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PhD Thesis

Explicit Computations of Ask Zeta Functions

OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Sultan Alzahrani

Supervisor: Tobias Rossmann

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Introduction

The origin of zeta functions goes back to Leonhard Euler, who introduced what we now call a zeta function as a function of a real variable

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where the infinite sum converges when $s > 1$. Euler also proved that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

which is called the **Euler decomposition** of $\zeta(s)$. This decomposition gave us what is known as the **local zeta function**

$$\zeta_p(s) = \frac{1}{1 - p^{-s}}$$

at the prime p. Bernhard Riemann extended the definition of Euler's zeta function to a complex variable. This function converges when $\mathrm{Re}(s) > 1$. It is known as the **Riemann zeta function**, which led to the development of the renowned Riemann hypothesis.

Later, Richard Dedekind generalized the Riemann zeta function to number fields. Let K be a number field with ring of integers $\mathcal{O}$. For a complex variable s, the **Dedekind zeta function**

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}} |\, \mathcal{O} : \mathfrak{a}|^{-s}$$

converges when $\mathrm{Re}(s) > 1$, where $\mathfrak{a}$ ranges over the nonzero ideals of $\mathcal{O}$. The Dedekind zeta function admits the Euler decomposition

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - |\, \mathcal{O} : \mathfrak{p}|^{-s}},$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$. The Riemann zeta function is recovered for $K = \mathbb{Q}$.

Gustave Lejeune Dirichlet modified the Riemann zeta function by multiplying each term of the sum by a term from a complex sequence $(a_n)_n$. Such a series

$$\sum_{n=1}^{\infty} a_n n^{-s}$$

is known as a **Dirichlet series**. For more historical context, see [20, Section 1.1].

A Dirichlet series can be given by a sequence $(a_n)_n$ where each term of the sequence enumerates a finite number of algebraic objects. For instance, let G be a finitely generated, torsion-free, and nilpotent group. Let $a_n(G)$ be the number of subgroups of G of finite index n. Then the Dirichlet series

$$\zeta_G(s) = \sum_{n=1}^{\infty} a_n(G) n^{-s}$$

is known as the **subgroup zeta function** of G.

Grunewald, Segal, and Smith [26] were the first to study (normal) subgroup zeta functions. They introduced zeta functions as a tool for invistegating asymptotic and arithmetic aspects of subgroup growth. Later, du Sautoy and Grunewald [19] studied analytic properties of (normal) subgroup zeta functions of nilpotent groups. Various types of zeta functions later emerged and were studied as zeta functions became important tools to study asymptotic group and ring theory. See the book [20] as well as the surveys [33, 46, 57, 58].

Let R be a ring. By a **matrix of linear forms** over R, we mean a matrix of the form $A(X) := A_1X_1 + \ldots + A_\ell X_\ell$ where $A_1, \ldots, A_\ell \in \mathrm{M}_{d\times e}(R)$ and $X = (X_1, \ldots, X_\ell)$ consists of algebraically independent variables over R (i.e. $X_1, \ldots, X_2$ do not satisfy a trivial polynomial over R). Such a matrix of linear forms $A(X)$ defines two types of modules. The first type is called the **image module** of $A(X)$, defined as $M(A(X); R) := \{A(x) : x \in R^\ell\}$. (This is a submodule of $\mathrm{M}_{d\times e}(R)$.) Define $[d] = \{1, 2, \ldots, d\}$ and write $A_k = [a_{kij}]$ for $1 \leq k \leq \ell$. The second type is called the **relation module** of $A(X)$, defined as $\mathrm{Rel}(A(X); R) := \{[x_{ij}] \in \mathrm{M}_{d\times e}(R) : \forall k \in [\ell], \sum_{(i,j)\in[d]\times[e]} a_{kij}x_{ij} = 0\}$.

We devote this thesis to the study of a particular type of zeta function: ask zeta functions associated with modules of matrices and matrices of linear forms (single matrices and matrix pencils, in particular). These zeta functions are obtained by averaging over the sizes of the kernels of matrices obtained from modules of matrices or module representations. Next, we go over some of these ingredients in greater detail.

Matrix pencils

A **matrix pencil** is a matrix of linear forms of the form $A_1X_1 + A_2X_2$ where the matrices A_1 and A_2 are linearly independent. Matrix pencils can be partitioned into classes through notions called equivalence and weak equivalence. These relations arise from elementary row and column operations as well as (in the case of weak equivalence) from a change of variables, see Definitions 1.1.2–1.1.3. A natural problem that arises over a given base field is selecting a representative matrix pencil in a "normal form" from each class. (We only consider this problem over infinite fields, not over general rings.)

This classification problem was introduced in the literature more than a century ago. Weierstrass was the first to introduce irrational forms of non-singular matrix pencils (see Section 2.1 for a definition). He defined these

forms in terms of the elementary divisors of the non-singular matrix pencil. Weierstrass's irrational normal form may contain entries in an extension of the base field. Kronecker considered irrational normal forms of the singular case of matrix pencils using Jordan normal forms. For Weierstrass and Kronecker matrix pencil forms, see [31, Section 3]. Later, Frobenius introduced rational forms for non-singular matrix pencils. Turnbull and Aitken [53, Chapter IX] gave a full account of the theory of matrix pencils using rational forms of singular and non-singular matrix pencils. See the historical notes [53, Section IX.16] as well as the survey [31]. In this thesis, we use their rational forms of matrix pencils since these forms can be obtained over any infinite field.

Ask zeta functions

Let $X = (X_1, \ldots, X_\ell)$ be algebraically independent variables over $\mathbb{Z}$. Let $A(X) \in \mathrm{M}_{d\times e}(\mathbb{Z}[X])$ be a matrix of linear forms. Let $\bar{A}_n(X)$ be the image of $A(X)$ under the natural map $\mathrm{M}_{d\times e}(\mathbb{Z}[X]) \to \mathrm{M}_{d\times e}((\mathbb{Z}/n\mathbb{Z})[X])$. Then the zeta function

$$\zeta^{\mathsf{ask}}_{A(X)}(s) = \sum_{n=1}^{\infty} \mathrm{ask}(\bar{A}_n(X))n^{-s}$$

is called the **ask zeta function** associated with $A(X)$ where $\mathrm{ask}(\bar{A}_n(X))$ is the average size of the kernel of $\bar{A}_n(x)$ for all $x \in (\mathbb{Z}/n\mathbb{Z})^\ell$, defined by

$$\mathrm{ask}(\bar{A}_n(X)) = \frac{1}{n^\ell} \sum_{x\in(\mathbb{Z}/n\mathbb{Z})^\ell} |\ker(\bar{A}_n(x))|.$$

Rossmann [47, 49] introduced ask zeta functions and studied the effects of so-called Knuth duality on these zeta functions. Later, Carnevale and Rossmann [11] studied the effect of passing to relation modules associated with "admissible partial colourings". Rossmann and Voll [50] studied and introduced ask zeta functions associated with graphs and hypergraphs. They also gave an explicit formula to compute ask zeta functions associated with hypergraphs.

Related types of zeta functions

Some other types of zeta functions, such as orbit-counting and conjugacy class zeta functions, can be instances of ask zeta functions in some cases.

For a simple type of conjugacy class zeta functions, let $G \leq \mathrm{GL}_d(\mathbb{Z})$. Define G_n to be the image of G under the natural map $\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{Z}/n\mathbb{Z})$. Then the **conjugacy class zeta function** of G is

$$\zeta^{\mathsf{cc}}_G(s) = \sum_{n=1}^{\infty} k(G_n)n^{-s},$$

where $k(G_n)$ denotes the number of conjugacy classes of G_n.

Similar conjugacy class zeta functions were introduced by du Sautoy [18]. They were further studied by Berman et al. [8] for Chevalley groups and Lins [14–16] for unipotent groups. For the correspondence between ask and

conjugacy class zeta functions, see [11,47,49,50]. Conjugacy class zeta functions have other names found in the literature such as "class number zeta functions" and "class counting zeta functions".

For $G \leq \mathrm{GL}_d(\mathbb{Z})$, the **orbit-counting zeta function** of G is

$$\zeta_G^{\mathsf{oc}}(s) = \sum_{n=1}^{\infty} |(\mathbb{Z}/n\mathbb{Z})^d/G|^{-s}.$$

Avni et al. computed a certain type of orbit-counting zeta functions, see [2, Theorem E]. Rossmann [47] investigated cases in which orbit-counting zeta functions are instances of ask zeta functions.

Main results

Our main results in this thesis consist of Theorems A, B, C, D, E, F, and G.

Conjugacy class zeta functions: Rossmann [47, Theorem 1.7] showed that conjugacy class zeta functions of particular types of Lie groups can be converted into ask zeta functions of Lie algebras. In Theorem A, we prove a formula for converting conjugacy class zeta functions of a particular type of saturable Lie groups into ask zeta functions of Lie algebras. Theorem A shows that the conjugacy class zeta functions of this type of saturable Lie groups are instances of ask zeta functions of Lie algebras. The formulas in Theorem A and [47, Theorem 1.7] allow us to study conjugacy class zeta functions of these types of groups using methods developed for computing ask zeta functions.

Antisymmetric matrices and Pfaffians: Rossmann [47, Section 4] gave explicit formulas for computing ask zeta functions of modules of matrices using sets of minors. In Section 6.1, we write an explicit formula for computing ask zeta functions of modules of antisymmetric matrices using Pfaffians. A similar method was used by Avni et al. [1, Section 3] to compute another type of zeta functions called representation zeta functions. Moreover, we use the formula in Section 6.1 to prove Theorem B which gives an explicit formula for ask zeta functions of image modules of antisymmetric matrices of linear forms with smooth Pfaffian hypersurfaces. The statement in Theorem B is analogous to [47, Theorem 7.1] which deals with image modules of matrices with smooth determinantal hypersurfaces. Moreover, a similar formula can be found in [55, Theorem 3] for the normal zeta function of class 2-nilpotent groups with smooth Pfaffian hypersurfaces. A type of group scheme can be constructed by antisymmetric matrices, see [50, Section 1.2]. The conjugacy class zeta functions of this group scheme are instance of ask zeta functions of the modules of these antisymmetric matrices, see [50, Proposition 1.1]. Therefore, the formula in Section 6.1 and Theorem B can be also used for studying the conjugacy class zeta functions of this group scheme.

