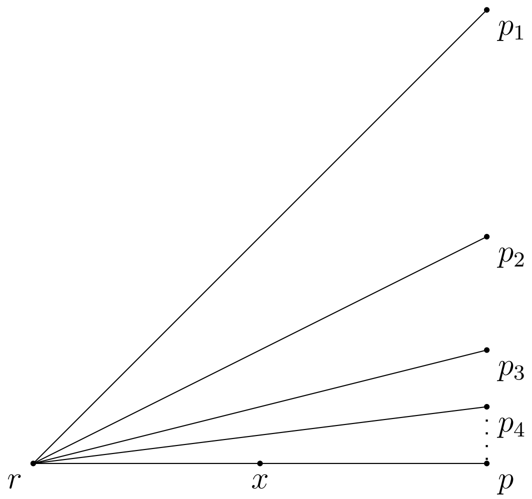


Figure 6.2: The point x is a non-block but not a distal point.

Bing's Theorem 5 of [12] guarantees each metric continuum has two or more non-block points. The problem is open can the theorem be sharpened to find distal points. The metric assumption is necessary since it is consistent the Hausdorff continuum $\mathbb{H}^*$ has no non-block points.

Conjecture 32. Suppose X is metric and $a \in X$. Some $p \in X - a$ is a-distal.

The result would immediately imply X has two distinct distal points. For once we find p we simply apply the result again with $a = p$ to find a second distal point.

The result would also sharpen Theorem 2.7 of Bobok et al [16] which says X is irreducible about the set of non-block points. Theorem 7.4 part (ii) of [8] even says X would be irreducible about $\{a\} \cup \{p \in X : p \text{ is } a\text{-distal}\}$.

The following is a stronger version of Conjecture 32. Despite the stronger conclusion we are unable to find a counterexample.

Conjecture 33. Suppose X is metric and $a \in X$. For each $b \in X - a$ some a-distal $c \in X$ has $b \leq_a c$.

Collapsing the Hierarchy

Bobok, Pyrih and Vejnar [16] have assigned the special types of non-cut point to a hierarchy with properties of higher index implying lower ones.

($P6$) Co-locally Connected Points
($P5$) Strong Non-Cut Points
($P4$) Non-Block Points
($P3$) Shore Points
($P2$) Non Strong Centers
($P1$) Non-Cut Points

We have already mentioned the classes ($P5$), ($P4$), ($P3$) and ($P1$). The remaining two classes are defined below.

Definition 34. ($P6$) We call the point $p \in X$ *co-locally connected* to mean each open neighborhood of p contains a further open neighborhood of p with connected complement.

Definition 35. ($P2$) We call the point $p \in X$ a *strong center* to mean there are disjoint open sets $U, V \subset X$ with $[u, p, v]$ for each $u \in U$ and $v \in V$.

For a strong non-cut point that is not co-locally connected see the continuum in Figure 6.3. This example and the following are based on examples from [16].

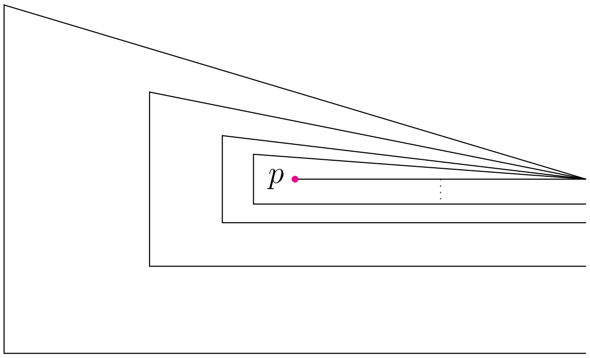


Figure 6.3: The point p has class ($P5$) but not ($P6$)

For a point of class $(P2)$ but not $(P3)$ consider the continuum in Figure 6.4. Let C be the Cantor set and $F \subset \mathbb{R}^2$ the fan over C with vertex $p \in F$. Choose a countable open basis $\mathcal{W} = \{W_1, W_2, \dots\}$ for C and choose an enumeration $\big\{(U_1, V_1), (U_2, V_2), \dots\big\}$ of $\mathcal{W} \times \mathcal{W}$. Choose distinct $a_1 \in U_1$ and $b_1 \in V_1$ and let the arc $A_1 \subset \mathbb{R}^2 \times [0,1]$ meet F only at the endpoints a_1 and b_1. Now choose $a_2 \in U_2$ and $b_2 \in V_2$ with a_1, b_1, a_2, b_2 distinct and let the arc $A_2 \subset \mathbb{R}^2 \times [0, 1/2]$ be disjoint from A_1 and meet F only at the endpoints a_2 and b_2. By induction define the arcs $A_3, A_4, \dots$ and continuum $X = F \cup A_1 \cup A_2 \cup \dots$. It follows from construction $p \in X$ has class $(P2)$ but not $(P3)$.

