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# ON THE QUIVER PRESENTATION OF THE DESCENT ALGEBRA OF THE SYMMETRIC GROUP

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ABSTRACT. We describe a presentation of the descent algebra of the symmetric group  $\mathfrak{S}_n$  as a quiver with relations. This presentation arises from a new construction of the descent algebra as a homomorphic image of an algebra of forests of binary trees, which can be identified with a subspace of the free Lie algebra. In this setting we provide a short new proof of the known fact that the quiver of the descent algebra of  $\mathfrak{S}_n$  is given by restricted partition refinement. Moreover, we describe certain families of relations and conjecture that for fixed  $n \in \mathbb{N}$  the finite set of relations from these families that are relevant for the descent algebra of  $\mathfrak{S}_n$  generates the ideal of relations of an explicit quiver presentation of that algebra.

## 1. INTRODUCTION

Let  $(W, S)$  be a finite Coxeter system and let  $k$  be a field of characteristic zero. For all  $J \subseteq S$  we denote the parabolic subgroup  $\langle J \rangle$  of  $W$  by  $W_J$  and the set of minimal length left coset representatives of  $W_J$  in  $W$  by  $X_J$ . In 1976 Solomon proved [19] that the elements  $x_J = \sum_{x \in X_J} x \in kW$  with  $J \subseteq S$  satisfy

$$(1) \quad x_J x_K = \sum_{L \subseteq S} c_{JKL} x_L$$

for certain integers  $c_{JKL}$  with  $J, K, L \subseteq S$ . This implies that the linear span  $\langle x_J \mid J \subseteq S \rangle$  is a *subalgebra* of  $kW$ . This algebra is called the *descent algebra* of  $W$  and is denoted by  $\Sigma(W)$ .

Solomon shows [19] that the structure constants  $c_{JKL}$  in (1) are the same constants appearing in the Mackey formula for the product of the permutation characters  $\text{Ind}_{W_J}^W 1$  and  $\text{Ind}_{W_K}^W 1$  in terms of the characters  $\text{Ind}_{W_L}^W 1$  with  $L \subseteq S$ . Therefore the map  $\theta : \Sigma(W) \rightarrow k\text{Irr}(W)$  given by  $x_J \mapsto \text{Ind}_{W_J}^W 1$  for all  $J \subseteq S$  is a homomorphism of  $k$ -algebras, where  $k\text{Irr}(W)$  is the character ring of  $W$  over  $k$ . Solomon also shows that  $\ker \theta$  is the radical of  $\Sigma(W)$ .

We identify  $k\text{Irr}(W)$  with the ring  $k^m$  under pointwise addition and multiplication, where  $m$  is the number of conjugacy classes of  $W$ . Then the map  $\theta$  above presents the semisimple algebra  $\Sigma(W) / \text{Rad } \Sigma(W)$  as a subalgebra of  $k^m$ . Since  $k^m$  is commutative, the simple  $\Sigma(W)$ -modules are all one-dimensional over  $k$ . This means that  $\Sigma(W)$  is a basic algebra which thereby admits a quiver presentation. See [1] for more information about basic algebras and quivers. The preceding discussion also shows that we can assume that  $k$  is the field  $\mathbb{Q}$  of rational numbers, since the permutation characters  $\text{Ind}_{W_J}^W 1$  take values in  $\mathbb{Z}$ .

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The aim of this paper is to calculate and study the quiver presentation of  $\Sigma(W)$  when  $W$  is the symmetric group  $\mathfrak{S}_n$  of degree  $n \geq 0$ . An elementary proof of formula (1) in this case was given by Atkinson [2] in 1986. The Coxeter generating set of  $\mathfrak{S}_n$  is  $S = \{1, 2, \dots, n-1\}$  where we identify each  $s \in S$  with the transposition exchanging the points  $s$  and  $s+1$ . We remark that the set  $X_J$  has a description in terms of the graphs of the elements of  $\mathfrak{S}_n$ . Here we regard  $w \in \mathfrak{S}_n$  as a function from  $\{1, 2, \dots, n\}$  to itself and the graph of  $w$  as the set of points  $\{(i, i.w) \mid 1 \leq i \leq n\}$ . Then  $X_J$  is the set of all  $w \in \mathfrak{S}_n$  such that  $i.w > (i+1).w$  holds only when  $i \in J$ . In other words, the only points where the graph of  $w$  is *descending* are in  $J$ . The name *descent algebra* derives from this interpretation.

The algebra  $\Sigma(\mathfrak{S}_n)$  plays a major role in the book by Blessenohl and Schocker [10] where the authors study the character theory of  $\mathfrak{S}_n$  through an extension of the map  $\theta$  above to  $k\mathfrak{S}_n$ . As in [10], this article takes the point of view of studying  $\Sigma(\mathfrak{S}_n)$  for all  $n \geq 0$  simultaneously by uniting families of objects indexed by  $n$  into single objects. The industry of studying  $\Sigma(\mathfrak{S}_n)$  through its quiver presentation begins in 1989 with Garsia and Reutenauer's description [13] of the quiver of  $\Sigma(\mathfrak{S}_n)$ . We derive this quiver in §7 using an algebra  $k\mathcal{L}_n$  that we describe below. Garsia and Reutenauer also calculate the Cartan invariants and the projective indecomposable modules of  $\Sigma(\mathfrak{S}_n)$ . Atkinson [3] derives these using elementary methods.

Bergeron and Bergeron [5, 7] partially describe the quiver of  $\Sigma(W)$  for  $W$  of type  $B_n$  in 1992 with their calculation of the idempotents of  $\Sigma(W)$ , which correspond with the vertices of the quiver. The full quiver in type  $B_n$  was calculated by Saliola [16] in 2008 using hyperplane arrangements.

In a somewhat different direction, but amounting to essentially the same information as a quiver presentation, the *module structure* of  $\Sigma(\mathfrak{S}_n)$  was calculated [8, 9] and later expanded by Schocker [18], where he showed that articles [8] and [9] essentially calculate the quiver of  $\Sigma(\mathfrak{S}_n)$ . One component of the module structure of  $\Sigma(W)$  is the length of its Loewy series, which was calculated for  $W$  of type  $D_n$  and  $n$  odd by Saliola [17] in 2010 after the calculation by Bonnafé and Pfeiffer in 2008 [11] for the remaining finite irreducible Coxeter groups.

A first step towards the calculation of the quiver for arbitrary Coxeter groups lies in Bergeron, Bergeron, Howlett, and Taylor's calculation [6] of a basis of idempotents of  $\Sigma(W)$  for any Coxeter group  $W$ , since these idempotents serve as the vertices of the quiver. Pfeiffer's article [15] builds on the idempotent construction above and shows how one can construct the quiver and the relations for the presentation of  $\Sigma(W)$ . Since Pfeiffer's construction provides the foundation for this article, we briefly summarize it in the following theorem.

**Theorem 1.** *Let  $(W, S)$  be a finite Coxeter system. Then there exist*

- a category  $\mathcal{A}$
- an action of the free monoid  $S^*$  on  $\mathcal{A}$  that partitions  $\mathcal{A}$  into orbits
- subsets  $\Lambda$  and  $\mathcal{E}$  of the set  $\mathcal{X}$  of orbits of  $\mathcal{A}$
- a linear map  $\Delta: k\mathcal{A} \rightarrow k\mathcal{P}$  (where  $\mathcal{P}$  is the power set of  $S$ )

such that

- $k\mathcal{X}$  is a subalgebra of  $k\mathcal{A}$  (where we identify the orbit of an element of  $\mathcal{A}$  with the sum of its elements in  $k\mathcal{A}$ )
- $\Lambda$  is a complete set of pairwise orthogonal primitive idempotents of  $k\mathcal{X}$
- $\lambda(k\mathcal{X})\lambda' \cap \mathcal{X}$  is a basis of the subspace  $\lambda(k\mathcal{X})\lambda'$  for all  $\lambda, \lambda' \in \Lambda$

- the pair  $(Q, \ker \Delta)$  is a quiver presentation of  $\Sigma(W)^{\text{op}}$  where  $Q$  is the quiver with vertices  $\Lambda$  and edges  $\mathcal{E}$ .

We briefly repeat the definitions of the constructions introduced in [Theorem 1](#) needed in this article. The category

$$\mathcal{A} = \left\{ (J; s_1, s_2, \dots, s_l) \mid \{s_1, s_2, \dots, s_l\} \subseteq J \subseteq S \text{ with } s_1, s_2, \dots, s_l \text{ distinct} \right\}$$

has a partial product  $\circ : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  defined by

$$(J; s_1, s_2, \dots, s_l) \circ (K; t_1, t_2, \dots, t_m) = (J; s_1, s_2, \dots, s_l, t_1, t_2, \dots, t_m)$$

if  $K = J \setminus \{s_1, s_2, \dots, s_l\}$ . The action of  $S^*$  on  $\mathcal{A}$  is given by

$$(2) \quad (J; s_1, s_2, \dots, s_l) \cdot t = (J^\omega; s_1^\omega, s_2^\omega, \dots, s_l^\omega)$$

for  $t \in S$  and  $(J; s_1, s_2, \dots, s_l) \in \mathcal{A}$  where  $\omega = w_J w_{J \cup \{t\}}$  and where  $w_J$  and  $w_{J \cup \{t\}}$  are the longest elements of the parabolic subgroups  $W_J$  and  $W_{J \cup \{t\}}$  respectively. The superscripts in (2) denote conjugation, so for example  $s_1^\omega = \omega^{-1} s_1 \omega$ .

We define a difference operator  $\delta$  on  $k\mathcal{A}$  by  $\delta(\mathbf{a}) = \mathbf{a}$  if  $l = 0$  or by  $\delta(\mathbf{a}) = \mathbf{b} - \mathbf{b} \cdot s_l$  if  $l > 0$  for all  $\mathbf{a} = (J; s_1, s_2, \dots, s_l)$  where  $\mathbf{b} = (J \setminus \{s_l\}; s_1, \dots, s_{l-1})$ . Then  $\Delta$  is defined by iterating  $\delta$  as many times as possible, so  $\Delta(\mathbf{a}) = \delta^l(\mathbf{a})$  for  $\mathbf{a} \in \mathcal{A}$  as above. Finally,  $\Lambda$  is the set of orbits of elements of the form  $(J; )$  and  $\mathcal{E}$  can be calculated using an algorithm.

Once the quiver provided by [Theorem 1](#) has been identified, it remains to calculate the relations of the presentation. While difficult in practice, this amounts in principle only to transferring  $\ker \Delta$  to  $kQ$ . Pfeiffer [14] has done this with his explicit quiver presentations of the descent algebras of the Coxeter groups of exceptional and non-crystallographic type. Other than these calculations, no quiver presentations of descent algebras are known. However, in contrast with the finite calculations in [14], this paper deals with the calculation of presentations of the algebras in the infinite family  $\{\Sigma(\mathfrak{S}_n) \mid n \geq 0\}$ .