Image modules: Rossmann [45] studied zeta functions enumerating submodules invariant given by a constant matrix. In Chapter 7, we give a generic and explicit (Theorem C) formula for the local ask zeta function of a module generated by a constant matrix. We call this type of modules the image modules associated with single matrices (i.e. $A(X) = A_1X_1$). We also obtain an explicit and generic formula for the local ask zeta functions of an image module of a row or column of linear forms which arises from the so-called Knuth duals, see [49, Section 4.2] and Section 1.4. Moreover, we write formulas for the global ask zeta function of an image module associated with a single matrix and its Knuth duals. In Theorem D, we give a generic formula for ask zeta functions of image modules associated with matrix pencils. A similar work has been done by Voll for the normal zeta functions of class 2-nilpotent groups, see [56, Theorem 2]. The generic formula in [56, Theorem 2] has similar shape as the formula in Theorem D. We note that the map ϕ_G in [56, Theorem 4] is represented by an antisymmetric matrix pencil.

Relation modules: We define a type of modules, called Relation modules associated with matrices of linear forms, motivated by the work of Carnevale and Rossmann [11] on relation modules associated with partial colorings. In Section 9.1, we replace partial colorings with a matrix of linear forms associated to a sequence of pairwise disjoint sets. This allows us to use the results of the so-called admissible partial colorings in [11, Corollary B] and our results on the so-called weak equivalence of matrices of linear forms to study ask zeta functions of relation modules associated to matrices of linear forms (single matrices and matrix pencils, in particular). Theorem E gives generic formulas for ask zeta functions of relation modules associated with single matrices. Furthermore, we prove Theorem F, which provides a formula for the effects of adding zero rows and columns to a matrix pencil on the ask zeta function of the relation module associated with that matrix pencil. The formula we obtained in Theorem F is analogous to the formula for the effects of adding generic rows and columns on ask zeta functions associated with hypergraphs, see [50, Proposition 5.24]. Moreover, we write generic formulas for ask zeta functions of relation modules associated with arbitrary matrix pencils (Theorem G).

Other results

Weak equivalence: We introduce a notion called weak equivalence between matrices of linear forms. We also proved that weakly equivalent matrices of linear forms give rise to the so-called isotopic module representations, see Section 1.3. Rossmann [49, Section 3.1] noted that isotopic module representations have the same ask zeta functions. In Section 4.4, we show that image modules and relation modules associated to weakly equivalent matrices of linear forms have the same ask zeta functions. Therefore, we use weak equivalence as a tool to study ask zeta functions of image modules and relation modules associated to matrices of linear forms.

Matrix pencils: The literature on rational forms of matrix pencils, found in [52, 53], is older than 80 years. We modernize their language that is used for obtaining rational forms of singular and non-singular matrix pencils. We also write algorithms to compute rational forms of both singular and non-singular matrix pencils. Each weak equivalence class can be represented by a rational form, and we prove that all modules within a weak equivalence class have the same generic ask zeta functions. The rational forms of matrix pencils are used in Chapter 8 and Chapter 10 to compute ask zeta functions of image modules and relation modules associated with matrix pencils, respectively.

Counting numbers of matrices: Bender [7] gave a formula for calculating the number of traceless matrices by rank over a finite field. In Section 9.2, we developed a formula for calculating the number of matrices by rank of a vector space after adding generic rows and columns. We use this formula to count the number of matrices by rank of relation vector spaces associated with single matrices which generalizes the formula for counting the number of traceless matrices by rank over a finite field. We also use this formula to write equivalent classes of relation modules associated to single matrices.

Strategy for computing ask zeta functions

Here, we sketch our strategy for computing ask zeta functions of modules of matrices. We use a formula to compute ask zeta functions of modules of matrices by integrating norms of set of minors of said modules over a compact discrete valuation ring. Corollary 4.4.1 shows that equivalent modules of matrices have the same ask zeta functions. Hence, we choose a suitable equivalence class with minors that can be computed using Proposition 4.5.1. In some cases, computing ask zeta functions of modules which arise from one of Knuth duals are less complicated than computing the original modules, e.g. see Lemma 10.5.2. Therefore, we obtain these modules and use Proposition 4.3.3 to obtain the ask zeta functions of the original modules. For computing the integral in Proposition 4.5.1 or Proposition 4.5.2, we use the strategy of "divide and compute" by dividing the domain of the integral into sets comes from rational cones. In each set, we break the polynomials of the minors into monomials using some statements derived from Hensel's lemma or using change of variables. Then we compute the generating functions of each cone. Finally, we use Corollary 3.5.6 to compute the integral of each set.

Overview of the thesis

The first four chapters are background material. In Chapter 1, we introduce weak equivalence of matrices of linear forms and recall some terminology about module representations and Knuth duals. We also prove that module representations which arise from weakly equivalent matrices of linear forms are isotopic. In Chapter 2, we recall the notion of rational forms of matrix pencils from [53]. We also give algorithms to obtain these rational forms. In Chapter

3, we recall some background on non-Archimedean local fields and valuation rings. We also write statements derived from Hensel's lemma and give integral computations using polyhedra which will be needed for our computations in the later chapters. In Chapter 4, we recall from [47] and [49] the notion of local and global ask zeta functions. Moreover, we prove that the modules associated with weakly equivalent matrices of linear forms have the same ask zeta functions. In Chapter 5, we recall from [18] and [47] the definition of the conjugacy class and orbit-counting zeta functions of groups. We also prove Theorem A. In Chapter 6, we give a formula for computing ask zeta functions of modules of antisymmetric matrices using the set of all nonzero principal Pfaffians. We also prove Theorem B. In Chapter 7, we give an explicit formula for the local (Theorem C) and global ask zeta functions of an image module associated with a single matrix and its Knuth duals. In Chapter 8, we prove a formula for the ask zeta function of an image module associated with a matrix pencil (Theorem D). In Chapter 9, we give a formula for computing the number of matrices by rank in a relation of vector spaces associated with a single matrix over a finite field and prove Theorem E. By studying ask zeta functions of relation modules of rational forms of matrix pencils, we write all the possible zeta functions of relation modules associated with matrix pencils in Chapter 10. We also prove Theorem F and Theorem G.

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Chapter 1

Modules of matrices and module representations

In this chapter, we define weak equivalence on matrices of linear forms and equivalence of modules of matrices. We also recall terminology from [49] about so-called module representations and Knuth duality. The former provide means to write modules of matrices up to equivalence classes. The latter provide techniques to study zeta functions, averaging over the sizes of the kernels known as ask zeta functions, associated with modules of matrices as we shall recall in Chapter 4. Throughout this chapter, we denote by R a ring (associative, commutative, and unital). The unit group of R is denoted by $R^\times$. When we speak of modules over R (denoted by R-module), we mean right modules.

1.1 Matrices of linear forms

For $\ell \geq 1$, let $A_1, \ldots, A_\ell \in \mathrm{M}_{d\times e}(R)$. Let $X = (X_1, \ldots, X_\ell)$ consist of algebraically independent variables over R. We say that $A(X) = A_1X_1 + \cdots + A_\ell X_\ell \in \mathrm{M}_{d\times e}(R[X_1, \ldots, X_\ell])$ is a **matrix of linear forms** . If $\ell = 1$, we call $A(X_1) = A_1X_1$ a **single matrix**. Moreover, if $\ell = 2$, we call $A(X_1, X_2) = A_1X_1 + A_2X_2$ a **matrix pencil** when $A_1, A_2 \in \mathrm{M}_{d\times e}(R)$ is linearly independent (i.e. $A_1\lambda_1 + A_2\lambda_2 = 0$ if and only if $\lambda_1 = \lambda_2 = 0$ for $\lambda_1, \lambda_2 \in R$). Let $B_1, \ldots, B_\ell \in \mathrm{M}_{d\times e}(R)$. Let $B(X) = B_1X_1 + \cdots + B_\ell X_\ell$ be another matrix of linear forms. Then $A(X) = B(X)$ if and only if $A_i = B_i$ for all $1 \leq i \leq \ell$.

Let S be an R-algebra. For $Z = (Z_1, \ldots, Z_\ell) \in S^\ell$, we define $A(Z) \in \mathrm{M}_{d\times e}(S)$ to be the matrix obtained from $A(X)$ by replacing each X_i by Z_i for all $1 \leq i \leq \ell$. In the following, we always assume $A(X) = \sum_{i=1}^{\ell} A_iX_i$ and $B(X) = \sum_{i=1}^{\ell} B_iX_i$.

Lemma 1.1.1. *Let $A(X), B(X) \in \mathrm{M}_{d\times e}(R[X])$ be two matrices of linear forms. Then $A(X) = B(X)$ if and only if $A(x) = B(x)$ for all $x \in R^\ell$.*

Proof. If $A(X) = B(X)$, then we have $A_i = B_i$ for all $1 \leq i \leq \ell$. Hence $A_1x_1 + \cdots + A_\ell x_\ell = B_1x_1 + \cdots + B_\ell x_\ell$ for all $x \in R^\ell$. Suppose now that $A(x) = B(x)$ for all $x \in R^\ell$. Let $(e_1, \ldots, e_\ell) \subset R^\ell$ be the standard basis (i.e. $e_1 =$

$(1,0,\ldots,0), e_2 = (0,1,0\ldots,0), \ldots, e_\ell = (0,\ldots,0,1))$. Since $A(e_i) = B(e_i)$ for all $1 \leq i \leq \ell$, we see that $A_i = B_i$ for all i. Therefore, $A(X) = B(X)$. □

Definition 1.1.2. Let R be a subring of a ring S. Let $A, B \in \mathrm{M}_{d\times e}(S)$ be two matrices over S. We say that A and B are **equivalent over** R if there exist $g \in \mathrm{GL}_d(R)$ and $h \in \mathrm{GL}_e(R)$ such that $gAh = B$.

Let $\Omega \subset \mathrm{M}_{d\times e}(R[X])$ be the set of all matrices of linear forms $A(X) = A_1X_1 + \cdots + A_\ell X_\ell$ with $A_1, \ldots, A_\ell \in \mathrm{M}_{d\times e}(R)$. Define a map

$$\Omega \times (\mathrm{GL}_\ell(R) \times \mathrm{GL}_d(R) \times \mathrm{GL}_e(R)) \to \Omega; \tag{1.1}$$

via $A(X)(f,g,h) = g^{-1}A(Xf^{-1})h$ for $A(X) \in \Omega$ and $(f,g,h) \in \mathrm{GL}_\ell(R) \times \mathrm{GL}_d(R) \times \mathrm{GL}_e(R)$. Here, we regard X as a row in $R[X]^\ell$. Since $f \in \mathrm{GL}_\ell(R)$, the product $Xf^{-1} \in R[X]$ is defined such that $[X_1, \ldots, X_\ell]f^{-1} = [f_1(X), \ldots, f_\ell(X)]$ where $f_i(X)$ is a polynomial in $R[X]$ for $1 \leq i \leq \ell$.