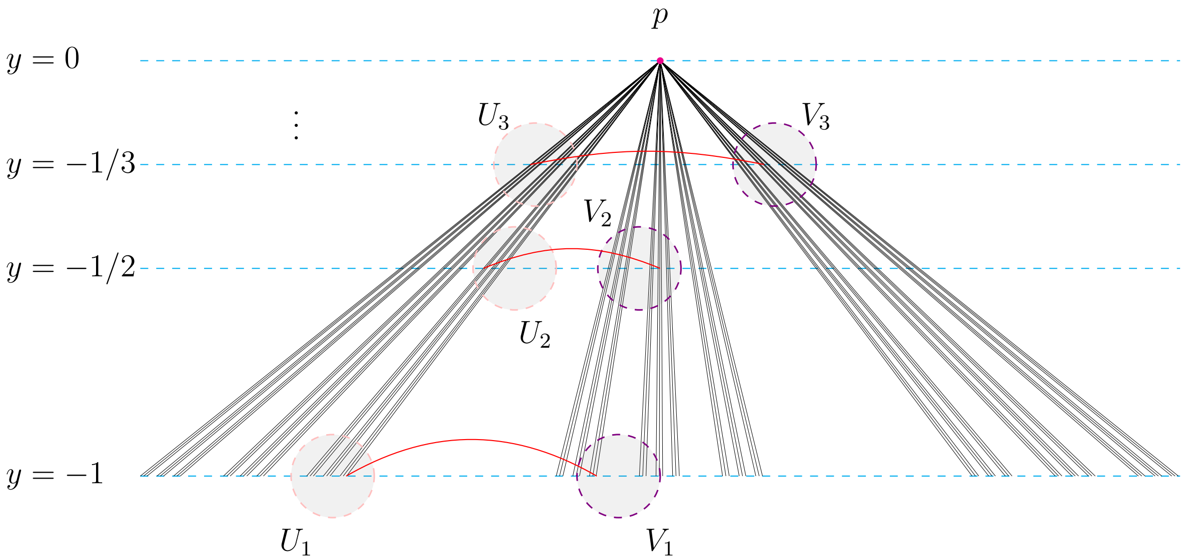


Figure 6.4: The point p has class $(P2)$ but not $(P3)$. The arcs $A_1, A_2, \dots$ are shown in red.

In general $(P6) \implies (P5) \implies \dots \implies (P1)$ for any given point. This suggests a number of classification problems.

Problem 36. Let $I, J \in \{1, \dots 6\}$ be fixed. Give a global description of the class of continua with $(PI) = (PJ)$.

Bobok et al show $(P6) = (P1)$ for X aposyndetic. Our main interest is characterising $(P5) = (P1)$ which means all weak cut points are cut points. For the weaker conclusion the proof of [16] Proposition 2.2 still works if we replace aposyndesis with the following.

Definition 37. We call X *semi-aposyndetic* at $x \in X$ with respect to $p \in X$ to mean $x \in S^\circ \subset S \subset X - p$ for some semicontinuum S. We call X *semi-aposyndetic from* p to mean semi-aposyndetic at each $x \in X$ with respect to p. We call X *semi-aposyndetic* to mean semi-aposyndetic from each $p \in X$.

Theorem 38. Suppose X is semi-aposyndetic from $p \in X$. The continuum components of $X - p$ are clopen and agree with the quasicomponents. Hence p has class $(P1)$ if and only if it has class $(P5)$. In particular $(P5) = (P1)$ for X semi-aposyndetic.