The following is an outline of this paper. The algebras and maps introduced in the outline are shown in the following diagram.

$$\begin{array}{ccccc} kQ_n & \xrightarrow{\iota} & k\mathcal{L}_n & \longrightarrow & kL_n \\ & & \downarrow E & & \downarrow E \\ & & kM_n & \longrightarrow & kM_n \xrightarrow{\pi} kN^* \\ & & & & \nearrow \Delta \end{array}$$

To calculate a presentation of  $\Sigma(\mathfrak{S}_n)$  we first develop a simpler description of  $\mathcal{A}$ . Namely, we show in §4 that each element of  $\mathcal{A}$  can be represented as a sequence of binary trees, or a *forest*. The category  $L_n$  in the diagram above is the category of forests corresponding with elements of  $\mathcal{A}$ . The definition and basic properties of forests are the subject of §3. We show in §5 that the monoid action of  $S^*$  on  $L_n$  amounts simply to rearrangement of the trees of a forest, so the  $S^*$ -orbit of an element of  $\mathcal{A}$  corresponds with the sum of all rearrangements of the corresponding forest. This action yields a set  $\mathcal{L}_n$  of orbit sums in  $kL_n$  corresponding with  $\mathcal{X}$  in [Theorem 1](#). We show in §6 that the map  $\Delta$  also has a simple description when we represent the elements of  $\mathcal{A}$  as forests. Namely, we introduce sets  $M_n$  and  $\mathcal{M}_n$  analogous to  $L_n$  and  $\mathcal{L}_n$  in §9 and we show in §10 that  $\Delta$  factors through  $kM_n$  as the composition of a natural map  $E : kL_n \rightarrow kM_n$  with the map  $\pi : kM_n \rightarrow kN^*$  that replaces all the nodes of a forest with the Lie bracket in the free associative algebra

$k\mathbb{N}^*$ . This allows us to identify  $\Sigma(\mathfrak{S}_n)^{\text{op}}$  with a quotient of  $k\mathcal{L}_n$  in [Theorem 7](#). We introduce a quiver  $Q_n$  in [§7](#) and show in [§8](#) that the path algebra of  $Q_n$  can be embedded into the algebra  $k\mathcal{L}_n$  of forest classes through the injective anti-homomorphism  $\iota$  shown in the diagram above. We also show in [§12](#) that  $Q_n$  is the ordinary quiver of  $\Sigma(\mathfrak{S}_n)$ . This means that  $\Sigma(\mathfrak{S}_n)$  can be identified with the quotient of the path algebra of  $Q_n$  by an ideal that can be explicitly calculated. We conjecture in [§13](#) that a generating set for this ideal can be produced through a simple procedure. Finally we calculate the presentation of  $\Sigma(\mathfrak{S}_8)$  in [§14](#), thus verifying our conjecture in this particular example.

## 2. COMPOSITIONS, PARTITIONS, AND REARRANGEMENT

Much of the charm of the theory developed in this paper stems from the reduction of complicated combinatorial operations to the simpler operation of rearrangement, which is the subject of this section. We denote the free monoid on a set  $\Omega$  by  $\Omega^*$ . This is the set of all formal products  $x_1x_2\cdots x_j$  where  $x_i \in \Omega$  for all  $1 \leq i \leq j$ . The binary operation on  $\Omega^*$  is not denoted. In this paper, an important instance of this construction occurs when  $\Omega$  is the set  $\mathbb{N}$  of natural numbers, which does *not* include 0. The elements of  $\mathbb{N}^*$  are called *compositions* and the numbers  $x_i$  in a composition  $x_1x_2\cdots x_j$  are called its *parts*.

The symmetric group  $\mathfrak{S}_j$  acts on compositions with  $j$  parts by

$$(x_1x_2\cdots x_j) \cdot \pi = x_{1 \cdot \pi^{-1}}x_{2 \cdot \pi^{-1}}\cdots x_{j \cdot \pi^{-1}}$$

for  $\pi \in \mathfrak{S}_j$ . This action is called the *Pólya action*. The orbits of the Pólya action on  $\mathbb{N}^*$  are called *partitions*. We represent a partition by any of its representatives when this causes no confusion.

## 3. TREES AND FORESTS

A (*binary*) *tree* is either a natural number or a diagram  $\widehat{X} \ Y$  where  $X$  and  $Y$  are trees. Trees of the first type are called *leaves* while trees of the second type are called *nodes*. A *labeled forest* is a sequence of trees whose nodes are labeled by natural numbers in such a way that the label of every node is greater than that of its parent if it has one, and each number  $1, 2, \dots, l$  is the label of exactly one node, where  $l$  is the number of nodes in the sequence. For example

$$(3) \quad \begin{array}{c} \widehat{3} \quad \widehat{1} \quad \widehat{4} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ 1 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \end{array}$$

is a labeled forest. Let  $Y$  be a labeled forest. The sequence of leaves of  $Y$  is called its *foliage* and is denoted by  $\underline{Y}$ . The sum of the leaves of a tree is called its *value*. The sequence of values of the trees of  $Y$  is called its *squash* and is denoted by  $\bar{Y}$ . The number of nodes in  $Y$  is called its *length* and is denoted by  $\ell(Y)$ . For example, if  $Y$  is the forest shown in (3) then  $\underline{Y} = 1213121$  and  $\bar{Y} = 353$  while  $\ell(Y) = 4$ .

Whenever two forests  $X$  and  $Y$  satisfy  $\underline{X} = \bar{Y}$  we define a product  $X \bullet Y$  by replacing the leaves of  $X$  with the trees of  $Y$ . For example, if  $X$  is the forest  $\widehat{3} \ 5 \ 3$  and  $Y$  is the forest shown in (3) then  $\underline{X} = 353 = \bar{Y}$  so that

$$(4) \quad \begin{array}{c} \widehat{1} \quad \widehat{5} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \widehat{4} \quad \widehat{2} \quad \widehat{3} \quad \widehat{1} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ 1 \quad 2 \quad 1 \quad 3 \quad 3 \quad 1 \end{array}$$

is the product  $X \bullet Y$ . Note that the node labels of  $Y$  must be incremented by  $\ell(X)$  to ensure that the product will also be a labeled forest.

All the definitions above can be made mathematically precise by defining the set of labeled trees  $\mathbb{L}$  to be the minimal set containing  $\mathbb{N}$  and also containing the tuple  $(X_1, i, X_2)$  whenever  $X_1, X_2 \in \mathbb{L}$  and  $i \in \mathbb{N}$ . A labeled tree of the form  $(X_1, i, X_2)$  should also satisfy the labeling condition  $i_1, i_2 < i$  where  $X_1 = (X_{11}, i_1, X_{12})$  and  $X_2 = (X_{21}, i_2, X_{22})$ . We define the squash of a labeled tree  $X \in \mathbb{L}$  by the formula

$$(5) \quad \bar{X} = \begin{cases} X & \text{if } X \in \mathbb{N} \\ \overline{X_1 + X_2} & \text{if } X = (X_1, i, X_2) \end{cases}$$

and we similarly define the foliage and length of  $X$ . Then a labeled forest is an element of the free monoid on  $\mathbb{L}$  which satisfies the labeling condition dealing with unique node labels from the original definition. The definition in (5) extends to labeled forests by  $X_1 X_2 \cdots X_j = \overline{X_1 X_2 \cdots X_j}$  where  $X_1, X_2, \dots, X_j \in \mathbb{L}$  and similarly for the foliage and length analogs of (5).

**Lemma 2.** *A labeled forest of length at least one can be uniquely factorized as a product of labeled forests of length one.*

*Proof.* Suppose that  $X = X_1 X_2 \cdots X_j$  is a labeled forest, where  $X_1, X_2, \dots, X_j$  are trees. Note that since 1 is the smallest node label of  $X$ , it must be the label of one of the trees  $X_1, X_2, \dots, X_j$ , say  $X_i$ . This means that  $X_i = \widehat{\begin{matrix} \uparrow \\ X_{i1} \ X_{i2} \end{matrix}}$  for some trees  $X_{i1}$  and  $X_{i2}$ . Let  $Y$  be the forest obtained from  $X_1 X_2 \cdots X_{i-1} X_{i1} X_{i2} X_{i+1} \cdots X_j$  by reducing the node labels by one and write  $x_1 x_2 \cdots x_{i-1} x_{i1} x_{i2} x_{i+1} \cdots x_j = \bar{Y}$ . We put

$$X' = \overset{x_1 x_2 \cdots x_{i-1}}{\quad} \widehat{\begin{matrix} \uparrow \\ x_{i1} \ x_{i2} \end{matrix}} \overset{x_{i+1} \cdots x_j}{\quad}$$

so that  $X = X' \bullet Y$ . Note that  $X'$  is the unique forest of length one with squash  $\bar{X}$  and foliage  $\bar{Y}$ . Repeating the procedure with  $Y$  in place of  $X$  yields the desired factorization by induction.  $\square$

For example, the forest in (4) can be factorized as

$$(6) \quad \left( \widehat{\begin{matrix} \uparrow \\ 3 \ 5 \end{matrix}} \ 3 \right) \bullet \left( 3 \ \widehat{\begin{matrix} \uparrow \\ 1 \ 4 \end{matrix}} \ 3 \right) \bullet \left( \overset{31}{\quad} \widehat{\begin{matrix} \uparrow \\ 3 \ 1 \end{matrix}} \ 3 \right) \bullet \left( \widehat{\begin{matrix} \uparrow \\ 1 \ 2 \end{matrix}} \ 1313 \right) \bullet \left( \overset{12131}{\quad} \widehat{\begin{matrix} \uparrow \\ 2 \ 1 \end{matrix}} \right).$$

The *value* of a forest is the sum of the values of its trees. For the purpose of constructing a quiver presentation of  $\Sigma(\mathfrak{S}_n)$  we restrict our attention to the set  $L_n$  of forests of value  $n \in \mathbb{N}$ . Then  $L_n$  is a *category*, that is, a monoid whose product is only partially defined. Taking  $X \bullet Y$  to be zero whenever  $\bar{X} \neq \bar{Y}$  makes  $kL_n$  into a  $k$ -algebra.

#### 4. EQUIVALENCE OF FORESTS WITH ALLEYS

Recall from §1 that

$$\mathcal{A} = \left\{ (J; s_1, s_2, \dots, s_l) \mid \{s_1, s_2, \dots, s_l\} \subseteq J \subseteq S \text{ with } s_1, s_2, \dots, s_l \text{ distinct} \right\}$$

and that the partial product  $\circ : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is defined by

$$(J; s_1, s_2, \dots, s_l) \circ (K; t_1, t_2, \dots, t_m) = (J; s_1, s_2, \dots, s_l, t_1, t_2, \dots, t_m)$$

if  $K = J \setminus \{s_1, s_2, \dots, s_l\}$ . The category  $\mathcal{A}$  is a combinatorial gadget used to construct quiver presentations of the descent algebras of finite Coxeter groups. The

elements of  $\mathcal{A}$  are called *alleys*. The number  $l$  is called the *length* of the alley  $\mathbf{a} = (J; s_1, s_2, \dots, s_l)$  and is denoted by  $\ell(\mathbf{a})$ . One can also view  $\mathbf{a}$  as the chain

$$(7) \quad J \supseteq J \setminus \{s_1\} \supseteq J \setminus \{s_1, s_2\} \supseteq \dots \supseteq J \setminus \{s_1, s_2, \dots, s_l\}$$

of subsets of  $\{1, 2, \dots, n\}$ . Then the product of two alleys corresponds with the concatenation of the corresponding chains whenever the concatenation is also a chain.