Definition 1.1.3. Let $A(X), B(X) \in \mathrm{M}_{d\times e}(R[X])$ be two matrices of linear forms. We say that $A(X)$ is **weakly equivalent** to $B(X)$ over R if there exist $g \in \mathrm{GL}_d(R)$, $h \in \mathrm{GL}_e(R)$, and $f \in \mathrm{GL}_\ell(R)$ such that $gA(Xf)h = B(X)$ where $Xf \in R[X]^\ell$.

Definition 1.1.3 allows us to write matrices of linear forms up to weak equivalence classes over a ring R.

Corollary 1.1.4. *Let $A(X), B(X) \in \mathrm{M}_{d\times e}(R[X])$ be two matrices of linear forms and $f \in \mathrm{GL}_\ell(R)$. If $A(Xf) = B(X)$, then $\{A(x) : x \in R^\ell\} = \{B(x) : x \in R^\ell\}$.*

Proof. Lemma 1.1.1 shows that $A(Xf) = B(X)$ if and only if $A(xf) = B(x)$ for each $x \in R^\ell$. Hence $\{A(x) : x \in R^\ell\} = \{B(x) : x \in R^\ell\}$. □

For two matrices $A \in \mathrm{M}_{d\times e}(R)$ and $B \in \mathrm{M}_{m\times n}(R)$, let $A\bar{\otimes}B$ be the Kronecker product such that if $A = [a_{ij}]_{\substack{1\leq i\leq d\\ 1\leq j\leq e}}$ then $A\bar{\otimes}B = [a_{ij}B]_{\substack{1\leq i\leq d\\ 1\leq j\leq e}}$ where

$$[a_{ij}B]_{\substack{1\leq i\leq d\\ 1\leq j\leq e}} = \begin{bmatrix} a_{11}B & \cdots & a_{1e}B \\ \vdots & \ddots & \vdots \\ a_{d1}B & \cdots & a_{de}B \end{bmatrix} \in \mathrm{M}_{dm\times en}(R).$$

Lemma 1.1.5. *Let $A(X), B(X) \in \mathrm{M}_{d\times e}(R[X])$ be two matrices of linear forms and $f \in \mathrm{GL}_\ell(R)$. If $A(Xf) = B(X)$, then*

$$[A_1, \ldots, A_\ell](f^T\bar{\otimes}I_e) = [B_1, \ldots, B_\ell],$$

where f^T is the transpose of f.

Proof. Write $f = [f_{ij}]_{1\leq i,j\leq \ell}$. Then

$$Xf = [X_1, \ldots, X_\ell]f = [X_1f_{11} + \cdots + X_\ell f_{\ell 1}, \ldots, X_1f_{1\ell} + \cdots + X_\ell f_{\ell\ell}].$$

Hence

$$\begin{aligned} A(Xf) &= (A_1 f_{11} + \cdots + A_\ell f_{1\ell})X_1 + \cdots + (A_1 f_{\ell 1} + \cdots + A_\ell f_{\ell\ell})X_\ell \\ &= B_1 X_1 + \cdots + B_\ell X_\ell = B(X). \end{aligned}$$

Thus $[A_1, \ldots, A_\ell][f_{ji} I_e]_{1 \le i,j \le \ell} = [B_1, \ldots, B_\ell]$. □

Corollary 1.1.6. *Let $A(X), B(X), C(X) \in \mathrm{M}_{d\times e}(R[X])$ be matrices of linear forms. For $f_1, f_2 \in \mathrm{GL}_\ell(R)$, if $A(Xf_1) = B(X)$ and $B(Xf_2) = C(X)$, then $A(Xf_2f_1) = C(X)$.*

Proof. By Lemma 1.1.5, we have $[A_1, \ldots, A_\ell](f_1^T \bar{\otimes} I_e) = [B_1, \ldots, B_\ell]$ and $[B_1, \ldots, B_\ell](f_2^T \bar{\otimes} I_e) = [C_1, \ldots, C_\ell]$. Thus

$$\begin{aligned} [C_1, \ldots, C_\ell] &= [A_1, \ldots, A_\ell](f_1^T \bar{\otimes} I_e)(f_2^T \bar{\otimes} I_e) = [A_1, \ldots, A_\ell](f_1^T f_2^T \bar{\otimes} I_e) \\ &= [A_1, \ldots, A_\ell]((f_2 f_1)^T \bar{\otimes} I_e). \end{aligned}$$

□

In the following proposition, we show that the action in (1.1) is a group action.

Proposition 1.1.7. *Let $\Omega \subset \mathrm{M}_{d\times e}(R[X])$ be the set of all matrices of linear forms. Then the map in (1.1) is a group action.*

Proof. For any $A(X) \in \Omega$, since $(I_\ell, I_d, I_e) \in \mathrm{GL}_\ell(R) \times \mathrm{GL}_d(R) \times \mathrm{GL}_e(R)$, we have $A(X)(I_\ell, I_d, I_e) = I_d A(XI_\ell) I_e = A(X)$. By Corollary 1.1.6

$$\begin{aligned} A(X)(f_1, g_1, h_1)(f_2, g_2, h_2) &= g_2^{-1} g_1^{-1} A(X f_2^{-1} f_1^{-1}) h_1 h_2 \\ &= (g_1 g_2)^{-1} A(X(f_1 f_2)^{-1}) h_1 h_2 \\ &= A(X)(f_1 f_2, g_1 g_2, h_1 h_2) \end{aligned}$$

for any $(f_1, g_1, h_1), (f_2, g_2, h_2) \in \mathrm{GL}_\ell(R) \times \mathrm{GL}_d(R) \times \mathrm{GL}_e(R)$. □

Proposition 1.1.7 shows that weak equivalence defines an equivalence relation on the set of all matrices of linear forms $A_1X_1 + \cdots + A_\ell X_\ell$ with $A_1, \ldots, A_\ell \in \mathrm{M}_{d\times e}(R)$.

A matrix of linear forms $A(X) \in \mathrm{M}_{d\times e}(R[X])$ defines two types of modules. The first type is the **image module** of $A(X)$ defined as

$$M(A(X); R) = \{A(x) : x \in R^\ell\} \subset \mathrm{M}_{d\times e}(R),$$

of which we show examples in Chapter 7 (image modules of single matrices) and Chapter 8 (image modules of matrix pencils). Let $\langle \cdot, \cdot \rangle : \mathrm{M}_{d\times e}(R) \times \mathrm{M}_{d\times e}(R) \to R$ be the bilinear form defined by $\langle A, B \rangle = \mathrm{tr}(A^T B)$ for $A, B \in \mathrm{M}_{d\times e}(R)$ where $\mathrm{tr}(\cdot)$ denotes the trace. We call $\langle \cdot, \cdot \rangle$ the **standard inner product**. The other type of modules is called the **relation module associated with** $A(X)$ defined as

$$\mathrm{Rel}(A(X); R) = \{x \in \mathrm{M}_{d\times e}(R) : \langle A_1, x \rangle = \cdots = \langle A_\ell, x \rangle = 0\}.$$

We think of $\mathrm{Rel}(A(X);R)$ as the module obtained by the orthogonal complement of $M(A(X);R)$ with respect to $\langle\cdot,\cdot\rangle$. We shall show examples of $\mathrm{Rel}(A(X);R)$ in Chapter 9 (relation modules associated with single matrices) and Chapter 10 (relation modules associated with matrix pencils). Relation modules associated with matrices of linear forms can be thought of as a generalization of relation modules associated with partial colorings [11].

Corollary 1.1.8. *Let $A(X), B(X) \in \mathrm{M}_{d\times e}(R[X])$ be two matrices of linear forms and $f \in \mathrm{GL}_\ell(R)$. If $A(Xf) = B(X)$, then $\mathrm{Rel}(A(X);R) = \mathrm{Rel}(B(X);R)$.*

Proof. Let $f = [f_{ij}]_{1\le i,j\le \ell}$. By Lemma 1.1.5, $B_i = A_1 f_{i1} + \cdots + A_\ell f_{i\ell}$. For $x \in \mathrm{M}_{d\times e}(R)$, if $x \in \mathrm{Rel}(A(X);R)$, then $\langle A_i, x\rangle = 0$ for $1 \le i \le \ell$. Hence $\langle B_i, x\rangle = \langle A_1 f_{i1} + \cdots + A_\ell f_{i\ell}, x\rangle = \langle A_1, x\rangle f_{i1} + \cdots + \langle A_\ell, x\rangle f_{i\ell} = 0$ for all i. Thus $x \in \mathrm{Rel}(B(X);R)$. Suppose now that $x \in \mathrm{Rel}(B(X);R)$. Then $\langle B_i, x\rangle = 0$ for all i. Put $t_i = \mathrm{tr}(A_i^T x)$ for all i. Then we obtain the linear equations system:

$$\begin{aligned} \langle B_1, x\rangle &= t_1 f_{11} + \cdots + t_\ell f_{1\ell} = 0 \\ \vdots \qquad & \qquad \vdots \qquad \vdots \\ \langle B_\ell, x\rangle &= t_\ell f_{\ell 1} + \cdots + t_\ell f_{\ell\ell} = 0 \end{aligned}$$

Since $f \in \mathrm{GL}_\ell(R)$, the linear equations system has one solution $(t_1,\ldots,t_\ell) = (0,\ldots,0)$. Thus $t_i = \langle A_i, x\rangle = 0$ for all i. Hence $x \in \mathrm{Rel}(A(X);R)$. □

In the next section, we shall explain the relevance between weak equivalence of matrices of linear forms and equivalence of modules associated with matrices of linear forms.

1.2 Equivalence of modules of matrices

For two modules of matrices M_1, M_2 over a ring R, we say that M_1 and M_2 are **equivalent** if there exist $g \in \mathrm{GL}_d(R)$ and $h \in \mathrm{GL}_e(R)$ such that $gM_1h = M_2$. In the following definition, we generalize the notion of equivalence of modules of matrices into equivalence over a ring extension $R \subset S$.