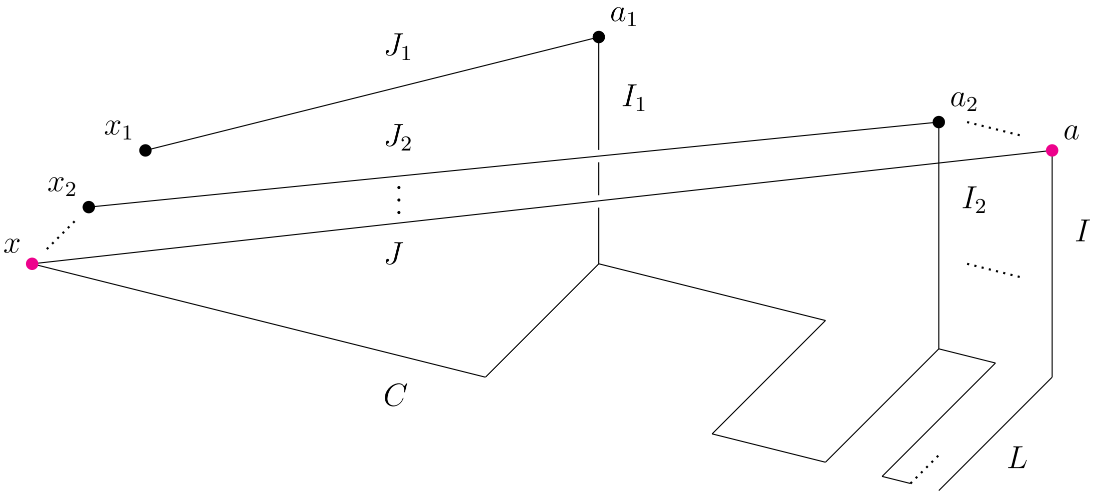


Figure 6.5: X is semi-aposyndetic but not aposyndetic from a.

To see Theorem 38 is indeed an improvement consider the continuum X in Figure 6.5. To form X begin with a $\sin(1/x)$ continuum on the xy plane with curve C and limiting arc L. Attach vertical unit arcs $I_1, I_2, \ldots$ to each peak of C and a vertical unit arc I to the limit of the peaks as shown. Let x be the endpoint of C. For each $n > 1$ select a point x_n on in the xy plane outside $C \cup L$ and distance $1/n$ from x. Let each a_n be the free endpoint of I_n and a the free endpoint of I. Attach pairwise disjoint arcs J_n from a_n to x_n and J from a to x as shown.

It is straightforward to see $a \in X$ is a non-cut point $(P1)$. We claim X is semi-aposyndetic but not aposyndetic from a while at the same time a has class $(P5)$ but not class $(P6)$.

First observe $X - a$ is an open semicontinuum. This simultaneously shows X is semi-aposyndetic from a and shows a is a strong non-cut point $(P5)$. To see aposyndicity fails observe each subcontinuum $K \subset X$ with x in the interior contains some tail of $x_1, x_2, \ldots$ and hence contains the corresponding tail of $a_1, a_2, \ldots$ which implies $a \in K$ by closedness. To see a is not co-locally connected $(P6)$ observe each open neighborhood U of a contains some tail of $a_1, a_2, \ldots$. It follows the corresponding closed sets $J_1 - U, J_2 - U, \ldots$ are components of $X - U$.

While being semi-aposyndetic is a sufficient condition for $(P5) = (P1)$ it is too strong to characterise such continua. For example let X be the *Cantor earring*. That means the union of circles in $\mathbb{R}^2$ with leftmost point at $p = (0, 0)$ and rightmost point in the Cantor set constructed on $[1, 2] \times \{0\}$. The continuum X is semi-aposyndetic at no x with respect to p but nevertheless has $(P1) = (P6) = X - p$.

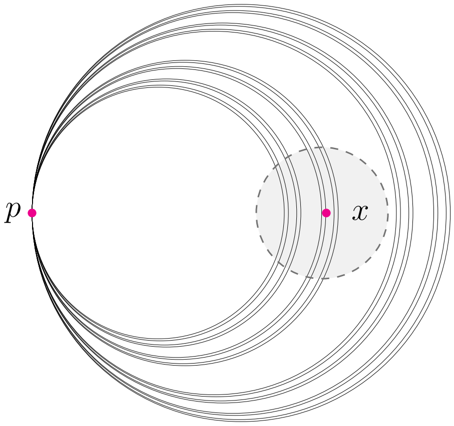


Figure 6.6: The *Cantor earring* is not semi-aposyndetic at x with respect to p

The difference here is the continuum is *busy* in a way that kills off most of the cut points but does not add aposyndicity. In fact the Cantor earring is *trivial* for the more general reason of being a *suspension*.

Definition 39. Suppose the compactum Z has basepoint $p \in Z$. The *suspension* ΣZ is the quotient space of $Z \times [-1, 1]$ where we treat the closed set

$\big(\{p\} \times [-1,1]\big) \cup \big(Z \times \{-1,1\}\big)$ as a point. Write $P \in \Sigma Z$ for that point and identify each $x \in Z$ with the point $(x,0) \in \Sigma Z$.

Example 40. Obtain the compactum Z by adding the unique isolated point p to the Cantor set. Then ΣZ is the Cantor earring.

There is no trouble to define aposyndesis for compacta rather than just continua (though for totally disconnected spaces we must be careful to observe no point is aposyndetic with respect to any other point). Likewise we can define the P-hierarchy for the points of a compactum (though we lose intuition for how lower class points make larger contributions to the global shape of the space).

For points x, y of the compactum Z write $(x,y)_Z$ to mean Z is aposyndetic at x with respect to y. The following properties of the suspension are straightforward to verify.

(1) $\Sigma Z - P$ is homeomorphic to $(Z - p) \times (-1,1)$.

(2) Each $x \in \Sigma Z - P$ is co-locally connected $(P6)$.