**Proposition 3.** *The category  $\mathcal{A}$  associated to the Coxeter group  $\mathfrak{S}_n$  is equivalent to  $L_n$  through a length-preserving functor.*

*Proof.* We identify the Coxeter generating set  $S$  of  $\mathfrak{S}_n$  with the set  $\{1, 2, \dots, n-1\}$ . If  $J \subseteq S$  with  $|J| = n-j$  then we write  $S \setminus J = \{t_1, t_2, \dots, t_{j-1}\}$  where  $t_1 < t_2 < \dots < t_{j-1}$ . We put  $t_0 = 0$  and  $t_j = n$  and let  $\varphi(J)$  be the composition  $q_1 q_2 \dots q_j$  where  $q_i = t_i - t_{i-1}$ . Then  $\varphi$  is a bijection between the subsets of  $S$  and the compositions of  $n$ .

Let  $H_m$  be the Hasse diagram of the relation  $\subseteq$  on the subsets of  $\{1, 2, \dots, m\}$  for  $m \geq 0$ . Then  $H_m$  is a quiver with a vertex for every subset of  $\{1, 2, \dots, m\}$  and an edge from  $J$  to  $K$  if  $J \subseteq K$  and  $|K \setminus J| = 1$ . Thanks to the description in (7) we can identify  $\mathcal{A}$  with the set of paths in  $H_{n-1}$ . Note that under this identification the length of an alley equals the length of the corresponding path.

Now consider the quiver  $H'_n$  which has a vertex for every composition of  $n$  and an edge from  $\mathbf{p}$  to  $\mathbf{q}$  if there exists a forest of length one with foliage  $\mathbf{p}$  and squash  $\mathbf{q}$ . Thanks to Lemma 2 we can identify  $L_n$  with the set of paths in  $H'_n$ . Note that under this identification the length of a forest equals the length of the corresponding path.

Next we observe that the vertices of  $H_{n-1}$  are in bijection with the vertices of  $H'_n$  through  $\varphi$  and that  $H_{n-1}$  has an edge from  $J$  to  $K$  if and only if  $H'_n$  has an edge from  $\varphi(J)$  to  $\varphi(K)$ . This means that the quivers  $H_{n-1}$  and  $H'_n$  are isomorphic as directed graphs so that  $\mathcal{A}$  and  $L_n$  are equivalent through a length-preserving functor, which we denote by  $\varphi$  in the following sections.  $\square$

For example, the alley  $(\{1, 2, 3, 4, 5, 6, 7, 9, 10\}; 3, 4, 7, 1, 10)$  corresponds with the path

$$\begin{aligned} \{2, 5, 6, 9\} &\rightarrow \{2, 5, 6, 9, 10\} \rightarrow \{1, 2, 5, 6, 9, 10\} \rightarrow \{1, 2, 5, 6, 7, 9, 10\} \\ &\rightarrow \{1, 2, 4, 5, 6, 7, 9, 10\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 9, 10\} \end{aligned}$$

in  $H_{10}$ , which in turn corresponds under  $\varphi$  with the path

$$1213121 \rightarrow 121313 \rightarrow 31313 \rightarrow 3143 \rightarrow 353 \rightarrow 83$$

in  $H'_{11}$  corresponding with the forest shown in (4) and factorized in (6).

## 5. ACTIONS AND ORBITS

If  $X = X_1 X_2 \dots X_j$  is a labeled forest where  $X_1, X_2, \dots, X_j$  are trees, then the trees  $X_1, X_2, \dots, X_j$  are called the *parts* of  $X$ . The Pólya action of  $\mathfrak{S}_j$  on compositions with  $j$  parts extends to an action on forests with  $j$  parts. If  $X$  is a forest with  $j$  parts, then we denote the sum of the elements in the same  $\mathfrak{S}_j$ -orbit as  $X$  by  $[X]$ .

For example, if  $X$  is the forest  $\begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 1 & 2 & 1 & 2 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array}$  then

$$\begin{aligned}
 [X] = & \begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 1 & 2 & 1 & 2 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array} + \begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 2 & 1 & 2 & 1 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array} + \begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 1 & 2 & 1 & 2 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array} + \begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 2 & 1 & 2 & 1 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array} \\
 & + \begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 2 & 1 & 2 & 1 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array} + \begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 3 & 1 & 2 & 1 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array} + \begin{array}{cccc} \nearrow & \nearrow & \nearrow & \nearrow \\ 3 & 1 & 2 & 1 \\ \searrow & \searrow & \searrow & \searrow \\ & 1 & 2 & 1 & 2 \end{array} .
 \end{aligned}$$

The set of orbit sums in  $kL_n$  is denoted by  $\mathcal{L}_n$ .

Suppose that  $X, Y \in L_n$  are such that  $\underline{X} = \underline{Y}$ . If  $X$  has  $i$  parts and  $Y$  has  $j$  parts, then any element  $\sigma \in \mathfrak{S}_i$  induces a permutation  $\tau \in \mathfrak{S}_j$  of the leaves of  $X$ . Namely,  $\tau$  is the permutation satisfying  $X \bullet \sigma \bullet Y \bullet \tau = (X \bullet Y) \bullet \sigma$ . This correspondence is an injective homomorphism when restricted to any subgroup of  $\mathfrak{S}_i$  that permutes only parts of  $X$  that have the same number of leaves. The stabilizer of  $X$  in  $\mathfrak{S}_i$  is such a subgroup, since it permutes only parts of length zero, the parts of positive length having distinct node labels. Therefore the stabilizer of  $X$  is isomorphic to a subgroup  $K$  of  $\mathfrak{S}_j$ . Now if  $H$  is the stabilizer of  $Y$  in  $\mathfrak{S}_j$  then

$$[X] \bullet [Y] = \sum_{t=1}^m [X \bullet Y \bullet \sigma_t] \in k\mathcal{L}_n$$

where  $\sigma_1, \sigma_2, \dots, \sigma_m$  are representatives of the double cosets of  $H, K$  in  $\mathfrak{S}_j$ . This proves the following proposition.

**Proposition 4.**  $k\mathcal{L}_n$  is a subalgebra of  $kL_n$ .

Alternately, [Proposition 4](#) follows with [Theorem 1](#) from [Proposition 5](#) below through the equivalence of  $L_n$  with  $\mathcal{A}$ .

Recall from [§1](#) that the free monoid  $S^*$  acts on  $\mathcal{A}$  by

$$(J; s_1, s_2, \dots, s_l) \cdot t = (J^\omega; s_1^\omega, s_2^\omega, \dots, s_l^\omega)$$

for  $t \in S$  and  $(J; s_1, s_2, \dots, s_l) \in \mathcal{A}$  where  $\omega = w_J w_{J \cup \{t\}}$  and where  $w_J$  and  $w_{J \cup \{t\}}$  are the longest elements of the parabolic subgroups  $W_J$  and  $W_{J \cup \{t\}}$  respectively. When  $\mathcal{A}$  is the category associated with  $\Sigma(\mathfrak{S}_n)$  we calculate the orbits of this action in the following proposition.

**Proposition 5.** *The orbits of the Pólya action on  $L_n$  correspond under the equivalence  $\varphi$  in [Proposition 3](#) with the  $S^*$ -orbits on  $\mathcal{A}$ .*

*Proof.* Let  $\mathbf{a} = (J; s_1, s_2, \dots, s_l) \in \mathcal{A}$  and let  $X = \varphi(\mathbf{a}) \in L_n$ . Let  $t_0, t_1, \dots, t_j$  be as in the proof of [Proposition 3](#). Note that if  $t \in J$  then  $\omega = w_J w_{J \cup \{t\}} = \text{id}_W$  so that  $\mathbf{a} \cdot t = \mathbf{a}$ . Otherwise assume that  $t = t_i$  for some  $1 \leq i \leq j-1$ . We claim that  $\varphi(\mathbf{a} \cdot t_i)$  is obtained from  $X$  by exchanging the parts in positions  $i$  and  $i+1$ . From this it will follow that  $\varphi(\mathbf{a} \cdot S^*) = \varphi(\mathbf{a}) \cdot \mathfrak{S}_j$ .

It is easy to see that conjugation by  $w_j$  reverses the elements in the block

$$B_g = \{t_g + 1, t_g + 2, \dots, t_{g+1} - 1\}$$

for all  $0 \leq g \leq j-1$ . Note that including  $t_i$  in  $J$  joins the blocks  $B_{i-1}$  and  $B_i$  into the block  $B_{i-1} \cup \{t_i\} \cup B_i$ . Then since conjugation by  $w_{J \cup \{t_i\}}$  again reverses all the blocks, the effect of conjugation by  $\omega$  is to shift  $B_{i-1}$  to the right of  $B_i$  while fixing the remaining blocks.

It follows from the definition of  $\varphi$  that if  $K \subseteq J$  then  $\varphi(K)$  is a refinement of  $\varphi(J)$ . In other words, if  $\varphi(J) = q_1 q_2 \cdots q_j$  where  $q_1, q_2, \dots, q_j \in \mathbb{N}$  then  $\varphi(K) = p_1 p_2 \cdots p_j$  where  $p_i$  is a composition of  $q_i$  for all  $1 \leq i \leq j$ . Then conjugating  $K$  by  $\omega$  corresponds under  $\varphi$  with exchanging the compositions  $p_i$  and  $p_{i+1}$  of  $\varphi(K)$ . Applying this observation to each vertex  $K$  of the path corresponding with  $\mathbf{a}$  in  $H_{n-1}$  we see that the path in  $H'_n$  corresponding with  $\varphi(\mathbf{a}.t_i)$  is obtained from the path corresponding with  $X = \varphi(\mathbf{a})$  by exchanging the compositions  $p_i$  and  $p_{i+1}$  of each vertex  $\varphi(K)$ . Therefore  $\varphi(\mathbf{a}.t_i)$  is obtained from  $X$  by exchanging the parts in positions  $i$  and  $i+1$ .  $\square$

## 6. DIFFERENCE OPERATORS

In this section we prove one of the main results of this paper, namely that  $\Sigma(\mathfrak{S}_n)$  is isomorphic to a quotient of  $k\mathcal{L}_n$ . For this purpose we define a difference operator  $\delta$  on  $k\mathcal{L}_n$  as follows. If  $\ell(X) = 0$  then we define  $\delta(X) = X$ . Otherwise suppose that  $X = X_1 X_2 \cdots X_j \in L_n$  where  $X_1, X_2, \dots, X_j$  are trees and that  $X_i$  is the node of  $X$  labeled 1. Then  $X_i = \widehat{X_{i1} X_{i2}}$  for some trees  $X_{i1}$  and  $X_{i2}$ . We define  $\delta(X)$  to be the element of  $k\mathcal{L}_n$  obtained from  $X$  by replacing  $X_i$  with the Lie bracket  $X_{i1}X_{i2} - X_{i2}X_{i1}$  and reducing the remaining node labels by one. In terms of the Pólya action, this means that  $\delta(X) = Y - Y.i$  where  $Y$  is the forest obtained from  $X$  by splitting the part  $\widehat{X_{i1} X_{i2}}$  in position  $i$  into  $X_{i1}X_{i2}$  and reducing the remaining node labels by one.

Recall from §1 that the difference operator  $\delta$  on  $k\mathcal{A}$  is defined by  $\delta(\mathbf{a}) = \mathbf{a}$  if  $l = 0$  or by  $\delta(\mathbf{a}) = \mathbf{b} - \mathbf{b}.s_1$  if  $l > 0$  for all  $\mathbf{a} = (J; s_1, s_2, \dots, s_l)$  where  $\mathbf{b} = (J \setminus \{s_1\}; s_2, \dots, s_l)$ . When  $\mathcal{A}$  is the category associated with  $\Sigma(\mathfrak{S}_n)$  this difference operator coincides with the one introduced above in the following sense.