Definition 1.2.1. For a ring extension $R \subset S$. Let $M_1, M_2 \subset \mathrm{M}_{d\times e}(R)$ be submodules. We say that M_1 and M_2 are **equivalent over** S (S-**equivalent**) if there exist $g \in \mathrm{GL}_d(S)$ and $h \in \mathrm{GL}_e(S)$ such that $gM_1h = M_2$. If R is an integral domain with field of fractions S, we say M_1 and M_2 are **rationally equivalent**.

Since most of our main results are focused on modules defined by matrices of linear forms, we shall next relate the equivalence between matrices of linear forms and the modules they define.

Lemma 1.2.2. *Let $A(X) \in \mathrm{M}_{d\times e}(R[X])$ be a matrix of linear forms. For any $g \in \mathrm{GL}_d(R)$ and $h \in \mathrm{GL}_e(R)$,*

$$g\,\mathrm{Rel}(A(X);R)h = \mathrm{Rel}(g^{-T}A(X)h^{-T};R).$$

Proof. Let $x \in \mathrm{M}_{d\times e}(R)$. Then $x \in \mathrm{Rel}(A(X); R)$ if and only if $\langle A_i, x\rangle = \mathrm{tr}(A_i^T x) = 0$ for $1 \leq i \leq \ell$. For each invertible matrix $h \in \mathrm{GL}_e(R)$, $\mathrm{tr}(h^{-1}xh) = \mathrm{tr}(x)$. Therefore, $\mathrm{tr}(A_i^T x) = \mathrm{tr}(h^{-1}A^T xh) = \mathrm{tr}(h^{-1}A_i^T g^{-1}gxh) = \mathrm{tr}((g^{-T}A_i h^{-T})^T gxh) = \langle g^{-T}A_i h^{-T}, gxh\rangle$. Hence $\langle A_i, x\rangle = \langle g^{-T}A_i h^{-T}, gxh\rangle = 0$ for all i if and only if $gxh \in \mathrm{Rel}(g^{-T}A(X)h^{-T}; R)$. □

Next, we show that weak equivalence of matrices of linear forms gives us equivalence between image modules and equivalence between relation modules associated with these matrices of linear forms.

Corollary 1.2.3. *Let $A(X), B(X) \in \mathrm{M}_{d\times e}(R[X])$ be two matrices of linear forms. If $A(X)$ and $B(X)$ are weakly equivalent, then:*

(i) $M(A(X); R)$ and $M(B(X); R)$ are equivalent over R, and

(ii) $\mathrm{Rel}(A(X); R)$ and $\mathrm{Rel}(B(X); R)$ are equivalent over R.

Proof. For (i), suppose that $A(X)$ and $B(X)$ are weakly equivalent. Hence there exist $g \in \mathrm{GL}_d(R), h \in \mathrm{GL}_e(R)$, and $f \in \mathrm{GL}_\ell(R)$ such that $gA(Xf)h = B(X)$ where $Xf \in R[X]^\ell$. By Lemma 1.1.4, we obtain $\{gA(x)h : x \in R^\ell\} = \{B(x) : x \in R^\ell\}$. Therefore, we have $M(gA(X)h; R) = M(B(X); R)$ if and only if $gM(A(X); R)h = M(B(X); R)$. For (ii), let $gA(Xf)h = B(X)$. By Corollary 1.1.6, we have $\mathrm{Rel}(gA(X)h; R) = \mathrm{Rel}(B(X); R)$. By Lemma 1.2.2, we obtain $g^{-T}\,\mathrm{Rel}(A(X); R)h^{-T} = \mathrm{Rel}(B(X); R)$. □

1.3 Module representations

Here, we recall from [49, Section 2.1] the notion of module representations. For two R-modules V and W, the set of all R-linear maps from V to W is denoted by $\mathrm{Hom}(V, W)$. For example, when $V = R^d$ and $W = R^e$, then there exists a nature isomorphism between $\mathrm{Hom}(R^d, R^e)$ and $\mathrm{M}_{d\times e}(R)$, see [23, Section 1.11].

Definition 1.3.1 ([49, Section 2.2])**.** Let M, V and W be R-modules. A **module representation** is a homomorphism $M \xrightarrow{\theta} \mathrm{Hom}(V, W)$. We denote by $M\theta$ the image of M under θ.

A matrix of linear forms $A(X) \in \mathrm{M}_{d\times e}(R[X])$ gives rise to a module representation

$$R^\ell \xrightarrow{A(\cdot)} \mathrm{Hom}(R^d, R^e), \quad x \mapsto A(x).$$

Other examples of module representations are the inclusion maps $M(A(X); R) \xrightarrow{\theta} \mathrm{Hom}(R^d, R^e)$ and $\mathrm{Rel}(A(X); R) \xrightarrow{\theta'} \mathrm{Hom}(R^d, R^e)$.

Definition 1.3.2 ([49, Section 2.2])**.** For two module representations, $M_1 \xrightarrow{\theta_1} \mathrm{Hom}(V_1, W_1)$ and $M_2 \xrightarrow{\theta_2} \mathrm{Hom}(V_2, W_2)$, an **isotopy** $\theta_1 \to \theta_2$ is a triple of module isomorphisms $M_1 \xrightarrow{\nu} M_2, V_1 \xrightarrow{\phi} V_2$, and $W_1 \xrightarrow{\psi} W_2$ such that $(a\theta_1)\psi = \phi((a\nu)\theta_2)$ for all $a \in M_1$. We use the notation $\theta_1 \approx \theta_2$ to indicate the existence of an isotopy between θ_1 and θ_2.

Next, we show that weak equivalence classes of matrices of linear forms corresponds to isotopy classes of the module representations arise from those matrices of linear forms.

Lemma 1.3.3. *Let $A(X)$ and $B(X)$ be matrices of linear forms. Then $A(X)$ and $B(X)$ are weakly equivalent if and only if $A(\cdot)$ and $B(\cdot)$ are isotopic.*

Proof. Let $A(\cdot)$ and $B(\cdot)$ are isotopic if and only if there exists a triple of isomorphisms $R^\ell \xrightarrow{\nu} R^\ell, R^d \xrightarrow{\phi} R^d$, and $R^e \xrightarrow{\psi} R^e$ such that $A(x)\psi = \phi B(x\nu)$ for all $x \in R^\ell$. By Lemma 1.1.1, the latter condition is equivalent to $\phi^{-1}A(X)\psi = B(X\nu)$ where $X = (X_1, \cdots, X_\ell)$ consists of independent variables over R. Now we regard ν, ϕ, and ψ as matrices over $R[X]$. Since $\nu \in \mathrm{GL}_\ell(R), \phi \in \mathrm{GL}_d(R)$ and $\psi \in \mathrm{GL}_e(R)$, we have $A(X)$ and $B(X)$ are weakly equivalent over R. □

For two modules of matrices M_1 and M_2 over a non-trivial ring R, if there exists an isotopy between the inclusion maps $M_1 \xrightarrow{\theta_1} \mathrm{Hom}(R^{d_1}, R^{e_1})$ and $M_2 \xrightarrow{\theta_2} \mathrm{Hom}(R^{d_2}, R^{e_2})$, then $d_1 = d_2$ and $e_1 = e_2$. The maps $R^{d_1} \xrightarrow{\phi} R^{d_2}$ and $R^{e_1} \xrightarrow{\psi} R^{e_2}$ can be isomorphisms if and only if $\mathrm{rank}(R^{d_1}) = \mathrm{rank}(R^{d_2})$ and $\mathrm{rank}(R^{e_1}) = \mathrm{rank}(R^{e_2})$; see [43, Theorem 1 and Theorem 2].

In the following lemma, we show that equivalence classes of modules and isotopic classes of their injective maps to $\mathrm{Hom}(R^d, R^e)$ are the same when dealing with modules of matrices.

Lemma 1.3.4. *For two modules of matrices $M_1, M_2 \in \mathrm{M}_{d\times e}(R)$, Let $M_1 \xrightarrow{\theta_1} \mathrm{Hom}(R^d, R^e)$ and $M_2 \xrightarrow{\theta_2} \mathrm{Hom}(R^d, R^e)$ be two injective maps. Then the images $M_1\theta_1$ and $M_2\theta_1$ are equivalent over R if and only if θ_1 and θ_2 are isotopic.*

Proof. Let θ_1 and θ_2 are isotopic. Then there exists a triple of isomorphisms $M_1 \xrightarrow{\nu} M_2, R^d \xrightarrow{\phi} R^d$, and $R^e \xrightarrow{\psi} R^e$ such that $\phi^{-1}(a\theta_1)\psi = (a\nu)\theta_2$ for all $a \in M_1$. Thus $\phi^{-1}(M_1\theta_1)\psi = M_2\theta_2$. We regard ν, ϕ, and ψ as matrices over $R[X]$. Since $\nu \in \mathrm{GL}_\ell(R), \phi \in \mathrm{GL}_d(R)$ and $\psi \in \mathrm{GL}_e(R)$, we have $M_1\theta_1$ and $M_2\theta_2$ are equivalent over R. Conversely, suppose that $M_1\theta_1$ and $M_2\theta_2$ are equivalent. Then there exist $\phi \in \mathrm{GL}_d(R)$ and $\psi \in \mathrm{GL}_e(R)$ such that $\phi^{-1}(M_1\theta_1)\psi = M_2\theta_2$. Then ϕ and ψ are isomorphic maps such that $R^d \xrightarrow{\phi} R^d$ and $R^e \xrightarrow{\psi} R^e$. Since θ_1 and θ_2 are injective maps, we obtain two isomorphic maps, say, $\delta_1 : M_1 \to M_1\theta_1$ and $\delta_2 : M_2 \to M_2\theta_2$. Let ν be the composite mapping from M_1 to M_2 such that $a\nu = (\phi^{-1}(a\delta_1)\psi)\delta_2^{-1} \in M_2$ for all $a \in M_1$. Then ν is isomorphic since ν is the composite of the isomorphic maps $\delta_1, \phi^{-1}, \psi$, and δ_2^{-1}. Since $(a\nu)\delta_2 = \phi^{-1}(a\theta_1)\psi \in M_2\theta_2$, then $\phi^{-1}(a\theta_1)\psi = (a\nu)\theta_2$. Thus $(a\theta_1)\psi = \phi((a\nu)\theta_2)$ for all $a \in M_1$. Therefore θ_1 and θ_2 are isotopic. □