(3) $(x,y)_X \implies (x,y)_{\Sigma Z}$ for each $x, y \in Z$.

(4) $(x,p)_X \iff (x,P)_{\Sigma Z}$ for each $x \in Z - p$.

Property (1) says the continuum components of $\Sigma Z - P$ are the sets $B \times (-1,1)$ for B the continuum components of $Z - p$. It follows from the definitions of the various types of non-cut points that $p \in Z$ has class (PI) exactly if $P \in \Sigma Z$ has class (PI). Together with Property (2) this shows taking suspensions preserves the class of continua with $(PI) = (PJ)$ while *absorbing* much of the interesting structure.

If this behaviour is considered trivial we need an internal way to check a given continuum is something like a suspension. One observation for the Cantor earring is the failure of aposyndesis is witnessed by the sort of subcontinuum shown in Figure 6.7.

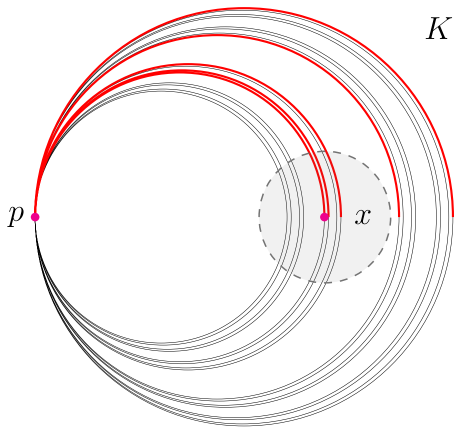


Figure 6.7: The subcontinuum K witnesses how $\neg(x,p)_X$

The subcontinuum K is a copy of the harmonic fan. Since $\neg(x,p)_X$ and $\neg(x,p)_K$ the subcontinuum K witnesses the non-aposyndesis. On the other hand K fails the witness the place in the hierarchy of some points of X. For example let $y \in K - x$ be an element of the limiting arc. Then $y \in K$ is weak cut point but $y \in X$ is a strong non-cut point. This motivates the definitions.

Definition 41. We say the embedding $K \to X$ *respects the hierarchy below* N to mean each $x \in K$ has class (PI) in K exactly if x has class (PI) in X for each $I \in \{1, \dots, N\}$.

Definition 42. Suppose $\neg(x,p)_X$ for fixed $x, p \in X$. Suppose moreover for each subcontinuum $K \subset X$ with $\neg(x,p)_K$ the embedding $K \to X$ respects the hierarchy below N. Then we call the pair (x,p) *N-nontrivial*. Suppose for all $a, b \in X$ with $\neg(a,b)_X$ the pair (a,b) is N-nontrivial. Then we call X *N-nontrivial*. Otherwise X is called *N-trivial*.

In fact the Cantor earring does not fully demonstrate the purpose of Definition 42. The definition is intended to take a suspension ΣX as input and take K as the embedded copy of X in ΣX to recognise the space is trivial.

For example let X be the harmonic fan with vertex p and ΣX the suspension (shown in Figure 6.8). Choose $x \neq p$ in the limiting arc. Since $\neg(x,p)_X$ Property (4) says $\neg(x,P)_{\Sigma X}$. It is easy to see $(x,P)_K$ for K embedded copy of X in ΣX.

Observe x weakly cuts K while Property (2) says x does not weakly cut ΣX. We conclude ΣX is 6-trivial.