**Proposition 6.**  $\varphi(\delta(\mathbf{a})) = \delta(\varphi(\mathbf{a}))$  for all alleys  $\mathbf{a} \in \mathcal{A}$ .

*Proof.* Let  $\mathbf{a}$  and  $\mathbf{b}$  be as above and let  $X = X_1 X_2 \cdots X_j = \varphi(\mathbf{a}) \in L_n$  where  $X_1, X_2, \dots, X_j$  are trees. The factorization  $\mathbf{a} = (J; s_1) \circ \mathbf{b}$  and the factorization  $X = X' \bullet Y$  in Lemma 2 imply that  $\varphi(J; s_1) = X'$  and  $\varphi(\mathbf{b}) = Y$  by unique factorization and length-preserving equivalence.

Now let  $t_1, \dots, t_{j-1}$  be as in Proposition 3. Then

$$\{1, 2, \dots, n-1\} \setminus (J \setminus \{s_1\}) = \{t_1, t_2, \dots, t_{i-1}, s_1, t_i, t_{i+1}, \dots, t_{j-1}\}$$

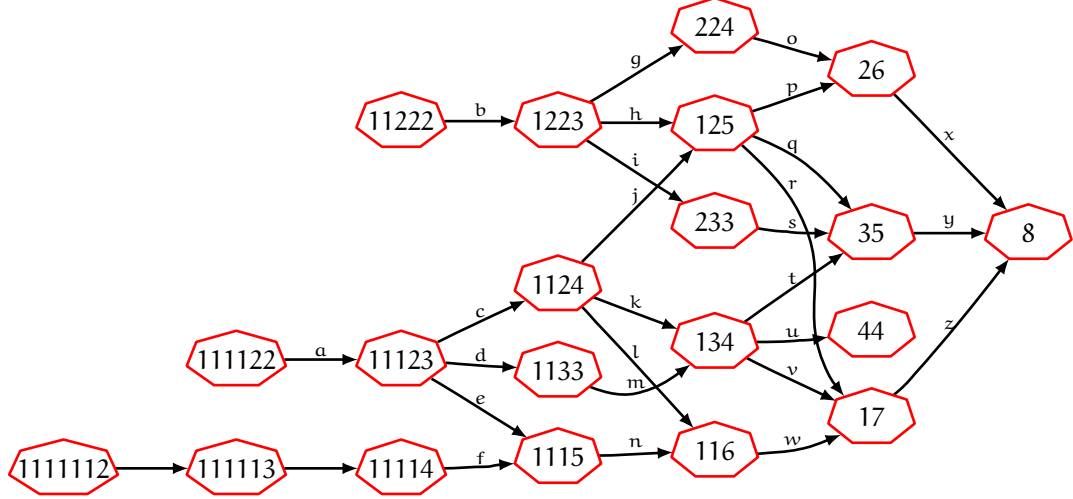
with  $t_1 < t_2 < \cdots < t_{i-1} < s_1 < t_i < t_{i+1} < \cdots < t_{j-1}$ . Since  $s_1$  is in position  $i$  of this list,  $\varphi(\mathbf{b}.s_1)$  is obtained from  $\varphi(\mathbf{b})$  by exchanging the trees in positions  $i$  and  $i+1$  by the proof of Proposition 5. Thus  $\delta(X) = Y - Y.i = \varphi(\mathbf{b} - \mathbf{b}.s_1) = \varphi(\delta(\mathbf{a}))$ .  $\square$

Iterating  $\delta$  as many times as possible determines another difference operator  $\Delta$  on  $k\mathcal{L}_n$  defined by  $\Delta(X) = \delta^{\ell(X)}(X)$  for all forests  $X$ . This is analogous to the difference operator  $\Delta$  on  $k\mathcal{A}$  defined by  $\Delta(\mathbf{a}) = \delta^{\ell(\mathbf{a})}(\mathbf{a})$  for all alleys  $\mathbf{a}$ . Note that applying  $\Delta$  to  $X \in L_n$  results in a  $\mathbb{Z}$ -linear combination of compositions of  $n$ .

**Theorem 7.**  $\Sigma(\mathfrak{S}_n)^{\text{op}}$  is isomorphic to  $k\mathcal{L}_n / \ker \Delta$ .

*Proof.*  $k\mathcal{X} / \ker \Delta$  is isomorphic to  $\Sigma(\mathfrak{S}_n)^{\text{op}}$  by Theorem 1 and  $k\mathcal{X}$  is isomorphic to  $k\mathcal{L}_n$  by Proposition 5. Then  $k\mathcal{X} / \ker \Delta$  is isomorphic to  $k\mathcal{L}_n / \ker \Delta$  since the maps  $\Delta$  on the two algebras coincide under  $\varphi$  by Proposition 6.  $\square$

FIGURE 1. The quiver  $Q_8$



[Theorem 7](#) gives a new construction of  $\Sigma(\mathfrak{S}_n)$  as a quotient of  $k\mathcal{L}_n$ . We show in the following sections that  $k\mathcal{L}_n$  in turn is a homomorphic image of the path algebra of a quiver.

7. THE QUIVER

Recall from [Lemma 2](#) that a labeled forest of length at least one can be uniquely factorized as a product of forests of length one. This property fails when we replace  $L_n$  with  $\mathcal{L}_n$ . For example, if we try to factorize  $\left[ \begin{array}{c} \hat{1} \quad \quad 3 \\ 1 \quad \hat{2} \quad 2 \\ \quad 1 \quad 2 \end{array} \right]$  as the product of  $\left[ \begin{array}{c} \hat{1} \quad 3 \\ 1 \quad 3 \end{array} \right]$  and  $\left[ \begin{array}{c} \hat{1} \quad 12 \\ 1 \quad 2 \end{array} \right]$  we find that the product

$$\left[ \begin{array}{c} \hat{1} \quad 3 \\ 1 \quad 3 \end{array} \right] \bullet \left[ \begin{array}{c} \hat{1} \quad 12 \\ 1 \quad 2 \end{array} \right] = \left[ \begin{array}{c} \hat{1} \quad \quad 3 \\ 1 \quad \hat{2} \quad 2 \\ \quad 1 \quad 2 \end{array} \right] + \left[ \begin{array}{c} \hat{1} \quad \quad 2 \\ 1 \quad 3 \quad 1 \quad 2 \end{array} \right]$$

has an extra term. This defect in factorization is the subject of [§12](#).

Nonetheless, the success of factorization in  $L_n$  suggests representing the algebra  $k\mathcal{L}_n$  as a path algebra. Namely, in the factorization of any labeled forest, the foliage of each factor equals the squash of the following factor, so we can regard each factor as an edge from its foliage to its squash.

Let  $Q_n$  be the quiver having the partitions of  $n$  as vertices and an edge from the vertex  $p$  to the vertex  $q$  whenever  $q$  can be obtained from  $p$  by replacing two *distinct* parts with their sum. In other words,  $Q_n$  is the Hasse diagram of the partitions of  $n$  under restricted partition refinement, in contrast with the Hasse diagram  $H'_n$  of the *compositions* of  $n$  under *ordinary* partition refinement introduced in the proof of [Proposition 3](#). For example, the quiver  $Q_8$  is shown in [Figure 1](#), omitting the vertices 1111111 and 2222, which are not incident with any edges. This quiver also appears in [\[4, p. 54\]](#).

We define a map  $\iota : Q_n \rightarrow k\mathcal{L}_n$  as follows. Recall from §2 that we regard a partition as the Pólya equivalence class of a composition, but that we represent a partition by any convenient representative. If  $p$  is a vertex of  $Q_n$  then we define  $\iota(p)$  to simply be  $p$  itself, now regarded as a class sum in  $\mathcal{L}_n$ . If  $e$  is the edge from a vertex  $p$  to another vertex  $q$ , then by rearranging the parts of  $p$  we have  $p = p_{11}p_{12}p_2p_3 \cdots p_j$  and  $q = p_1p_2p_3 \cdots p_j$  for some  $j \in \mathbb{N}$  and some  $p_{11}, p_{12}, p_2, p_3, \dots, p_j \in \mathbb{N}$  where  $p_{11} < p_{12}$  and  $p_1 = p_{11} + p_{12}$ . We put  $\iota(e) = \left[ \begin{array}{c} \widehat{\phantom{p}} \\ p_{11} \ p_{12} \end{array} \quad p_2 p_3 \cdots p_j \right]$ . Note that  $\iota$  satisfies  $\iota(xy) = \iota(y)\iota(x)$  whenever one of  $x$  or  $y$  is a vertex and the other is an incident vertex or edge. This proves the following proposition.

**Proposition 8.** *The map  $\iota$  extends to an anti-homomorphism  $\iota : kQ_n \rightarrow k\mathcal{L}_n$ .*

We show in Corollary 12 that  $\iota$  is injective and in Proposition 22 that  $Q_n$  is the ordinary quiver of  $\Sigma(\mathfrak{S}_n)$ . One of the main ingredients in the proof of Proposition 22 is the following technical lemma.

**Lemma 9.** *If  $e$  is an edge of  $Q_n$  then  $\iota(e) \notin \ker \Delta$ .*

*Proof.* Suppose  $\iota(e) = \left[ \begin{array}{c} \widehat{\phantom{p}} \\ a \ b \end{array} \quad q_1 q_2 \cdots q_j \right]$  and that  $0 \leq i \leq j$  is such that  $q_1 \leq q_2 \leq \cdots \leq q_i \leq a < q_{i+1} \leq \cdots \leq q_j$ . Then the term  $q_1 q_2 \cdots q_i a b q_{i+1} \cdots q_j$  of  $\Delta \left( \begin{array}{c} \widehat{\phantom{p}} \\ q_1 q_2 \cdots q_i \ a \ b \end{array} \quad q_{i+1} \cdots q_j \right)$  has at most one descending subsequence, namely  $b q_{i+1}$ . However, all the terms of  $\Delta(\iota(e))$  appearing with negative coefficients have the descending subsequence  $b a$  which is different from  $b q_{i+1}$  since  $a < q_{i+1}$ . Thus  $\Delta(\iota(e))$  cannot be zero.  $\square$

In an effort both to simplify notation and to shift emphasis from the individual groups  $\mathfrak{S}_n$  to the family  $\bigcup_{n \in \mathbb{N} \cup \{0\}} \mathfrak{S}_n$  of groups, we define

$$Q = \prod_{n \in \mathbb{N} \cup \{0\}} Q_n \quad L = \prod_{n \in \mathbb{N} \cup \{0\}} L_n \quad \mathcal{L} = \prod_{n \in \mathbb{N} \cup \{0\}} \mathcal{L}_n$$

and regard  $\iota$  as a map  $kQ \rightarrow k\mathcal{L}$ .

## 8. THE BRANCH MONOID

Let  $\mathcal{B}$  be the set of symbols  $\langle a \mid b \rangle$  for all  $a, b \in \mathbb{N}$  with  $a < b$ . We call the free monoid  $\mathcal{B}^*$  the *branch monoid* and we write the element  $\langle a_1 \mid b_1 \rangle \langle a_2 \mid b_2 \rangle \cdots \langle a_l \mid b_l \rangle$  of  $\mathcal{B}^*$  as  $\langle a_1 \ a_2 \ \cdots \ a_l \mid b_1 \ b_2 \ \cdots \ b_l \rangle$  to simplify notation. The notation is meant to reflect the fact that the elements of  $\mathcal{B}^*$  can be used to build forests as we now describe.