1.4 Knuth duality

Knuth duals were first introduced for hypercubes in [35, Section 4.1]. Later the operations $\circ, \bullet$, and $\vee$, which generalize Knuth duals on module representations, were introduced in [49, Section 4.2]. We recall from [49, Section 4.2] the

definition of Knuth duals on module representations. Let $(\cdot)^* = \mathrm{Hom}(\cdot, R)$ denote the dual module. Let $M \xrightarrow{\theta} \mathrm{Hom}(V, W)$ be a module representation over R. For $a \in M, x \in V$ and $\omega \in W^*$, the module representations $\theta^\circ, \theta^\bullet$ and $\theta^\vee$ are **Knuth duals** of the module representation θ defined as follows:

$$\begin{aligned} V &\xrightarrow{\theta^\circ} \mathrm{Hom}(M, W);\ \ x \mapsto (a \mapsto x(a\theta)), \\ W^* &\xrightarrow{\theta^\bullet} \mathrm{Hom}(V, M^*);\ \ \omega \mapsto (x \mapsto (a \mapsto (x(a\theta))\omega)), \textit{and} \\ M &\xrightarrow{\theta^\vee} \mathrm{Hom}(W^*, V^*);\ \ a \mapsto (\omega \mapsto (x \mapsto (x(a\theta))\omega)). \end{aligned}$$

In the following definition, we will define operations $\circ, \bullet$ and $\vee$ on a matrix of linear forms $A(X_1, \ldots, X_\ell)$.

Definition 1.4.1. For $A_1, \ldots, A_\ell \in \mathrm{M}_{d\times e}(R)$, let $A(X_1, \ldots, X_\ell) = A_1X_1 + \cdots + A_\ell X_\ell = [\sum_{k=1}^{\ell} a_{kij}X_k]_{\substack{1\le i\le d \\ 1\le j\le e}} \in \mathrm{M}_{d\times e}(R[X_1, \ldots, X_\ell])$ be a matrix of linear forms where a_{kij} is the (i, j) entry of A_k for $1 \le k \le \ell$. Then define $A^\circ, A^\bullet$ and $A^\vee$ as follows:

(i) $A^\circ(X_1, \ldots, X_d) = [\sum_{i=1}^{d} a_{kij}X_i]_{\substack{1\le k\le \ell \\ 1\le j\le e}} \in \mathrm{M}_{\ell\times e}(R[X_1, \ldots, X_d])$,

(ii) $A^\bullet(X_1, \ldots, X_e) = [\sum_{j=1}^{e} a_{kij}X_j]_{\substack{1\le i\le d \\ 1\le k\le \ell}} \in \mathrm{M}_{d\times \ell}(R[X_1, \ldots, X_e])$, and

(iii) $A^\vee(X_1, \ldots, X_\ell) = [\sum_{k=1}^{\ell} a_{kij}X_k]_{\substack{1\le j\le e \\ 1\le i\le d}} \in \mathrm{M}_{e\times d}(R[X_1, \ldots, X_\ell])$.

Note that $A^\vee(X)$ is the transpose of $A(X)$.

Example 1.4.2. Let $A(X_1, X_2, X_3, X_4, X_5, X_6) = \begin{bmatrix} a_1X_1 & a_2X_2 & a_3X_3 \\ a_4X_4 & a_5X_5 & a_6X_6 \end{bmatrix}$. By applying definition 1.4.1, we obtain the following matrices of linear forms:

(i) $A^\circ(X_1, X_2) = \begin{bmatrix} a_1X_1 & 0 & 0 \\ 0 & a_2X_1 & 0 \\ 0 & 0 & a_3X_1 \\ a_4X_2 & 0 & 0 \\ 0 & a_5X_2 & 0 \\ 0 & 0 & a_6X_2 \end{bmatrix}$,

(ii) $A^\bullet(X_1, X_2, X_3) = \begin{bmatrix} a_1X_1 & a_2X_2 & a_3X_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4X_1 & a_5X_2 & a_6X_3 \end{bmatrix}$, and

(iii) $A^\vee(X_1, X_2, X_3, X_4, X_5, X_6) = \begin{bmatrix} a_1X_1 & a_4X_4 \\ a_2X_2 & a_5X_5 \\ a_3X_3 & a_6X_6 \end{bmatrix} = A(X_1, \ldots, X_6)^T$.

Let $A(X)$ be a matrix of linear forms. Let $A(\cdot)$ be the module representation arising from $A(X)$ such that

$$R^\ell \xrightarrow{A(\cdot)} \operatorname{Hom}(R^d, R^e), \quad x \mapsto A(x).$$

The following proposition shows the correspondence between Knuth duals on a module representation arise from a matrix of linear forms $A(X)$ and the module representations arising from $A^\circ(X), A^\bullet(X)$, and $A^\vee(X)$.

Proposition 1.4.3 ([49, Proposition 4.12(i)])**.** *Let $R^\ell \xrightarrow{A(\cdot)} \operatorname{Hom}(R^d, R^e)$ be a module representation given by a matrix of linear forms $A(X)$. Then*

(i) $A(\cdot)^\circ = A^\circ(\cdot)$,

(ii) $A(\cdot)^\bullet \approx A^\bullet(\cdot)$, *and*

(iii) $A(\cdot)^\vee \approx A^\vee(\cdot)$.

Chapter 2

Rational forms of matrix pencils

In this chapter, we recall from [53, Chapter IX] and [52] the notions of singular and non-singular matrix pencils and rational forms of matrix pencils. We also describe algorithms to obtain rational forms of singular and non-singular matrix pencils. We assume that K is an arbitrary infinite field throughout this chapter. Other forms such as Weierstrass forms of non-singular matrix pencils and Kronecker forms of singular matrix pencils can be used over algebraically closed infinite fields, see [31, Section 3].

2.1 Matrix pencils

For linearly independent $A_1, A_2 \in \mathrm{M}_{d\times e}(K)$, Let $A(X_1, X_2) = A_1X_1 + A_2X_2 \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ where X_1 and X_2 are algebraically independent variables over K. Then we say that $A(X_1, X_2)$ is a **singular matrix pencil** if $d \neq e$ or $\det(A(x_1, x_2)) = 0$ for all $x_1, x_2 \in K$. Otherwise, we say that $A(X_1, X_2)$ is a **non-singular matrix pencil** when the determinant of $A(X_1, X_2)$ does not vanish as a polynomial (i.e. $\det(A(X_1, X_2))$ is not a zero polynomial). Moreover, a matrix pencil $A(X_1, X_2)$ is called **regular** if one, or both, of its coefficients A_1 or A_2 is a non-singular matrix.

Recall from [42, Proposition I.2.4] that a polynomial $F(X) \in K[X_1, \ldots, X_\ell]$ is zero if and only if $F(x_0) = 0$ for all $x_0 \in K^\ell$. Hence, the determinant of $A(X_1, X_2)$ vanishes if and only if $\det(A(x_1, x_2)) = 0$ for all $x_1, x_2 \in K$. In this chapter, we exclude finite fields, since there are some cases when a matrix pencil $A(X_1, X_2)$ can be defined as both a singular and a non-singular matrix pencil over finite fields. That is the case when $\det(A(X_1, X_2))$ does not vanish as a polynomial, and $\det(x_1, x_2) = 0$ for all $x_1, x_2 \in K$.

Example 2.1.1. Consider the matrix pencil

$$A(X_1, X_2) = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_1 + X_2 & 0 \\ 0 & 0 & X_2 \end{bmatrix} \in \mathrm{M}_3(\mathbb{F}_2[X_1, X_2]).$$

Then $\det(A(X_1, X_2)) = X_1^2X_2 + X_1X_2^2$, and $\det(A(x_1, x_2)) = 0$ for all $x_1, x_2 \in \mathbb{F}_2$.

A matrix pencil $A(X_1, X_2) = A_1X_1 + A_2X_2 \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ can be converted to a polynomial matrix via $X_1^{-1}A(X_1, X_2) = A(Y) = A_1 + A_2Y \in \mathrm{M}_{d\times e}(K[Y])$ where $Y = \frac{X_2}{X_1}$. Alternatively, $X_2^{-1}A(X_1, X_2) = A(Z) = A_1Z + A_2 \in \mathrm{M}_{d\times e}(K[Z])$ where $Z = \frac{X_1}{X_2}$. If a matrix pencil $A(X_1, X_2)$ is non-singular, we can write

$$\det(A(X_1, X_2)) = a_0X_1^n + a_1X_1^{n-1}X_2 + \ldots + a_nX_2^n, \tag{2.1}$$

where $a_0, \ldots, a_n \in K$. In this case, we have n or less distinct roots of $\det(A(X_1, X_2))$. By converting a non-singular matrix pencil $A(X_1, X_2)$ to a polynomial matrix $A(Z)$ via $Z = \frac{X_1}{X_2}$, we obtain

$$\det(A(Z)) = a_0Z^n + a_1Z^{n-1} + \cdots + a_n.$$

Definition 2.1.2 ([29, Proposition VII.2.11])**.** Let R be a principal ideal domain. For a matrix $A \in \mathrm{M}_{d\times e}(R)$, there exist $g \in \mathrm{GL}_d(R)$ and $h \in \mathrm{GL}_e(R)$ such that

$$gAh = \mathrm{diag}(a_1, \ldots, a_n, 0, \ldots, 0), \tag{2.2}$$

where a_i's are nonzero elements in R satisfies $a_i|a_{i+1}$ for $1 \leq i \leq n$. We call the canonical form in (2.2) the **Smith normal form** of A and we call a_i's the **invariant factors** of A.