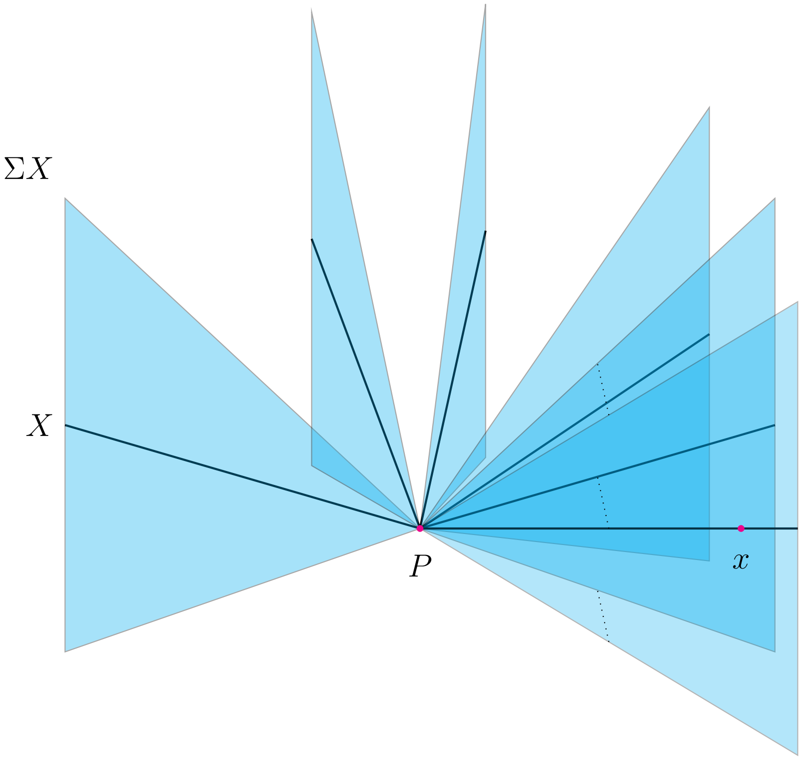


Figure 6.8: Suspension of the harmonic fan X with vertex p.

The new definitions allow us to ask more nuanced questions related to Problem 36.

Question 43. Is every 6-nontrivial continuum that has $(P6) = (P1)$ aposyndetic?

Question 44. Is every 5-nontrivial continuum that has $(P5) = (P1)$ semi-aposyndetic?

Bases for Continua

We say the continuum X is *spanned* by the subset $B \subset X$ to mean no proper subcontinuum of X contains B. Moreover if no proper subset of B spans X we call B a *basis* off X.

The simple triod is an example of a space with a finite basis. For a space with an infinite basis consider the harmonic fan with vertex r and endpoints $p_1, p_2, \ldots, p$. Then $\{p_1, p_2, \ldots\}$ is the unique basis for the space.

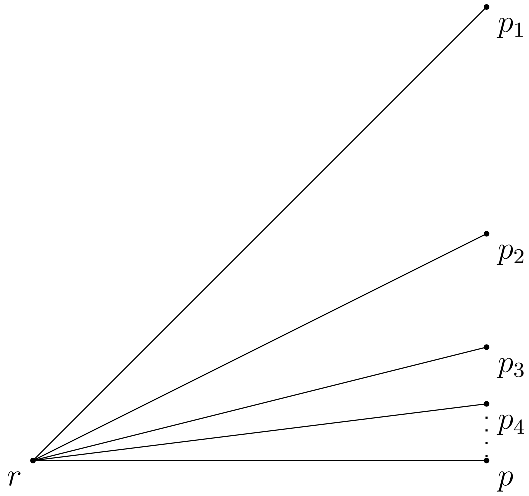


Figure 6.9: $\{p_1, p_2, \ldots\}$ is an infinite basis.

The topic of bases for continua came to attention from the paper of Leonel [33] which has since become the impetus for this project. Leonel proves for X irreducible about the finite set $B = \{b_1, b_2, \ldots, b_n\}$ that all b_m are shore points. There is an error in the proof as can be seen taking $X = [0, 1]$ and $B = \{0, 1/2, 1\}$. To patch up the result we require B be a basis. In that case we can show more.

Lemma 45. Suppose $B \subset X$ is a basis and $a \in B$. Each $b \in B - a$ is a-distal.

Proof. It follows from B being a basis $B - b \subset K \subset X - b$ for some subcontinuum $K \subset X$. Suppose b fails to be $\le_a$-maximal. That means there is a point $c \in X$ with $b <_a c$. Since $b \notin K$ we have $c \notin K$. By definition $b <_a c$ means some subcontinuum $L \subset X$ has $\{a,b\} \subset L \subset X-c$. Thus $B \subset L \cup K \subset X-c$ contrary to the assumption that B spans X. We conclude b is a-distal as required. $\square$

Lemma 31 gives the following

Corollary 46. Elements of any basis are non block points.

Similar arguments solve the next problem for finite bases. For the infinite case the problem is open.

Conjecture 47. Suppose $B \subset X$ is a basis and $a \in X$ not an element of any basis. Each $b \in B$ is a-distal.

Having a basis is a very strong condition for X. For example the next lemma implies the circle has no basis.

Lemma 48. Suppose $B \subset X$ is a basis. Then B is discrete.

Proof. Suppose to the contrary that $b \in B$ is a limit point. We claim X is spanned by $B-b$ and hence B is not a basis. For suppose K is a subcontinuum containing $B-b$. Since K is closed it contains its limit points and hence the limit points of $B-b$. We conclude $b \in K$. This implies $B \subset K$ and hence $K = X$. But since K was arbitrary this shows $B-b$ spans X as required. $\square$

Suppose D is a discrete subset of the circle. Then D is not dense so its closure is a proper closed subset. Since each proper closed subset of the circle is contained in a proper subcontinuum we see there is no basis.

For a space with many distinct bases consider X the *topologist's skipping rope* obtained by gluing together two $\sin(1/x)$ continua at the endpoints. For any choice of a in the left limiting spine and b in the right it is easy to see $\{a,b\}$ is a basis. Any other basis $\{a',b'\}$ is *the same* in that $X-\kappa(a)$ and $X-\kappa(b)$ each include exactly one element of $\{a',b'\}$.