If  $X$  is a labeled forest then let  $X \cdot \langle a \mid b \rangle$  be the sum of all forests that can be obtained from  $X$  by replacing a leaf  $a + b$  with  $\begin{array}{c} \widehat{\phantom{p}} \\ a \ b \end{array}$  where  $l = \ell(X) + 1$ . If  $P$  is a path in  $Q$  with source  $p$  then we define  $P \cdot \langle a \mid b \rangle$  to be the path obtained from  $P$  by appending the edge from  $a b q_1 \cdots q_j$  to  $p$  if  $p$  has a part  $a + b$ , where  $q_1, \dots, q_j \in \mathbb{N}$  are the remaining parts of  $p$ . We put  $P \cdot \langle a \mid b \rangle = 0$  if  $p$  has no part  $a + b$ . Then  $\mathcal{B}^*$  acts on  $kL$  and on  $kQ$  by extending the definitions above by linearity. From the definitions we have

$$(8) \quad (P_1 P_2) \cdot B = (P_1 \cdot B) P_2 \quad \text{and} \quad (X_1 \bullet X_2) \cdot B = X_1 \bullet X_2 \cdot B$$

for  $P_1, P_2 \in kQ$  and  $X_1, X_2 \in kL$  and  $B \in \mathcal{B}^*$ . If  $p$  is a partition containing a part  $a + b$  then

$$(9) \quad \iota(p) \cdot \left\langle \begin{array}{c} a \\ b \end{array} \right\rangle = \left[ \begin{array}{c} \widehat{1} \\ a \quad b \end{array} \quad q_1 q_2 \cdots q_j \right] = \iota \left( p \cdot \left\langle \begin{array}{c} a \\ b \end{array} \right\rangle \right)$$

where  $q_1, q_2, \dots, q_j$  are the remaining parts of  $p$ . On the other hand, both  $\iota(p) \cdot \left\langle \begin{array}{c} a \\ b \end{array} \right\rangle$  and  $p \cdot \left\langle \begin{array}{c} a \\ b \end{array} \right\rangle$  are zero if  $p$  has no part  $a + b$ . Now if  $P$  is a path in  $Q$  with source  $p$ , then using (8) and (9) we have

$$\iota(P) \cdot B = \iota(pP) \cdot B = (\iota(P) \bullet \iota(p)) \cdot B = \iota(P) \bullet \iota(p \cdot B) = \iota((p \cdot B)P) = \iota(P \cdot B)$$

for all  $B \in k\mathcal{B}^*$ . This proves the following proposition.

**Proposition 10.** *The map  $\iota$  is a homomorphism of  $k\mathcal{B}^*$ -modules.*

The branch monoid provides a convenient language for specifying paths in  $Q$ . Namely, we can uniquely specify any path  $P$  as  $p \cdot B$  where  $p$  is the destination of  $P$  and  $B$  is an element of  $\mathcal{B}^*$ . Furthermore, the element  $B$  is related to  $\iota(P)$  in the way described in the following lemma.

**Lemma 11.** *Let  $P = p \cdot \left\langle \begin{array}{c} a_1 \quad a_2 \quad \cdots \quad a_l \\ b_1 \quad b_2 \quad \cdots \quad b_l \end{array} \right\rangle$  be a path in  $Q$ . Then the node  $\widehat{Z_1 \quad Z_2}^i$  of every term of  $\iota(P)$  satisfies  $\overline{Z_1} = a_i$  and  $\overline{Z_2} = b_i$  for all  $1 \leq i \leq l$ .*

*Proof.* The assertion holds by definition if  $l$  equals zero or one. Otherwise let  $P' = p \cdot \left\langle \begin{array}{c} a_1 \quad a_2 \quad \cdots \quad a_{l-1} \\ b_1 \quad b_2 \quad \cdots \quad b_{l-1} \end{array} \right\rangle$  so that  $P = P' \cdot \left\langle \begin{array}{c} a_l \\ b_l \end{array} \right\rangle$  and  $\iota(P) = \iota(P') \cdot \left\langle \begin{array}{c} a_l \\ b_l \end{array} \right\rangle$  by Proposition 10. Then  $\iota(P)$  is obtained from  $\iota(P')$  by replacing a leaf  $a_l + b_l$  in every term with  $\widehat{a_l \quad b_l}^i$ . Thus the node labeled  $l$  of every term of  $\iota(P)$  satisfies the assertion, while the other nodes satisfy the assertion by induction.  $\square$

**Corollary 12.** *The anti-homomorphism  $\iota$  is injective.*

*Proof.* By Lemma 11 the images of distinct paths are supported on disjoint subsets of  $\mathcal{L}$ .  $\square$

## 9. UNLABELED FORESTS

To compute the kernel of  $\Delta : k\mathcal{L} \rightarrow k\mathbb{N}^*$  it will be helpful to introduce an algebra through which  $\Delta$  factors. Then the kernel of  $\Delta$  can be assembled from the kernels of its factors. Let  $M$  be the category of *unlabeled forests*, which are simply sequences of binary trees whose leaves are natural numbers. The definitions of the foliage, squash, length, value, and product of unlabeled forests can be easily adapted from the definitions for labeled forests, as can the Pólya action and the action of  $k\mathcal{B}^*$  on  $M$ . Then

$$M = \coprod_{n \in \mathbb{N} \cup \{0\}} M_n \quad \text{and} \quad \mathcal{M} = \coprod_{n \in \mathbb{N} \cup \{0\}} \mathcal{M}_n$$

where  $M_n$  is the category of unlabeled forests of value  $n$  and  $\mathcal{M}$  and  $\mathcal{M}_n$  are the sets of Pólya class sums in  $kM$  and  $kM_n$ .

There is a map  $E : L \rightarrow M$  given by erasing the node labels of a forest. If  $X$  is a labeled forest with  $j$  parts, then we denote by  $\alpha_X$  the index of the stabilizer of  $X$  in  $\mathfrak{S}_j$  in the stabilizer of  $E(X)$  in  $\mathfrak{S}_j$ .

**Lemma 13.** *If  $X \in L$  then  $E[X] = \alpha_X [E(X)]$ .*

For example, if  $X = \widehat{1} \widehat{21} \widehat{2}$  then  $[X] = \widehat{1} \widehat{21} \widehat{2} + \widehat{2} \widehat{21} \widehat{1}$  while  $[E(X)] = \widehat{1} \widehat{21} \widehat{2}$  so that  $E[X] = 2[E(X)]$ .

Recall that the product in  $\mathbb{L}$  of two forests is formed by replacing the leaves in one forest with the trees of the other. Since this process depends on the foliage and squash but not the node labels of the two forests, we observe that up to node label erasure, the same products are formed with or without the node labels. This means that  $E$  is a functor and that the induced map  $E : k\mathbb{L} \rightarrow k\mathbb{M}$  is an algebra homomorphism. Then since the restriction of  $E$  to the subalgebra  $k\mathcal{L}$  has image in  $k\mathbb{M}$  by [Lemma 13](#) we have the following result.

**Proposition 14.** *The map  $E : k\mathcal{L} \rightarrow k\mathbb{M}$  given by erasing node labels is an algebra homomorphism.*

As with labeled forests, the definition of unlabeled forests can be made mathematically precise by defining the set of unlabeled trees  $\mathbb{M}$  to be the minimal set containing  $\mathbb{N}$  and also containing the tuple  $(X_1, X_2)$  whenever  $X_1, X_2 \in \mathbb{M}$ . Then for example, the map  $E$  can be defined by

$$E(X) = \begin{cases} X & \text{if } X \in \mathbb{N} \\ (E(X_1), E(X_2)) & \text{if } X = (X_1, i, X_2) \in \mathbb{L} \end{cases}$$

and similarly for the other functions of unlabeled forests.

## 10. ALIGNMENT

Let  $\mathbb{M}$  be the free magma generated by  $\mathbb{N}$ . We denote the product of two elements  $X$  and  $Y$  of  $\mathbb{M}$  by  $\widehat{X Y}$ . Although we could introduce a symbol for this operation, after several iterations, it becomes more legible to simply represent elements of  $\mathbb{M}$  as binary trees. We define the ideals

$$\begin{aligned} \mathbb{N} &= \left\langle \widehat{X Y} + \widehat{Y X} \mid X, Y \in \mathbb{M} \right\rangle \\ \mathbb{J} &= \left\langle \widehat{X \widehat{Y Z}} + \widehat{Y \widehat{Z X}} + \widehat{Z \widehat{X Y}} \mid X, Y, Z \in \mathbb{M} \right\rangle \end{aligned}$$

of  $k\mathbb{M}$  and recall that  $k\mathbb{M}/(\mathbb{N} + \mathbb{J})$  defines the free Lie algebra over  $k$  generated by  $\mathbb{N}$ .

Since the elements of  $\mathbb{M}$  correspond with elements of  $\mathbb{M}$  that have exactly one part, we can identify arbitrary elements of  $\mathbb{M}$  with the elements of the free monoid  $\mathbb{M}^*$ . Under this identification, the category  $\mathbb{M}$  has, in addition to the product  $\bullet$ , another product coming from concatenation in  $\mathbb{M}^*$ . Let  $\mathcal{N}$  and  $\mathcal{J}$  be the ideals of  $k\mathbb{M}$  with respect to concatenation generated by  $\mathbb{N}$  and  $\mathbb{J}$  respectively.

Let  $\pi : k\mathbb{M} \rightarrow k\mathbb{N}^*$  be defined by  $\pi(x) = x$  for  $x \in \mathbb{N}$  and  $\pi(\widehat{X Y}) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$  for  $X, Y \in \mathbb{M}$ . Then  $\pi$  extends to a monoid algebra homomorphism  $\pi : k\mathbb{M} \rightarrow k\mathbb{N}^*$  and the kernel of  $\pi$  is the ideal  $\mathcal{N} + \mathcal{J}$  generated by the kernel  $\mathbb{N} + \mathbb{J}$  of  $\pi : k\mathbb{M} \rightarrow k\mathbb{N}^*$ . Recall that the map  $\Delta$  replaces the nodes of a labeled tree with Lie brackets in the order specified by the node labels. The relationship between  $\Delta$  and  $\pi$  is given in the following lemma.

**Lemma 15.**  $\Delta = \pi \circ E$

*Proof.* Let  $X \in L$ . Then  $\Delta(X) = \pi(E(X))$  by definition if  $X$  has length zero. Suppose otherwise that  $X = X_1 X_2 \cdots X_j$  where  $X_1, X_2, \dots, X_j$  are trees and suppose that the node of  $X$  labeled 1 is  $X_i$  so that  $X_i = \widehat{X_{i1} X_{i2}}$  for some trees  $X_{i1}$  and  $X_{i2}$ . Then

$$\begin{aligned} \pi(E(X)) &= \pi(E(X_1)) \cdots \pi(E(X_i)) \cdots \pi(E(X_j)) \\ &= \pi(E(X_1)) \cdots \pi(E(X_{i1}X_{i2} - X_{i2}X_{i1})) \cdots \pi(E(X_j)) \\ &= \pi(E(X_1 \cdots (X_{i1}X_{i2} - X_{i2}X_{i1}) \cdots X_j)) \\ &= \pi(E(\delta(X))). \end{aligned}$$

Then  $\pi(E(\delta(X))) = \Delta(\delta(X)) = \Delta(X)$  by induction since  $\delta(X)$  has shorter length than  $X$ .  $\square$

A forest  $X$  is called *aligned* if  $\overline{Z_1} < \overline{Z_2}$  for all nodes  $\widehat{Z_1 Z_2}$  of  $X$ . Since the product of two aligned forests is aligned, the category  $M^+$  of aligned unlabeled forests is a subcategory of  $M$  and

$$M^+ = \coprod_{n \in \mathbb{N} \cup \{0\}} M_n^+ \quad \text{and} \quad \mathcal{M}^+ = \coprod_{n \in \mathbb{N} \cup \{0\}} \mathcal{M}_n^+$$

where  $M_n^+$  is the category of aligned unlabeled forests of value  $n$  and  $M^+$  and  $\mathcal{M}_n^+$  are the sets of class sums in  $kM^+$  and  $kM_n^+$ . We similarly define the corresponding sets of aligned labeled forests  $L^+, L_n^+, \mathcal{L}^+, \mathcal{L}_n^+$ . Our first observation about aligned forests is that the image of  $\iota$  is aligned.