For a matrix pencil $A(X_1, X_2)$, we can reduce $A(X_1, X_2)$ into a diagonal form, denoted by $\mathrm{D}_A(X_1, X_2)$, by using the Smith normal form of $A(Y)$ (where $Y = \frac{X_2}{X_1}$) or $A(Z)$ (where $Z = \frac{X_1}{X_2}$). Indeed, since $K[Z]$ is a principal ideal domain, there exist $g(Z) \in \mathrm{GL}_d(K[Z])$ and $h(Z) \in \mathrm{GL}_e(K[Z])$ such that

$$g(Z)A(Z)h(Z) = \mathrm{diag}(a_1(Z), \ldots, a_n(Z), 0, \ldots, 0), \tag{2.3}$$

where $a_i(Z)$'s are nonzero polynomial in $K[Z]$ satisfies $a_i(Z)|a_{i+1}(Z)$ for $1 \leq i \leq n$. Then we can multiply the polynomial matrix in (2.3) by X_2^m, for a large enough m, to obtain the diagonal form:

$$\mathrm{D}_A(X_1, X_2) = \mathrm{diag}(a_1(X_1, X_2), \ldots, a_n(X_1, X_2), 0, \ldots, 0), \tag{2.4}$$

where $a(X_1, X_2)$'s are nonzero homogeneous polynomials in $K[X_1, X_2]$. We call $a(X_1, X_2)$ the **invariant factors** of the matrix pencil $A(X_1, X_2)$, see [52, Section 1].

Lemma 2.1.3. *Let $A_1, A_2 \in \mathrm{M}_n(K)$. Let $A(X_1, X_2) = A_1X_1 + A_2X_2$ be a non-singular matrix pencil. Then $A(X_1, X_2)$ is weakly equivalent to a regular matrix pencil.*

Proof. Let $\det(A(X_1, X_2)) = a_0X_1^n + a_1X_1^{n-1}X_2 + \ldots + a_nX_2^n$. If A_1 or A_2 is a non-singular matrix, then $A(X_1, X_2)$ is regular. Suppose that both A_1 and A_2 are singular matrices. Put $f = \begin{bmatrix} 1 & 0 \\ z_0 & 1 \end{bmatrix}$ then $[X_1, X_2]f = [X_1 + z_0X_2, X_2]$ where $z_0 \in K^\times$. Thus $A((X_1, X_2)f) = A(X_1 + z_0X_2, X_2) = A_1X_1 + A_2'X_2$ where $A_2' = A_1z_0 + A_2$. Consider the polynomial matrix $A(Z) = A_1Z + A_2$ where $Z = \frac{X_1}{X_2}$. Then $A_2' = A(z_0)$. Since $A(Z)$ is a non-singular polynomial matrix, then there exist $z_0 \in K$ such that $A(z_0) \neq 0$. In this case, $\det(A_2') \neq 0$. Thus, A_2' is a non-singular matrix. Hence $A(X_1 + z_0X_2, X_2)$ is a regular matrix pencil. □

2.2 Rational forms of non-singular matrix pencils

Here, we recall from [53, Section IX.2] the rational forms of non-singular matrix pencils on arbitrary infinite fields.

For $n > 0$, let $F(X) = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + X^n$ be a monic polynomial over K. The **companion matrix** of the polynomial $F(X)$, denoted by $\mathrm{CM}(F)$, is a matrix of the form

$$\mathrm{CM}(F) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}.$$

Theorem 2.2.1 ([52, Section VI.4])**.** *Let K be any field. For each matrix $A \in \mathrm{M}_n(K)$, there exist $u \in \mathrm{GL}_n(K)$ such that*

$$u^{-1}Au = \mathrm{diag}(C_1, \ldots, C_w), \tag{2.5}$$

where each $C_i = \mathrm{CM}(F_i)$ for some monic polynomial $F_i(X) \in K[X]$ for $1 \leq i \leq w$. The polynomials $F_i(X)$'s are uniquely determined and satisfy $F_1(X)|F_2(X)|\cdots|F_w(X)$.

We call the canonical form in (2.5) the **rational canonical form** of the matrix A.

Lemma 2.2.2 ([52, Section IX.2])**.** *Let $A(X_1, X_2) \in \mathrm{M}_n(K[X_1, X_2])$ be a non-singular matrix pencil. Then $A(X_1, X_2)$ is weakly equivalent to a matrix pencil of the form*

$$\mathrm{R}X_1 + I_nX_2, \tag{2.6}$$

where R *is a matrix in a rational canonical form.*

By swapping X_1 and X_2, Lemma 2.2.2 can also be written as

$$I_nX_1 + \mathrm{R}X_2,$$

where R is a matrix in a rational canonical form.

Definition 2.2.3. We call the form in (2.6) a **rational form of the non-singular matrix pencil** $A(X_1, X_2)$, denoted by $\mathrm{R}_A(X_1, X_2)$. Moreover, let $\mathrm{R} = \mathrm{diag}(C_1, \ldots, C_w)$ where each $C_i = \mathrm{CM}(F_i)$ for some monic polynomial $F_i(X)$ of degree n_i for $1 \leq i \leq w$ and $F_1(X)|\ldots|F_w(X)$. Then we call $F_i(X_1, X_2) = \det(C_iX_1 + I_{n_i}X_2)$, for all i, the **rational invariant factors** of the non-singular matrix pencil $A(X_1, X_2)$.

The following algorithm can be used when the matrix pencil $A(X_1, X_2)$ is non-singular.

Algorithm 2.1 Rational form of non-singular matrix pencil

Input: a non-singular pencil $A(X_1, X_2) = A_1X_1 + A_2X_2 \in \mathrm{M}_n(K[X_1, X_2])$
Output: a rational form of $\mathrm{R}_A(X_1, X_2) = \mathrm{R}X_1 + I_nX_2$

1: **loop**
2: **if** A_2 is non-singular **then**
3: compute the rational canonical form R of $A_2^{-1}A_1$
4: $\mathrm{R}_A(X_1, X_2) := \mathrm{R}X_1 + I_nX_2$
5: **else if** A_1 is non-singular **then**
6: swap $A_1 \leftrightarrow A_2$
7: $A(X_1, X_2) := A_1X_1 + A_2X_2$
8: **else** A_1 and A_2 are singular matrices
9: find z_0 such that $\det(A_1z_0 + A_2) \neq 0$
10: $A(X_1 = X_1 + z_0X_2, X_2 = X_2)$ is regular
11: $A_1 := A(X_1 = 1, X_2 = 0)$
12: $A_2 := A(X_1 = 0, X_2 = 1)$
13: $A(X_1, X_2) := A_1X_1 + A_2X_2$
14: **return** $\mathrm{R}_A(X_1, X_2)$

Remark 2.2.4. Since $\det(A(Z)) = a_0Z^n + a_1Z^{n-1} + \ldots + a_n \neq 0$ has at most n distinct roots and K is an infinite field, we can enumerate z_0 in line 9 over K until the condition $\det(A_1z_0 + A_2) \neq 0$ is true.

Example 2.2.5. Consider the matrix pencil

$$A(X_1, X_2) = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_1 + X_2 & 0 \\ 0 & 0 & X_2 \end{bmatrix} \in \mathrm{M}_3(\mathbb{Q}[X_1, X_2]).$$

Let $f = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $[X_1, X_2]f = [X_1 + X_2, X_2]$. Thus $A((X_1, X_2)f) = A(X_1 + X_2, X_2) = A_1X_1 + A_2'X_2$ where $A_2' = A_1 + A_2$. We have

$$A(X_1 + X_2, X_2) = \begin{bmatrix} X_1 + X_2 & 0 & 0 \\ 0 & X_1 + 2X_2 & 0 \\ 0 & 0 & X_2 \end{bmatrix}$$

is a regular matrix pencil since

$$A_2' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a non-singular matrix. Hence $(A_2')^{-1}A(X_1 + X_2, X_2) = (A_2')^{-1}A_1X_1 + I_nX_2$. Computing by Sage [51], we obtain $R_A(X_1, X_2) = \mathrm{R}X_1 + I_nX_2$ where

$$\mathrm{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

2.3 Minimal row and column dependences

Let $A(X_1, X_2)$ be a singular matrix pencil. The method, for writing a rational form in Section 2.5, requires obtaining the so-called minimal row and column dependences. In this section, we recall from [53, Section XI.5] the definition of minimal row and column dependences. We also write algorithms for obtaining an explicit row or column dependence.

Theorem 2.3.1 ([52, Theorem 1])**.** *Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a singular matrix pencil. Then:*

(i) If $d \geq e$, then there exists a nonzero row $u \in K[X_1, X_2]^d$ such that $uA(X_1, X_2) = 0$ and the nonzero components of u are homogeneous of the same degree.

(ii) If $d \leq e$, then there exists a nonzero column $v \in K[X_1, X_2]^e$ such that $A(X_1, X_2)v = 0$ and the nonzero components of v are homogeneous of the same degree.

Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a singular matrix pencil. we say that u as in Theorem 2.3.1(i) is a **row dependence** of $A(X_1, X_2)$. Similarly, we say that v as in Theorem 2.3.1(ii) is a **column dependence** of $A(X_1, X_2)$. The **order** of a row (or a column) dependence is the degree of the nonzero components in that row (or column) dependence. Moreover, we say that a row (or a column) dependence is a **minimum row dependence** (or **minimum column dependence**) if that row (or column) dependence has the smallest order.

Proposition 2.3.2. *Let $A(X_1, X_2)$ be a singular matrix pencil. Let u be a minimum row (or column) dependence of order m. Then one of the components of u contains a term of the form aX_1^m where $a \in K^\times$. Similarly, one of the components of u contains a term of the form bX_2^m where $b \in K^\times$.*

Proof. Suppose that u is a minimum row (or column) dependence of order m and does not contain a component with a term of the form aX_1^m. Then there exist a row (or a column) u_0, say, such that $u = X_2u_0$. Then u_0 is also a row dependence $u_0A(X_1, X_2) = 0$ (or column dependence $A(X_1, X_2)u_0 = 0$) whose nonzero components are homogeneous polynomials of degree $m-1$ which contradicts that u is a minimum row (or column) dependence. □

Next, we recall from [52] a method to determine the minimal order m of row and column dependences.

Definition 2.3.3 ([52, Section 3])**.** For $A_1, A_2 \in \mathrm{M}_{d\times e}(K)$, let $A(X_1, X_2) = A_1X_1 + A_2X_2$ be a singular matrix pencil. Define the block matrices $\mathrm{D}_m(A)$ and $\mathrm{E}_m(A)$, for $m = 1, 2, \ldots$, by the following:

$$\mathrm{D}_1(A) = \begin{bmatrix} A_1, A_2 \end{bmatrix}, \mathrm{D}_2(A) = \begin{bmatrix} A_1 & A_2 & 0 \\ 0 & A_1 & A_2 \end{bmatrix}, \mathrm{D}_3(A) = \begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & A_1 & A_2 \end{bmatrix}, \ldots$$

$$\mathrm{E}_1(A) = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \mathrm{E}_2(A) = \begin{bmatrix} A_1 & 0 \\ A_2 & A_1 \\ 0 & A_2 \end{bmatrix}, \mathrm{E}_3(A) = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & A_1 & 0 \\ 0 & A_2 & A_1 \\ 0 & 0 & A_2 \end{bmatrix}, \ldots$$

such that $\mathrm{D}_m(A)$ is a matrix of dm rows and $e(m+1)$ columns; $\mathrm{E}_m(A)$ is a matrix of $d(m+1)$ rows and em columns. Define $\delta_m(A) = dm - \mathrm{rank}(\mathrm{D}_m(A))$ and $\eta_m(A) = em - \mathrm{rank}(\mathrm{E}_m(A))$.