Recently David Harper [28] has generalised this fact in his doctoral thesis.

Definition 49 (Harper)**.** Suppose $B = \{b_1, b_2, \ldots, b_n\}$ is a basis for the metric continuum X. For each $m \leq n$ obtain the continuum X_n by collapsing $B - b_m$ to a point. Call that point b^m. Define the subset $\Lambda(m) = X_n - \kappa(b^m)$ and let $\lambda(m)$ be the corresponding subset of X.

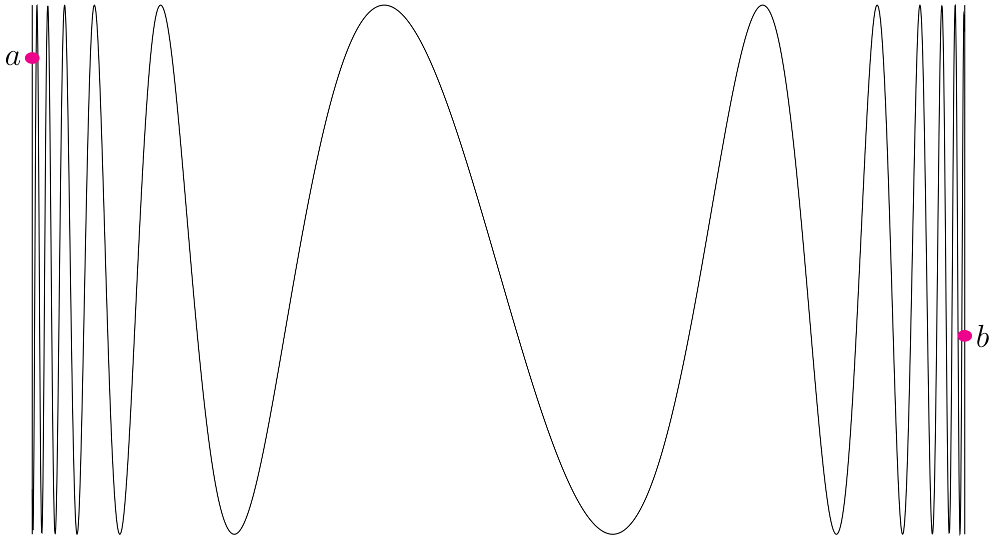


Figure 6.10: Any such pair $\{a, b\}$ is a basis for the *topologist's skipping rope*

Each $\lambda(m)$ is nonempty because B is a basis. For the metric case $\lambda(m)$ is connected by [43] Theorem 11.4. That proof includes a subtle application of hereditary normality of a metric space hence does not generalise to Hausdorff continua. However the result still holds using how $p \in X$ is in the closure of each continuum component of $X - p$.

Theorem 50 (Harper Corollary 3.2.14)**.** Suppose $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are bases for the metric continuum X. Then $n = m$. Moreover $\lambda(r)$ coincide when defined for each basis and each $\lambda(r)$ includes exactly one element of each basis.

Harper's method generalises to show any two bases $B = \{b_1, b_2, \ldots, b_n\}$ and A for X have $|A| = n$. Thus no continuum admits both a finite and an infinite basis. Since compact metric spaces are second countable every discrete subset hence every basis has cardinality $\aleph_0$ or less. This proves metric continua have a well-defined *dimension*.

Theorem 51. (Corollary to Harper) Suppose X is a metric continuum. Exactly one of the following holds.

(1) X admits no basis.

(2) Each basis $B \subset X$ has $|B| = n$ for some fixed $n \in \mathbb{N}$.

(3) Each basis $B \subset X$ has $|B| = \aleph_0$.

While Hausdorff continua can fail to be second countable, we still know the cardinality of a basis cannnot exceed the weight of the space. Lemma 52 shows this condition is sharp.

The construction is just a *high cardinality* version of the following process: First embed the given continuum X as the remainder of a $\sin(1/x)$ type continuum. Then arrange a harmonic fan of countably many copies of that continuum joined at the endpoint.

Lemma 52. Let α be the weight of the continuum X. For any cardinal $\beta \leq \alpha$ there exists an embedding of X into a continuum Y with weight α and a basis of size β.

Proof. Let $D \subset X$ be a dense subset of minimal cardinality. Clearly $|D| \leq \alpha$. Without loss of generality D is an ordinal. Write $\mathcal{S}$ and $\mathcal{L}$ respectively for the set of successor and limit ordinals below D. Once we have shown $|\mathcal{S}| = |D|$ we can choose a bijection $D \to \mathcal{S}$ and get a dense subset $\{d_\gamma : \gamma \in \mathcal{S}\} \subset X$.