**Lemma 16.**  $\iota(kQ) \subseteq k\mathcal{L}^+$

*Proof.* We observe that  $\iota(e)$  is aligned for each edge  $e$  of  $Q$  as a result of the requirement  $p_{11} < p_{12}$  in the definition of  $\iota$ . Then since  $\iota$  is an anti-homomorphism by [Proposition 8](#) it follows that the image of every element of  $kQ$  under  $\iota$  is aligned.  $\square$

**Lemma 17.** *If  $X \in M$  and no node  $\widehat{Z_1 Z_2}$  of  $X$  satisfies  $Z_1 = Z_2 \in \mathbb{N}$  then there exist  $A \in kM^+$  and  $Y \in \mathcal{N} + \mathcal{J}$  such that  $A = X + Y$ .*

*Proof.* If  $X$  is aligned, then we can take  $A = X$  and  $Y = 0$ . Otherwise let  $Z = \widehat{Z_1 Z_2}$  be a node of  $X$  for which  $\overline{Z_1} \geq \overline{Z_2}$ . We define an auxiliary element  $X' \in kM$  as follows. If  $\overline{Z_1} > \overline{Z_2}$  then let  $X'$  be the forest obtained from  $X$  by exchanging  $Z_1$  with  $Z_2$  so that  $X + X' \in \mathcal{N}$ . Otherwise suppose that  $\overline{Z_1} = \overline{Z_2}$ . Observe that one of  $Z_1$  or  $Z_2$  has positive length by hypothesis. If  $\ell(Z_2) > 0$  then  $Z = \widehat{Z_1 \begin{matrix} / \\ Z_{21} \\ \backslash \\ Z_{22} \end{matrix}}$  for some trees  $Z_{21}$  and  $Z_{22}$ . Let  $X'$  be obtained from  $X$  by replacing  $Z$  with  $\begin{matrix} / \\ Z_{22} \\ \backslash \\ Z_1 \end{matrix} Z_{21} + \begin{matrix} / \\ Z_{21} \\ \backslash \\ Z_{22} \end{matrix} Z_1$  so that  $X + X' \in \mathcal{J}$ . If  $\ell(Z_1) > 0$  then we can apply both replacements above to define an element  $X'$  such that  $X + X' \in \mathcal{N} + \mathcal{J}$ .

Observe that each term of  $X'$  has fewer nodes  $\widehat{U_1 U_2}$  with  $\overline{U_1} \geq \overline{U_2}$  than  $X$ . Then by induction  $A' = X' + Y'$  for some  $A' \in kM^+$  and some  $Y' \in \mathcal{N} + \mathcal{J}$ . Taking  $A = -A'$  and  $Y = -X - X' - Y'$  gives the result.  $\square$

The element  $A$  in Lemma 17 is called an *aligned rendering* of  $X$ . An aligned rendering of a forest need not be unique. For example, the forest  has aligned renderings

$$(10) \quad \begin{array}{c} 3 \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} 6 - \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} 6 \quad \text{and} \quad \begin{array}{c} 2 \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} 6 - \begin{array}{c} 1 \quad 3 \quad 2 \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} 6 - \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} 6 + \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} 6$$

obtained by applying the replacements in Lemma 17 to different nodes.

### 11. THE PATH ASSOCIATED TO A FOREST

Continuing the example at the beginning of §7 we recall that  $Q$  was constructed on the basis of the unique factorization of labeled forests. However, when mapping the path algebra of  $Q$  back to the algebra of labeled forests, we replaced the factors in such a factorization with their Pólya classes, which are more useful in light of our interest in  $\Sigma(\mathfrak{S}_n)$  but which break the factorization, as the example shows.

Specifically we associated the path  $P = p \cdot \langle \begin{smallmatrix} 1 & 1 \\ 3 & 2 \end{smallmatrix} \rangle$  to the class  $\left[ \begin{array}{c} \widehat{1} \quad \widehat{2} \quad 3 \\ \widehat{1} \quad 1 \quad 2 \end{array} \right]$  where  $p$

is the partition 34 and we found that  $\iota(P) = \left[ \begin{array}{c} \widehat{1} \quad \widehat{2} \quad 3 \\ \widehat{1} \quad 1 \quad 2 \end{array} \right] + \left[ \begin{array}{c} \widehat{1} \quad \widehat{2} \\ \widehat{1} \quad 3 \quad 1 \quad 2 \end{array} \right]$ . Applying

the same procedure instead to  $\left[ \begin{array}{c} \widehat{1} \quad \widehat{2} \\ \widehat{1} \quad 3 \quad 1 \quad 2 \end{array} \right]$  results in the same path  $P$ , so again the factorization fails. Motivated by this example, the purpose of this section is to precisely define the path associated to a labeled forest and to calculate its image under  $\iota$ . In §12 we show how the failure of factorization in  $\mathcal{L}$  can be resolved.

Consider the following transformations of a labeled forest.

- (1) exchanging two subtrees  $U$  and  $V$  for which  $\overline{U} = \overline{V}$  and the node labels of the parents of  $U$  and  $V$ , if they exist, are *smaller* than the node labels of  $U$  and  $V$ , if they exist
- (2) exchanging two parts of the forest

Note that both moves produce another labeled forest. We write  $X \sim Y$  for  $X, Y \in \mathcal{L}$  if  $Y$  can be obtained from  $X$  by applying a sequence of moves (1) or (2). Then  $\sim$  is an equivalence relation on  $\mathcal{L}$  that induces an equivalence relation on  $\mathcal{L}$ . Note that if  $X \sim Y$  then  $X$  is aligned if and only if  $Y$  is aligned. For example, the forests

$$(11) \quad \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 4 \quad 2 \\ \diagup \quad \diagdown \\ 1 \quad 2 \quad 1 \quad 3 \end{array} \quad \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 4 \quad 3 \\ \diagup \quad \diagdown \\ 1 \quad 3 \quad 1 \quad 4 \quad 2 \end{array} \quad \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 3 \quad 1 \\ \diagup \quad \diagdown \\ 1 \quad 3 \quad 1 \quad 2 \quad 4 \end{array} \quad \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 3 \quad 1 \\ \diagup \quad \diagdown \\ 1 \quad 1 \quad 2 \quad 3 \quad 4 \end{array}$$

are related by  $\sim$ .

As in the example at the beginning of this section, we can associate a path in  $Q$  to an aligned labeled forest through the map  $P : \mathcal{L}^+ \rightarrow kQ$  defined as follows. Suppose that  $X$  is an aligned labeled forest and let  $p$  be the composition  $\overline{X}$  regarded as a vertex of  $Q$ . Let  $a_1, b_1, a_2, b_2, \dots, a_l, b_l \in \mathbb{N}$  be such that the node  $\begin{array}{c} \widehat{i} \\ \widehat{z}_1 \quad \widehat{z}_2 \end{array}$  of  $X$  satisfies  $\overline{z}_1 = a_i$  and  $\overline{z}_2 = b_i$  for all  $1 \leq i \leq l$  where  $l = \ell(X)$ . Then we define  $P(X) = p \cdot \langle \begin{smallmatrix} a_1 & a_2 & \dots & a_l \\ b_1 & b_2 & \dots & b_l \end{smallmatrix} \rangle$ .

We observe that applying  $P$  to forests related to one another by  $\sim$  produces the same path. In particular, applying  $P$  to forests in the same Pólya class produces

the same path. Therefore we can define  $P[X] = P(X)$  for all  $X \in L^+$ . Finally, applying  $P$  to any terms of the image under  $\iota$  of any path  $P$  produces the same path by [Lemma 11](#), which must therefore be  $P$ . For example, if  $X$  is any of the forests in [\(11\)](#) then  $P[X] = p \cdot \langle \begin{smallmatrix} 4 & 3 & 1 & 1 \\ 7 & 4 & 3 & 2 \end{smallmatrix} \rangle$  where  $p$  is the partition containing the single part eleven.

The map  $P$  can also be formulated recursively as follows. Suppose again that  $X$  is an aligned labeled forest. If  $X$  has length zero, then we can regard  $X$  as a vertex of  $Q$  and take  $P(X) = X$ . Otherwise we define  $P(X) = P(Y) e$  where  $X', Y$  are as in [Lemma 2](#) and  $e$  is the edge of  $Q$  from  $X'$  to  $\overline{X'}$ . Note that  $[X'] = \iota(e)$  so that  $\iota$  and  $P$  are inverses of one another when restricted to elements of length one. The same is true of elements of length zero. The following lemma deals with the composition  $\iota \circ P$  in general.

**Lemma 18.** *If  $X \in L^+$  then  $\iota(P[X]) = \sum_{[U] \sim [X]} [U]$ .*

*Proof.* As mentioned above  $\iota(P[X]) = [X]$  if  $X$  has length zero or one. Otherwise let  $X', Y$  be as in [Lemma 2](#). Assuming by induction that  $\iota(P[Y]) = \sum_{[V] \sim [Y]} [V]$  we have

$$(12) \quad \iota(P[X]) = \iota(P[X' \bullet Y]) = \left[ \begin{array}{c} x_1 x_2 \cdots x_{i-1} \quad \widehat{\begin{array}{c} \phantom{x} \\ x_{i1} \quad x_{i2} \end{array}} \quad x_{i+1} \cdots x_j \end{array} \right] \bullet \sum_{[V] \sim [Y]} [V].$$

Note that all the terms  $[U]$  of [\(12\)](#) satisfy  $[U] \sim [X]$ . Conversely, suppose that  $[U] \sim [X]$ . We can assume that  $U$  can be obtained from  $X$  by exchanging a single pair of subtrees of the same squash since  $\sim$  is the reflexive and transitive closure of the set of all such pairs of forests. If the exchange moves the node labeled 1 then it must exchange it with another part of  $X$  since 1 is the smallest node label in  $X$ . Then  $[X] = [U]$ . Otherwise  $U = X' \bullet V$  for some forest  $V$  such that  $V \sim Y$ . This shows that  $[U]$  is a term of [\(12\)](#).  $\square$

## 12. PROOF OF THE QUIVER

In this section we prove that  $Q_n$  is the ordinary quiver of the algebra  $\Sigma(\mathfrak{S}_n)$ . We begin with a construction that produces an element  $F(X) \in L^+$  such that  $E(F(X)) = X$  for all  $X \in M^+$ . While this can be done by simply labeling the nodes of  $X$  in any legitimate way, the labeling provided by  $F$  is convenient in the proofs of the following results. If  $X$  has length zero, then  $X$  is also in  $L^+$  and we take  $F(X) = X$ . Otherwise suppose that  $X = X_1 X_2 \cdots X_j$  where  $X_1, X_2, \dots, X_j$  are trees. Let  $i$  be minimal such that  $\ell(X_i) > 0$  and let  $X_{i1}, X_{i2}$  be trees such that  $X_i = \widehat{\begin{array}{c} \phantom{x} \\ x_{i1} \quad x_{i2} \end{array}}$ . Let  $Y$  be the forest obtained from  $X_1 X_2 \cdots X_{i-1} X_{i1} X_{i2} X_{i+1} \cdots X_j$  by reducing all the node labels by one and write  $x_1 x_2 \cdots x_{i-1} x_{i1} x_{i2} x_{i+1} \cdots x_j = \overline{Y}$ . Then defining

$$F(X) = \left( \begin{array}{c} x_1 x_2 \cdots x_{i-1} \quad \widehat{\begin{array}{c} \phantom{x} \\ x_{i1} \quad x_{i2} \end{array}} \quad x_{i+1} \cdots x_j \end{array} \right) \bullet F(Y)$$

we have  $E(F(X)) = X$  by induction. Note that the nodes of any part of  $F(X)$  are labeled in prefix order and are smaller than those in the following part. For



parent of  $V$  would have a larger node label than  $U$ . We conclude that the leftmost subtree permuted by the cycle is a leaf and that the subtrees of positive length all move to the left, resulting in a forest which under  $E$  is lexicographically smaller than  $X$ .  $\square$

Assembling the results above yields the following main results of this section.