For a matrix pencil $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K)$, we always have $\delta_i(A), \eta_i(A) \geq 0$ for all $i \geq 1$.

Lemma 2.3.4. *Let u be a row dependence of order m for a singular matrix pencil $A(X_1, X_2)$. Then $\delta_{m+1}(A) > 0$. Similarly, if v is a column dependence of order m for a singular matrix pencil $A(X_1, X_2)$, then $\eta_{m+1}(A) > 0$.*

Proof. By Theorem 2.3.1, we can write $u = [u_1, \ldots, u_d]$ where u_i is either 0 or a homogeneous polynomial of order m for $1 \leq i \leq d$. Therefore, we write $u_i = a_{i0}X_1^m + a_{i1}X_1^{m-1}X_2 + \cdots + a_{im}X_2^m$ where $a_{ij} \in K$ for $0 \leq j \leq m$. Define $\bar{u} = [\bar{u}_0, \ldots, \bar{u}_m] \in K^{d(m+1)}$ where each $\bar{u}_j = [a_{1j}, \ldots, a_{dj}]$. Since $u = \bar{u}_0X_1^m + \bar{u}_1X_1^{m-1}X_2 + \cdots + \bar{u}_mX_2^m$ and $uA(X_1, X_2) = 0$, then

$$(\bar{u}_0X_1^m + \bar{u}_1X_1^{m-1}X_2 + \cdots + \bar{u}_mX_2^m)(A_1X_1 + A_2X_2) = 0.$$

Thus we obtain the following chain of vector equations:

$$0 = \bar{u}_0A_1, \bar{u}_0A_2 = -\bar{u}_1A_1, \ldots, \bar{u}_{m-1}A_2 = -\bar{u}_mA_1, \bar{u}_mA_2 = 0.$$

Hence $\bar{u}\mathrm{D}_{m+1}(A) = 0$. Therefore u is a row dependence of $\mathrm{D}_{m+1}(A)$. Thus $\mathrm{rank}(\mathrm{D}_{m+1}(A)) < m+1$ and $\delta_{m+1}(A) > 0$. Similarly, we prove $\eta_{m+1}(A) > 0$ if we have a column dependence of order m. □

Lemma 2.3.4 shows that the existence of $\delta_i(A) > 0$ or $\eta_i(A) > 0$ if $A(X_1, X_2)$ is a singular matrix pencil. Thus we can use the following theorem for obtaining a minimum row or column dependence.

Theorem 2.3.5 ([52, Theorem 2])**.** *Let $A(X_1, X_2)$ be a singular matrix pencil. If $\delta_{m+1}(A)$ is the first nonzero integer in the sequence $\delta_1, \delta_2, \ldots$, then m is the order of minimum row dependence. Similarly, the minimum order of column dependence m is given by the first nonzero integer $\eta_{m+1}(A)$.*

For $A_1, A_2 \in \mathrm{M}_{d\times e}(K)$, let $A(X_1, X_2) = A_1X_1 + A_2X_2$ be a singular matrix pencil with a minimal row dependence u of order m. By [52, Section 4], the row dependence u can be written explicitly as

$$u = \bar{u}_0X_1^m + \bar{u}_1X_1^{m-1}X_2 + \cdots + \bar{u}_mX_2^m \neq 0, \tag{2.7}$$

where each coefficient $\bar{u}_i$, for $0 \leq i \leq m$, are row vectors in K^d. The row vectors $\bar{u}_i$'s can be determined by a nonzero solution of the chain of vector equations:

$$0 = \bar{u}_0A_1, \bar{u}_0A_2 = -\bar{u}_1A_1, \ldots, \bar{u}_{m-1}A_2 = -\bar{u}_mA_1, \bar{u}_mA_2 = 0. \tag{2.8}$$

In particular, if $m = 0$, then we determined $\bar{u}_0A_1 = \bar{u}_0A_2 = 0$. Thus $u = \bar{u}_0$ since $\bar{u}_0A(X_1, X_2) = 0$. In general, let $\bar{u} = [\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_m] \in K^{dm}$ be a nonzero row vector, where $\bar{u}_0, \ldots, \bar{u}_m \in K^d$ are row vectors, such that $\bar{u}D_m(A) = 0$. Since $\mathrm{rank}(\mathrm{D}_{m+1}(A)) < dm$, a nonzero row vector $\bar{u}$ exists. Hence, we obtain the chain of vector equations in (2.8). By writing u as in (2.7), we obtain $uA(X_1, X_2) = 0$. Similarly, let $A(X_1, X_2)$ have a minimal column dependence v of order m. Then v can be written as

$$v = \bar{v}_0X_1^m + \bar{v}_1X_1^{m-1}X_2 + \cdots + \bar{v}_mX_2^m \neq 0, \tag{2.9}$$

where each coefficient $\bar{v}_i$, for $0 \leq i \leq m$, is a column vector in K^e. Similarly, the column vectors $\bar{v}_i$'s can be determined by a nonzero solution of the chain of vector equations:

$$0 = A_1\bar{v}_0, A_2\bar{v}_0 = -A_1\bar{v}_1, \ldots, A_2\bar{v}_{m-1} = -A_1\bar{v}_m, A_2\bar{v}_m = 0. \tag{2.10}$$

Remark 2.3.6. We can also obtain the rows $\bar{u}_0, \ldots, \bar{u}_m \in K^d$ in (2.8) once we obtain a nonzero row vector $\bar{u} = [\bar{u}_0, \ldots, \bar{u}_m] \in K^{dm}$ from the left kernel of $\mathrm{D}_{m+1}(A)$. Similarly, we can obtain the columns $\bar{v}_1, \ldots, \bar{v}_m \in K^e$ once we obtain a nonzero column $\bar{v} = [\bar{v}_0, \ldots, \bar{v}_m]^T \in K^{em}$ which lies in the right kernel of $\mathrm{E}_{m+1}(A)$.

Example 2.3.7. Let $A(X_1, X_2) \in \mathrm{M}_4(\mathbb{Q}[X_1.X_2])$ be a singular matrix pencil such that

$$A(X_1, X_2) = \begin{bmatrix} X_1 & X_1 + X_2 & 0 & 0 \\ X_2 & X_1 & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ X_1 + X_2 & 0 & 0 & 0 \end{bmatrix}.$$

Computing by Sage [51], the ranks of $\mathrm{D}_1(A), \mathrm{D}_2(A), \mathrm{D}_3(A)$ are $4, 8, 11$, respectively. Hence $\delta_3(A) = 1 > 0$ is the first nonzero in the sequence of $\delta_i(A)$ for $1 \leq i \leq d$. By Theorem 2.3.5, we have $m = 3 - 1 = 2$ is the order of the minimal row dependence. The following row vector is a nonzero row in the left kernel of $\mathrm{D}_3(A)$:

$$\bar{u} = [\bar{u}_0, \bar{u}_1, \bar{u}_2] = [1, -1, 0, -1, 1, -2, 0, 1, 0, -1, 0, 1].$$

Hence, we write $\bar{u}_0 = [1, -1, 0, -1], \bar{u}_1 = [1, -2, 0, 1]$, and $\bar{u}_2 = [-1, 1, 0, 1]$. By Equation (2.7), we obtain $u = [X_1^2 + X_1X_2, -(X_1 + X_2)^2, 0, -X_1^2 + X_1X_2 + X_2^2]$.

Similarly, the ranks of $\mathrm{E}_1(A), \mathrm{E}_2(A), \mathrm{E}_3(A)$ are $4, 7, 10$, respectively. Hence $\eta_2(A) = 1 > 0$ is the first nonzero in the sequence of $\eta_i(A)$ for $1 \leq i \leq e$. By Theorem 2.3.5, we have $m = 1$ is the order of the minimal column dependence. The following column is a nonzero column in the right kernel of $\mathrm{E}_2(A)$:

$$\bar{v} = [\bar{v}_0, \bar{v}_1]^T = [0, 0, 0, 1, 0, 0, -1, 0]^T.$$

Hence, we write $\bar{v}_0 = [0, 0, 0, 1]^T$ and $\bar{v}_1 = [0, 0, -1, 0]^T$. By Equation (2.9), $v = [0, 0, 0, -X_2, X_1]^T$.

The following algorithms are used to obtain an explicit row and column dependence for a singular matrix pencil $A(X_1, X_2)$.

Algorithm 2.2 Minimum row dependence

Input: a singular matrix pencil $A(X_1, X_2) = A_1X_1 + A_2X_2$
Output: a minimum row dependence of $A(X_1, X_2)$

1: find the least $m \geq 0$ such that $\delta_{m+1}(A) > 0$
2: find a nonzero row $\bar{u} = [\bar{u}_0, \ldots, \bar{u}_m]$ such that $\bar{u}D_{m+1}(A) = 0$
3: **return** $\bar{u}_0X_1^m + \bar{u}_1X_1^{m-1}X_2 + \cdots + \bar{u}_mX_2^m$

Algorithm 2.3 Minimum column dependence

Input: a singular matrix pencil $A(X_1, X_2) = A_1X_1 + A_2X_2$
Output: a minimum column v dependence of $A(X_1, X_2)$

1: find the least $m \geq 0$ such that $\eta_{m+1}(A) > 0$
2: find a nonzero column $\bar{v}^T = [\bar{v}_0, \ldots, \bar{v}_m]^T$ such that $E_{m+1}(A)\bar{v} = 0$
3: **return** $\bar{v}_0X_1^m - \bar{v}_1X_1^{m-1}X_2 + \cdots + \bar{v}_mX_2^m$

2.4 Canonical minimal submatrices

In this section, we recall from [53, Section IX.6] terminology about the so-called canonical minimal submatrices and how they occur in singular matrix pencils.