To that end write $D = \mathcal{S} \cup \mathcal{L}$ and observe $|D| = |\mathcal{S}| + |\mathcal{L}| = \max\{|\mathcal{S}|, |\mathcal{L}|\}$. Thus we need only show $|\mathcal{L}| \leq |\mathcal{S}|$. The inequality is witnessed by the injective map $\mathcal{L} \to \mathcal{S}$ given by $\gamma \mapsto \gamma + 1$.

Let $\mathbb{I}$ be the long arc over D with endpoints 0 and ∞. Identify X with the subset $X \times \{\infty\}$ of $X \times \mathbb{I}$. For each $\gamma \in \mathcal{S}$ choose an irreducible subcontinuum

$I(\gamma) \subset X$ from d_0 to d_γ. Let each $J(\gamma) \subset X \times \mathbb{I}$ be the subcontinuum of $X \times \{0_\gamma\}$ corresponding to $I(\gamma)$. Write $Z = \left(\bigcup\{J(\gamma) : \gamma \in \mathcal{S}\}\right) \cup X$. Taking the closure of Z amounts to adding a closed subset of each $X \times \{0_\lambda\}$ for $\lambda \in \mathcal{L}$.

The parity of an ordinal is defined by induction. The ordinal 0 is even; the successor of any even(odd) ordinal is odd(even); and limit ordinals are even. By gluing the odd and even $J(d)$ at the endpoints similar to the constructions in Chapter 5 we turn the closure of Z into a continuum which we denote by F. Clearly F has weight not exceeding $\alpha^2 = \alpha$ and is irreducible from x to y for any $x \in X$ and $y = (d_0, 0) \in X \times \{0\}$.

Let B be a discrete space of cardinality β and ωB the one-point compactification. To obtain Y from $F \times \omega B$ we treat the set $\{(y, b) : b \in \omega B\}$ as a single point. Clearly Y has weight equal to $\alpha \times \beta = \alpha$ and admits the basis $\{(x, b) : b \in B\}$ of size β. $\square$

Corollary 53. For any cardinals $\beta \leq \alpha$ there is a continuum of weight α with a basis of size β.

The problem is open whether Hausdorff continua have a well-defined *dimension*.

Question 54. Suppose the Hausdorff continuum X has infinite bases A and B. Does it follow that $|A| = |B|$?

It is slightly dangerous to refer to the cardinality of a continuum's smallest basis as that continuum's dimension – because the *dimension* of a space already has several meanings in topology. For example the small and large inductive dimensions and the Lebesgue covering dimension as defined and studied in [24].

All the examples in this section – simple triod, harmonic fan, and topologist's skipping rope – are 1-dimensional with respect to the inductive and Lebesgue dimensions. Indeed the three notions coincide for compact metric spaces. These examples were chosen only for convenience and simplicity. For a continuum with basis dimension 2 but topological dimension n consider a spiral over the n-cell.

These spirals are unsatisfactory examples because they are n-dimensional only over the n-cell. They are 1-dimensional at every other point. We would like examples that are *globally* n-dimensional. To make this precise we introduce the definitions.

Definition 55. Suppose $\mathcal{U}$ is an open cover of the topological space T. We call the open cover $\mathcal{V}$ of T a *refinement* of $\mathcal{U}$ to mean each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.

Definition 56. For the topological space T we write $\text{dim}(T) \leq n$ to mean each open cover $\mathcal{U}$ of T admits a refinement $\mathcal{V}$ with each $x \in T$ contained in at most $n+1$ distinct elements of $\mathcal{V}$. The *Lebesgue covering dimension* of T is the number $\text{dim}(T) = \min\{n \in \mathbb{N} : \text{dim}(T) \leq n\}$.

Definition 57. For the topological space T and $x \in T$ we write $\text{dim}(T, x) \leq n$ to mean each open neighborhood of x contains a closed neighborhood B of x with $\text{dim}(B) \leq n$. The *local dimension* of X at x is the number $\text{dim}(T, x) = \min\{n \in \mathbb{N} : \text{dim}(T, x) \leq n\}$.

With the new terminology we can give an example of a continuum with basis size 2 and local dimension 2 at each point. Since the proof is quite involved and off-topic we only sketch the details. It is a special case of the *resolution* construction invented by Fedorchuk [26] and formalised by Watson [60].

The example should be thought of as a generalisation of the $\sin(1/x)$ continuum, which arises by taking the arc $[0, 1]$, replacing the endpoint 1 with a vertical segment, and wrapping the leftover $[0, 1)$ around that vertical segment. In our example we simultaneously replace every point of $[0, 1]$ with a unit square. This amounts to equipping the cube $[0, 1]^3$ with an unusual topology.