**Proposition 20.**  $E \circ \iota : kQ \rightarrow k\mathcal{M}^+$  is surjective.

*Proof.* Let  $X$  be a nondecreasing element of  $\mathcal{M}^+$  and put  $P = P[F(X)]$ . Then  $\iota(P) = \sum_{[U] \sim [F(X)]} [U]$  by Lemma 18 so that taking  $\mathcal{Y} = E(\iota(P) - [F(X)])$  we have  $[Y] < [X]$  for each term  $[Y]$  of  $\mathcal{Y}$  by Lemma 19. Repeating the argument for all the terms of  $\mathcal{Y}$  and subtracting the result from  $P$  results in an element of  $kQ$  mapping to  $[X]$  under  $E \circ \iota$  by induction.  $\square$

For example, applying Proposition 20 to  $\left[ \begin{array}{ccc} & \hat{1} & \\ 1 & \hat{2} & 3 \\ & 1 & 2 \end{array} \right]$  produces the path  $p \cdot \langle \begin{smallmatrix} 1 & 1 \\ 3 & 2 \end{smallmatrix} \rangle - p \cdot \langle \begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix} \rangle$  where  $p$  is the partition 34.

**Corollary 21.**  $\iota$  is surjective modulo  $\ker \Delta$ .

*Proof.* Let  $X \in L$ . We will show that some element of  $kQ$  maps under  $\iota$  to an element of  $k\mathcal{L}^+$  congruent to  $[X]$  modulo  $\ker \Delta$ . If  $X$  has a node  $\hat{i}$  for which  $Z_1 = Z_2 \in \mathbb{N}$  then  $[X] \in \ker \Delta$  and we can take  $P = 0$ . Otherwise by Lemma 17 applied to all the terms of  $E[X]$  there exist  $A \in k\mathcal{M}^+$  and  $\mathcal{Y} \in \mathcal{N} + \mathcal{J}$  such that  $E[X] = A + \mathcal{Y}$ . Applying  $F$  we have  $[X] \equiv F(A) \pmod{\ker \Delta}$ . By Proposition 20 we have  $P \in kQ$  such that  $E(\iota(P)) = A$  so that  $\iota(P) - F(A) \in \ker E \subseteq \ker \Delta$ . It follows that  $\iota(P) \equiv F(A) \equiv [X] \pmod{\ker \Delta}$ .  $\square$

**Proposition 22.**  $Q_n$  is the ordinary quiver of  $\Sigma(\mathfrak{S}_n)$ .

*Proof.* Let  $I = \iota^{-1}(\ker \Delta)$  so that  $kQ_n/I \cong \iota(kQ_n)/\ker \Delta$  since  $\iota$  is injective by Corollary 12. But  $\iota(kQ_n)/\ker \Delta \cong k\mathcal{L}_n/\ker \Delta$  by Corollary 21 and  $k\mathcal{L}_n/\ker \Delta \cong \Sigma(\mathfrak{S}_n)^{\text{op}}$  by Theorem 7. Let  $R$  be the Jacobson radical of  $kQ_n$ . Then  $R$  is generated by all paths of  $Q_n$  of positive length. Since  $Q_n$  is the ordinary quiver of any quotient of  $kQ_n$  by an ideal contained in  $R^2$  by [1, Lemma 3.6] it suffices to show that  $I \subseteq R^2$ .

Let  $P$  be any element of  $I$ . By multiplying  $P$  on the left and on the right by various vertices of  $Q_n$  we can split  $P$  into a sum of elements of  $I$  all of whose terms have the same source and destination. We therefore assume that all the terms of  $P$  have the same source and destination and hence the same length. If this length were zero or one, then  $P$  would be a multiple of a vertex or an edge. But  $\Delta(\iota(p)) = p \neq 0$  for all vertices  $p$ , while  $\Delta(\iota(e)) \neq 0$  for all edges  $e$  by Lemma 9. Therefore  $P \in R^2$ .  $\square$

### 13. THE RELATIONS

In this section we state our conjecture on the relations for the quiver presentation of  $\Sigma(\mathfrak{S}_n)$ . Let  $\mathcal{R} \subseteq k\mathcal{B}^*$  be the set of elements

$$(14) \quad \left\langle \begin{array}{c} a \ c \\ b \ d \end{array} \right\rangle - \left\langle \begin{array}{c} c \ a \\ d \ b \end{array} \right\rangle \quad \text{where} \quad a + b \notin \{c, d\} \quad \text{and} \quad c + d \notin \{a, b\}$$

and the elements

$$(15) \quad \left\langle \begin{array}{c} \mathbf{a} \ \mathbf{c} \ \mathbf{x} \\ \mathbf{b} \ \mathbf{d} \ \mathbf{y} \end{array} \right\rangle + \left\langle \begin{array}{c} \mathbf{x} \ \mathbf{a} \ \mathbf{c} \\ \mathbf{y} \ \mathbf{b} \ \mathbf{d} \end{array} \right\rangle - \left\langle \begin{array}{c} \mathbf{a} \ \mathbf{x} \ \mathbf{c} \\ \mathbf{b} \ \mathbf{y} \ \mathbf{d} \end{array} \right\rangle - \left\langle \begin{array}{c} \mathbf{c} \ \mathbf{x} \ \mathbf{d} \\ \mathbf{d} \ \mathbf{y} \ \mathbf{b} \end{array} \right\rangle$$

where  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$  satisfy the condition in (14) and either

- (1)  $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} \in \{\mathbf{x}, \mathbf{y}\}$  or
- (2)  $\mathbf{x} + \mathbf{y} \in \{\mathbf{a}, \mathbf{b}\} \cap \{\mathbf{c}, \mathbf{d}\}$ .

The elements of  $\mathcal{R}$  are called *branch relations*. The following proposition shows that the branch relations produce relations when applied to vertices of  $Q$ .

**Proposition 23.** *If  $R \in \mathcal{R}$  then  $\mathbf{p} \cdot R \in \ker(E \circ \iota)$  for all vertices  $\mathbf{p}$  of  $Q$ .*

*Proof.* Suppose  $R = \left\langle \begin{array}{c} \mathbf{a} \ \mathbf{c} \\ \mathbf{b} \ \mathbf{d} \end{array} \right\rangle - \left\langle \begin{array}{c} \mathbf{c} \ \mathbf{a} \\ \mathbf{d} \ \mathbf{b} \end{array} \right\rangle$  where  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{N}$  satisfy the condition in (14). Then for any partition  $\mathbf{p}$  we have  $\iota(\mathbf{p} \cdot \left\langle \begin{array}{c} \mathbf{a} \ \mathbf{c} \\ \mathbf{b} \ \mathbf{d} \end{array} \right\rangle) = \left[ \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{q}_1 \cdots \mathbf{q}_j \right]$  and  $\iota(\mathbf{p} \cdot \left\langle \begin{array}{c} \mathbf{c} \ \mathbf{a} \\ \mathbf{d} \ \mathbf{b} \end{array} \right\rangle) = \left[ \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{q}_1 \cdots \mathbf{q}_j \right]$  if  $\mathbf{p}$  has parts  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{c} + \mathbf{d}$  where  $\mathbf{q}_1, \dots, \mathbf{q}_j$  are the remaining parts of  $\mathbf{p}$ , while both expressions are zero if  $\mathbf{p}$  has no part  $\mathbf{a} + \mathbf{b}$  or no part  $\mathbf{c} + \mathbf{d}$ . This shows that  $E(\iota(\mathbf{p} \cdot R)) = 0$  for all vertices  $\mathbf{p}$ .

Now let  $R$  be the element in (15) and suppose  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{x}, \mathbf{y} \in \mathbb{N}$  satisfy condition (1) of the definition of  $\mathcal{R}$ . Specifically, we assume that  $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} = \mathbf{x}$  although the argument can be easily modified if  $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} = \mathbf{y}$ . In each of the cases that

- (1)  $\mathbf{p}$  has at least one part  $\mathbf{x} + \mathbf{y}$  and exactly one part  $\mathbf{x}$
- (2)  $\mathbf{p}$  has at least one part  $\mathbf{x} + \mathbf{y}$  and two or more parts  $\mathbf{x}$
- (3)  $\mathbf{p}$  has no part  $\mathbf{x} + \mathbf{y}$  or no part  $\mathbf{x}$

the image of  $\mathbf{p} \cdot R$  can be calculated explicitly. In the third case all four terms of  $\mathbf{p} \cdot R$  are zero. In the second case we take  $\mathbf{p}$  to be the partition with parts  $\mathbf{x} + \mathbf{y}, \mathbf{x}, \mathbf{x}, \mathbf{q}_1, \dots, \mathbf{q}_j$  for any  $\mathbf{q}_1, \dots, \mathbf{q}_j \in \mathbb{N}$ . We calculate

$$\begin{aligned} \mathbf{p} \cdot \left\langle \begin{array}{c} \mathbf{x} \ \mathbf{a} \ \mathbf{c} \\ \mathbf{y} \ \mathbf{b} \ \mathbf{d} \end{array} \right\rangle &\xrightarrow{E \circ \iota} \left[ \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{y}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{x} \mathbf{q}_1 \cdots \mathbf{q}_j \right] + \left[ \widehat{\mathbf{c}} \widehat{\mathbf{d}} \widehat{\mathbf{y}} \widehat{\mathbf{a}} \widehat{\mathbf{b}} \ \mathbf{x} \mathbf{q}_1 \cdots \mathbf{q}_j \right] + \left[ \widehat{\mathbf{x}} \widehat{\mathbf{y}} \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{q}_1 \cdots \mathbf{q}_j \right] \\ \mathbf{p} \cdot \left\langle \begin{array}{c} \mathbf{a} \ \mathbf{x} \ \mathbf{c} \\ \mathbf{b} \ \mathbf{y} \ \mathbf{d} \end{array} \right\rangle &\xrightarrow{E \circ \iota} \left[ \widehat{\mathbf{c}} \widehat{\mathbf{d}} \widehat{\mathbf{y}} \widehat{\mathbf{a}} \widehat{\mathbf{b}} \ \mathbf{x} \mathbf{q}_1 \cdots \mathbf{q}_j \right] + \left[ \widehat{\mathbf{x}} \widehat{\mathbf{y}} \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{q}_1 \cdots \mathbf{q}_j \right] \\ \mathbf{p} \cdot \left\langle \begin{array}{c} \mathbf{c} \ \mathbf{x} \ \mathbf{a} \\ \mathbf{d} \ \mathbf{y} \ \mathbf{b} \end{array} \right\rangle &\xrightarrow{E \circ \iota} \left[ \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{y}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{x} \mathbf{q}_1 \cdots \mathbf{q}_j \right] + \left[ \widehat{\mathbf{x}} \widehat{\mathbf{y}} \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{q}_1 \cdots \mathbf{q}_j \right] \\ \mathbf{p} \cdot \left\langle \begin{array}{c} \mathbf{a} \ \mathbf{c} \ \mathbf{x} \\ \mathbf{b} \ \mathbf{d} \ \mathbf{y} \end{array} \right\rangle &\xrightarrow{E \circ \iota} \left[ \widehat{\mathbf{x}} \widehat{\mathbf{y}} \widehat{\mathbf{a}} \widehat{\mathbf{b}} \widehat{\mathbf{c}} \widehat{\mathbf{d}} \ \mathbf{q}_1 \cdots \mathbf{q}_j \right] \end{aligned}$$

so that  $E(\iota(\mathbf{p} \cdot R)) = 0$ . The first case is similar to the second and the calculation in the case that  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{x}, \mathbf{y}$  satisfy condition (2) of the definition of  $\mathcal{R}$  is similar to the calculation above.  $\square$

For unlabeled trees  $X, Y, Z$  we denote  $\widehat{X} \widehat{Y} \widehat{Z} + \widehat{Z} \widehat{X} \widehat{Y} + \widehat{Y} \widehat{Z} \widehat{X}$  by  $j(X, Y, Z)$ .

Suppose that  $A = \sum_{i=1}^m A_i$  is an aligned rendering of  $j(X, Y, Z)$  where  $A_1, A_2, \dots, A_m$  are aligned unlabeled trees. If it exists,  $A$  may have more or fewer than three terms and satisfies  $A - j(X, Y, Z) \in \ker \pi$  by Lemma 17. But since  $j(X, Y, Z) \in \ker \pi$  we have  $A \in \ker \pi$ . For any unlabeled forests  $U$  and  $V$  we denote by  $UV$  the forest whose parts are the parts of  $U$  followed by the parts of  $V$ . This applies in particular

when  $V$  is a composition, which we regard as a forest of length zero. For any composition  $q$  we apply [Proposition 20](#) to each class  $[A_i q]$ . This produces an element  $P \in kQ$  such that  $E(\iota(P)) = \sum_{i=1}^m [A_i q]$ . Then  $P \in \ker(\Delta \circ \iota)$ .

Let  $\mathcal{S}$  be the set of elements  $P \in kQ$  for which  $E(\iota(P)) = \sum_{i=1}^m [A_i q]$  for some composition  $q$  and some aligned rendering  $\sum_{i=1}^m A_i$  of an element of the form

- (1)  $j(x, y, z)$  where  $x < y < z$  are natural numbers such that  $x + y \neq z$ , or
- (2)  $j(\widehat{x_1 x_2}, y, z)$  where  $x_1 < x_2$  and  $y < z$  are natural numbers such that  $x_1 + x_2 \in \{y, z, y + z\}$ .

Then the elements of  $\mathcal{S}$  are relations for the quiver presentation of  $\Sigma(\mathfrak{S}_n)$ . Observe that elements of the form (1) have only one possible aligned rendering, while elements of the form (2) have only one “useful” aligned rendering. For example,

the term  $\overset{6}{\swarrow} \overset{1}{\swarrow} \overset{2}{\searrow} \overset{3}{\searrow}$  of  $j(\widehat{1\ 2}, 3, 6)$  has the two aligned renderings shown in (10)

but only the second can be used to construct a relation, since the terms of the first aligned rendering cancel the other terms of  $j(\widehat{1\ 2}, 3, 6)$ .

We conjecture that the relations above generate the ideal of relations for the quiver presentation of  $\Sigma(\mathfrak{S}_n)$  in the following way.

**Conjecture 24.** *The descent algebra  $\Sigma(\mathfrak{S}_n)$  has a presentation as the path algebra  $kQ_n$  subject to the relations  $\mathcal{S} \cap kQ_n$  and  $p.R$  for all partitions  $p$  of  $n$  and all  $R \in \mathcal{R}$ . In particular, the relations all have length two or three.*

We have verified [Conjecture 24](#) through a computer calculation for  $n \leq 15$ . In fact, we have implemented a procedure in [GAP \[12\]](#) which calculates minimal projective resolutions over the algebra  $A = kQ_n / \ker(\Delta \circ \iota)$  of the simple module  $(A / \text{Rad } A) p$  for all partitions  $p$  of  $n$ . One result of the calculation is a minimal generating set of  $\ker(\Delta \circ \iota)$  which can be used to confirm that the presentation in [Conjecture 24](#) is correct for small  $n$ .

[Table 1](#) shows the minimal number of relations for the presentation of  $\Sigma(\mathfrak{S}_n)$  for  $n \leq 15$ . The table also shows the numbers of branch and Jacobi relations. Note that when  $n \geq 10$  the total number of branch and Jacobi relations exceeds the size of the minimal generating set. Nonetheless, the ideal generated by the branch and Jacobi relations is exactly  $\ker(\Delta \circ \iota)$  in every case shown in [Table 1](#).

#### 14. EXAMPLE

As an example of [Conjecture 24](#) we calculate the presentation for  $\Sigma(\mathfrak{S}_8)$ . The quiver for this presentation is shown in [Figure 1](#). To calculate the branch relations, we list all  $R \in \mathcal{R}$  and apply them to all vertices  $p$  of  $Q_8$ . Those resulting in nonzero relations are shown in the column labeled  $P$  of [Table 2](#). To calculate the Jacobi relations, we list all tuples  $x, y, z$  and  $x_1, x_2, y, z$  satisfying the conditions in the definition of  $\mathcal{S}$ . For each partition  $q = q_1 q_2 \cdots q_j$  completing  $x + y + z$  or  $x_1 + x_2 + y + z$  to a composition  $p$  of  $n$  we find an element  $P \in kQ$  for which  $E(\iota(P)) = \sum_{i=1}^m [A_i q]$  where  $\sum_{i=1}^m A_i$  is an aligned rendering of  $j(x, y, z)$  or  $j(\widehat{x_1 x_2}, y, z)$ . These relations are also shown in [Table 2](#).

As mentioned in [§13](#) we have verified through a computer calculation that the quotient of  $kQ_8$  by the ideal generated by the elements in [Table 2](#) is isomorphic to  $\Sigma(\mathfrak{S}_8)$ .

TABLE 1. Numbers of Relations

n	Branch	Jacobi	Minimal
6	0	1	1
7	1	3	4
8	4	7	11
9	10	14	24
10	22	29	48
11	44	51	90
12	86	89	160
13	152	146	270
14	265	240	444
15	441	369	705

TABLE 2. Relations for  $\Sigma(\mathfrak{S}_8)$ 

p	R	P
35	$\langle \begin{smallmatrix} 1 & 1 \\ 2 & 4 \end{smallmatrix} \rangle - \langle \begin{smallmatrix} 1 & 1 \\ 4 & 2 \end{smallmatrix} \rangle$	$jq - kt$
134	$\langle \begin{smallmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{smallmatrix} \rangle + \langle \begin{smallmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \end{smallmatrix} \rangle - 2\langle \begin{smallmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{smallmatrix} \rangle$	$adm - 2ack$
35	$\langle \begin{smallmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \end{smallmatrix} \rangle + \langle \begin{smallmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \end{smallmatrix} \rangle - 2\langle \begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{smallmatrix} \rangle$	$bis - 2bhq$
44	$\langle \begin{smallmatrix} 1 & 1 & 1 \\ 3 & 3 & 2 \end{smallmatrix} \rangle + \langle \begin{smallmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \end{smallmatrix} \rangle - 2\langle \begin{smallmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \end{smallmatrix} \rangle$	$dmu - 2cku$
116	$j\left(\widehat{\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}}, 1, 2\right)$	$acl - aen$
17	$j(1, 2, 4)$	$jr + kv - lw$
17	$j\left(\widehat{\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}}, 1, 3\right)$	$2ckv + clw - dmv - enw$
26	$j\left(\widehat{\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}}, 1, 2\right)$	$bgo - bhq$
8	$j(1, 2, 5)$	$px + qy - rz$
8	$j\left(\widehat{\begin{smallmatrix} 1 & 3 \\ 1 & 3 \end{smallmatrix}}, 1, 3\right)$	$mtu - mvz$
8	$j\left(\widehat{\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}}, 2, 3\right)$	$gox - hpx - 2hgy - isy$

## REFERENCES

- [1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.
- [2] M. D. Atkinson. A new proof of a theorem of Solomon. *Bull. London Math. Soc.*, 18(4):351–354, 1986.
- [3] M. D. Atkinson. Solomon’s descent algebra revisited. *Bull. London Math. Soc.*, 24(6):545–551, 1992.

- [4] Thorsten Bauer. Über die Struktur der Solomon-Algebren: ein Zugang über einen Differential- und Integralkalkül. *Bayreuth. Math. Schr.*, (63):1–102, 2001.
- [5] F. Bergeron and N. Bergeron. A decomposition of the descent algebra of the hyperoctahedral group. I. *J. Algebra*, 148(1):86–97, 1992.
- [6] F. Bergeron, N. Bergeron, R. B. Howlett, and D. E. Taylor. A decomposition of the descent algebra of a finite Coxeter group. *J. Algebraic Combin.*, 1(1):23–44, 1992.
- [7] François Bergeron and Nantel Bergeron. Orthogonal idempotents in the descent algebra of  $B_n$  and applications. *J. Pure Appl. Algebra*, 79(2):109–129, 1992.
- [8] Dieter Blessenohl and Hartmut Laue. On the descending Loewy series of Solomon’s descent algebra. *J. Algebra*, 180(3):698–724, 1996.
- [9] Dieter Blessenohl and Hartmut Laue. The module structure of Solomon’s descent algebra. *J. Aust. Math. Soc.*, 72(3):317–333, 2002.
- [10] Dieter Blessenohl and Manfred Schocker. *Noncommutative character theory of the symmetric group*. Imperial College Press, London, 2005.
- [11] C. Bonnafé and G. Pfeiffer. Around Solomon’s descent algebras. *Algebr. Represent. Theory*, 11(6):577–602, 2008.
- [12] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.
- [13] A. M. Garsia and C. Reutenauer. A decomposition of Solomon’s descent algebra. *Adv. Math.*, 77(2):189–262, 1989.
- [14] Götz Pfeiffer. Quiver presentations for descent algebras of exceptional type. 2008.
- [15] Götz Pfeiffer. A quiver presentation for Solomon’s descent algebra. *Adv. Math.*, 220(5):1428–1465, 2009.
- [16] Franco V. Saliola. On the quiver of the descent algebra. *J. Algebra*, 320(11):3866–3894, 2008.
- [17] Franco V. Saliola. The Loewy length of the descent algebra of type D. *Algebr. Represent. Theory*, 13(2):243–254, 2010.
- [18] Manfred Schocker. The descent algebra of the symmetric group. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 145–161. Amer. Math. Soc., Providence, RI, 2004.
- [19] Louis Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.

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