Example 58. Write $I = [0, 1]$ and for each $p \in I$ choose a continuous map $f_p : (I - p) \to I^2$ with $f_p(U - p) = I^2$ for each open neighborhood U of p. For ease of notation write for example $p \times U$ instead of $\{p\} \times U$.

We define each each $(p,q) \in I \times I^2$ to have neighborhood basis of sets

$$U \otimes_p V = \big(p \times U\big) \cup \Big(\bigcup \{w \times I^2 : f_p(w) \in U \text{ and } w \in V\}\Big)$$

as $U \subset I^2$ ranges over all open neighborhoods of q and V ranges over all open subsets of I.

See Theorem 3.1.33 of [60] for the proof the resulting space X is compact Hausdorff. Taking $U = I^2$ we see the projection $\pi : X \to I$ onto the first coordinate is continuous. It is straightforward to show each fibre of π is homeomorphic to $[0,1]$. In particular the fibres are connected and π is monotone. This implies X is a continuum.

The proof each $p \times I^2$ is a rung – that means each subcontinuum of X is either disjoint from, contained in, or contains it – is similar to the proof for the limiting arc of the $\sin(1/x)$ continuum. It follows X is irreducible from each point of $0 \times I^2$ to each point of $1 \times I^2$. Hence there is a basis of size 2.

We first show dim$(X) = 2$. To that end suppose we have an open cover

$$U_1 \otimes_{p(1)} V_1 \ , \ U_2 \otimes_{p(2)} V_2 \ , \ \ldots , \ U_n \otimes_{p(n)} V_n$$

For simplicity assume all $p(i)$ are distinct. By reindexing if necessary we can have $p(1) < p(2) < \ldots < p(n)$. There are numbers $0 = q_0 < q_1 < \ldots < q_n = 1$ with each $p(i) \in (q_{i-1}, q_i)$. Let $\varphi : X \to \widetilde{X}$ be the quotient map that collapses the set $p \times I$ to a point for each $p \neq p(1), p(2), \ldots, p(n)$. Observe the sets $U_i \otimes_{p(i)} V_i$ are φ-saturated. That means they are unions of fibres of φ. It follows the sets $\varphi\big(U_i \otimes_{p(i)} V_i\big) \subset \widetilde{X}$ are open.

It can be shown each $\varphi\big([q_{i-1}, q_i] \times I^2\big)$ is homeomorphic to the subspace

$$X_i = \Big\{\big(x, f_{p(i)}(x)\big) : x \in [q_{i-1}, q_i] - p\Big\} \cup \big(p \times I^2\big)$$

of $[q_{i-1}, q_i] \times I$ under the product topology. It can also be shown X_i is homeomorphic to a spiral over the 2-cell hence is 2-dimensional. Then

Theorem 3.1.4 and Proposition 3.1.7 of [24] say the union $X_1 \cup \ldots \cup X_n = \widetilde{X}$ is also 2-dimensional.

Hence the open cover $\{\varphi\big(U_i \otimes_{p(i)} V_i\big) : i \leq n\}$ of $\widetilde{X}$ has a refinement where no four distinct elements intersect. Pulling back this refinement to X we see dim$X \leq 2$. To see dim$X = 2$ observe each $U \otimes_p V$ contains some copy $q \times I^2$ of the unit square. Since $q \times I^2$ is 2-dimensional Theorem 3.1.4 of [24] says $2 \leq$ dim$X$.

Finally we observe each open set $U \otimes_p V$ contains some closed set $[a, b] \times I$ which is 2-dimensional by the same proof as for X. Since each closed subset of X has dimension no more than dim$X = 2$ we conclude X has local dimension exactly 2.

We close by remarking the example above is non metric. To see this fix distinct $x, y \in I^2$ and observe for a basis to separate the points (p, x) and (p, y) it must contain a set of the form $U \otimes_p V$. But since there are uncountably many choices for p the basis must be uncountable.

However X is both separable and first countable. To see separability take countable dense subsets $\{a_1, a_2, \ldots\} \subset I$ and $\{b_1, b_2, \ldots\} \subset I^2$ and consider the countable subset $\{(a_i, b_j) : i, j \in \mathbb{N}\}$ of X. To see first countability consider the neighborhood basis $B(q, 1/n) \otimes_p B(p, 1/n)$ of the point $(p, q) \in X$.

The fact that X is non metric leaves many open problems about dimensions of irreducible continua. For example

Question 59. What are the local inductive dimensions of the example above? We define the local large and small inductive dimensions by replacing the Lebesgue covering dimension dim in Definition 57 with the small and large inductive dimension repectively.

Question 60. Does there exist an irreducible metric continuum with topological dimensions equal to 2?

Question 61. How do the various notions of dimension behave in general under taking resolutions of metric continua?

One good place to start looking for answers might be the paper [37] which is about mappings of continua onto the arc where each fibre is a rung.

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