Definition 2.4.1 ([53, Section IX.6])**.** A row $\omega_{(m)}$ over $K[X_1, X_2]$ is called a **row of apolarity** if it is of the form

$$\omega_{(m)} = [X_2^m, -X_1X_2^{m-1}, \ldots, (-X_1)^m, 0, \ldots, 0], m \geq 0.$$

In particular, $\omega_{(0)} = [1, 0, \ldots, 0]$. Similarly, we call $\omega_{(m)}^T$ a **column of apolarity**.

Definition 2.4.2 ([53, Section IX.6])**.** Let $A(X_1, X_2)$ be a singular matrix pencil over $K[X_1, X_2]$. A submatrix $L_m(X_1, X_2)$ of the form

$$L_m(X_1, X_2) = \begin{bmatrix} X_1 & 0 & \ldots & 0 \\ X_2 & X_1 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & X_2 & X_1 \\ 0 & \ldots & 0 & X_2 \end{bmatrix} \in \mathrm{M}_{(m+1)\times m}([X_1, X_2])$$

is called a **canonical minimal submatrix** .

Note that the canonical minimal submatrix $L_0 \in \mathrm{M}_{1\times 0}(K[X_1, X_2])$ is a zero row. Therefore, we write

$$\mathrm{diag}(L_0, A(X_1, X_2)) = \begin{bmatrix} 0 \\ A(X_1, X_2) \end{bmatrix}.$$

Next, we show that for any change of variables occurs in $L_m(X_1, X_2)$, we can use non-singular transformations to regain the same canonical form.

Lemma 2.4.3. *Let $f \in \mathrm{GL}_2(K)$. Then $L_m(X_1, X_2)$ and $L_m((X_1, X_2)f)$ are equivalent over K.*

Proof. From [13, Section 2], the group $\mathrm{GL}_2(K)$ can be generated by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ y_0 & 1 \end{bmatrix}, y_0 \in K \right\} \cup \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}, y_1, y_2 \in K^\times \right\}.$$

Suppose that $f = \begin{bmatrix} 1 & 0 \\ y_0 & 1 \end{bmatrix}$ where $y_0 \in K$. Then $[X_1, X_2]f = [X_1 + y_0X_2, X_2]$. Let $\bar{\mathsf{g}}_i, \in \mathrm{GL}_{i+1}(K)$ and $\bar{\mathsf{h}}_i \in \mathrm{GL}_i(K)$ for $1 \le i \le m$ such that

$$\bar{\mathsf{g}}_1 = \begin{bmatrix} 1 & -y_0 \\ 0 & 1 \end{bmatrix}, \bar{\mathsf{g}}_2 = \begin{bmatrix} 1 & -y_0 & 0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots, \bar{\mathsf{g}}_m = \begin{bmatrix} 1 & -y_0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -y_0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix},$$

$$\bar{\mathsf{h}}_2 = \begin{bmatrix} 1 & y_0 \\ 0 & 1 \end{bmatrix}, \bar{\mathsf{h}}_3 = \begin{bmatrix} 1 & y_0 & y_0^2 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots, \bar{\mathsf{h}}_m = \begin{bmatrix} 1 & y_0 & y_0^2 & \cdots & y_0^{m-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t_0^2 \\ 0 & \cdots & 0 & 1 & y_0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

In particular, if $i = 1$, then $\bar{\mathsf{h}}_1 = 1$. Put $\mathsf{g}_i = \mathrm{diag}(I_{m-i}, \bar{\mathsf{g}}_i)$ and $\mathsf{h}_i = \mathrm{diag}(I_{m-i}, \bar{\mathsf{h}}_i)$ where $1 \le i \le m$. Let $\mathsf{g} = \mathsf{g}_m \ldots \mathsf{g}_1$ and $\mathsf{h} = \mathsf{h}_1 \ldots \mathsf{h}_m$. One can check that $\mathsf{g}L_m((X_1 + y_0X_2, X_2)\mathsf{h} = L_m(X_1, X_2)$. If $f = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then

$[X_1, X_2]f = [X_2, -X_1]$. Let $\mathcal{J}_j \in \mathrm{GL}_j(K)$ for $1 \leq j \leq m+1$ such that

$$\mathcal{J}_1 = 1, \mathcal{J}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ldots, \mathcal{J}_{m+1} = \begin{bmatrix} 0 & \cdots & 0 & 0 & -1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & (-1)^m & 0 & \cdots & 0 \\ (-1)^{m+1} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then one can check that $\mathcal{J}_{m+1} L_m(X_2, -X_1) \mathcal{J}_m = L_m(X_1, X_2)$. Moreover, if $f = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ where $y_1, y_2 \in K^\times$, then $[X_1, X_2]f = [y_1 X_1, y_2 X_2]$. Let $\mathcal{G}_{m+1} \in \mathrm{GL}_{m+1}(K)$ and $\mathcal{H}_m \in \mathrm{GL}_m(K)$ such that

$$\mathcal{G}_{m+1} = \begin{bmatrix} \frac{1}{y_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{y_2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{y_1}{y_2^2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \frac{y_1^{m-1}}{y_2^m} \end{bmatrix} \text{ and } \mathcal{H}_m \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{y_2}{y_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \frac{y_2^{m-1}}{y_1^{m-1}} \end{bmatrix}.$$

Then we can check that $\mathcal{G}_{m+1} L_m(y_1 X_1, y_2 X_2) \mathcal{H}_m = L_m(X_1, X_2)$. Now consider the group action in (1.1). Since $\mathrm{GL}_2(K)$ is generated by S, write $f = f_1 \cdots f_s \in \mathrm{GL}_2(K)$ where $f_k \in S$ for $1 \leq k \leq s$. By induction on s, if $s = 1$, then $f = f_1 \in S$. We proved that there exist $g_1 \in \mathrm{GL}_{m+1}(K)$ and $h_1 \in \mathrm{GL}_m(K)$ such that $L(X_1, X_2)(f_1^{-1}, g_1^{-1}, h_1) = g_1 L((X_1, X_2)f)h_1 = L(X_1, X_2)$ for all $f \in S$. Fix $s > 1$. Suppose that there exist $g_s \in \mathrm{GL}_{m+1}(K)$ and $h_s \in \mathrm{GL}_m(K)$ such that $L(X_1, X_2)(f^{-1}, g_s^{-1}, h_s) = L(X_1, X_2)$. For $s+1$, let $f_N = f_1 \cdots f_s$ and $f = f_N f_{s+1}$. Then there exist $g_1, g_s \in \mathrm{GL}_{m+1}(K)$ and $h_1, h_s \in \mathrm{GL}_m(K)$ such that

$$\begin{aligned} L(X_1, X_2)(f_N^{-1}, g_s^{-1}, h_s)(f_{s+1}^{-1}, g_1^{-1}, h_1) &= L(X_1, X_2)(f_N^{-1} f_{s+1}^{-1}, g_s^{-1} g_1^{-1}, h_s h_1) \\ &= L(X_1, X_2)(f^{-1}, (g_1 g_s)^{-1}, h_s h_1) \\ &= L(X_1, X_2). \end{aligned}$$

□

For simplicity, we shall use the notation L_m instead of $L_m(X_1, X_2)$ for canonical minimal submatrices. The following lemmas are essential to obtain a rational form for a singular matrix, as we shall see in Section 2.5.

Lemma 2.4.4 ([52, Lemma IX.6.I])**.** *Let $A(X_1, X_2) \in \mathrm{M}_{d \times e}(K[X_1, X_2])$ be a singular matrix pencil with row dependence u of order m. Then there exist $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that*

$$g^{-1} A(X_1, X_2) h = \begin{bmatrix} L_m & 0 \\ BX_2 & \bar{A}(X_1, X_2) \end{bmatrix}, \tag{2.11}$$

where $B \in \mathrm{M}_{(d-m-1)\times(m)}(K)$ *and* $\bar{A}(X_1, X_2) \in \mathrm{M}_{(d-m-1)\times(e-m)}(K[X_1, X_2])$ *is a matrix pencil. In particular, if* $m = 0$*, then we can assume* $h = I_e$ *so that*

$$g^{-1}A(X_1, X_2) = \begin{bmatrix} 0 \\ \bar{A}(X_1, X_2) \end{bmatrix}.$$

We use the following algorithm from the proof of [52, Lemma IX.6.I]. For a matrix pencil $A(X_1, X_2)$, we let $A(X_1, X_2)^{(i,j)}$ denote the submatrix pencil obtained by removing the first i rows and j columns. Similarly, for a row u, we let $u^{(\ell)}$ denote the row derived from u by omitting the first ℓ^{th} components.

Algorithm 2.4 Singular matrix pencil form in (2.11)

Input: a singular matrix pencil $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ with $d \geq e$
Output: a singular matrix pencil as in (2.11), which is equivalent to $A(X_1, X_2)$

1: using Algorithm 2.2 construct a minimum row dependence u of $A(X_1, X_2)$
2: **if** $m = 0$ **then**
3: find $g \in \mathrm{GL}_d(K)$ such that $ug = \omega_{(0)}$ and $h := I_e$
4: **else** $m > 0$
5: $\bar{A}(X_1, X_2) := A(X_1, X_2)$
6: $\ell := 0$
7: **repeat**
8: find $g_\ell \in \mathrm{GL}_d(K)$ such that the first component of $(ug_\ell)^{(\ell)}$ is $(-X_1)^\ell X_2^{m-\ell}$ and casts out any term contains $(-X_1)^\ell X_2^{m-\ell}$ from the remaining components in ug_ℓ
9: find $h_\ell \in \mathrm{GL}_e(K)$ such that $(g_\ell^{-1}\bar{A}(X_1, X_2)h_\ell)^{(\ell,\ell)}$ has a first row of the form $[aX_1, 0, \ldots, 0]$
10: find $f_\ell \in \mathrm{GL}_d(K)$ such that $(f_\ell g_\ell^{-1}\bar{A}(X_1, X_2)h_\ell)^{(\ell,\ell)}$ has X_1 in the leading entry and casts out any term contains X_1 in the first column except the leading entry
11: $u := ug_\ell f_\ell^{-1}$ and $\bar{A}(X_1, X_2) := f_\ell g_\ell^{-1}\bar{A}(X_1, X_2)h_\ell$
12: $\ell := \ell + 1$
13: **until** $\ell > m$
14: $g := g_0 f_0^{-1} \cdots g_m f_m^{-1}$, and $h := h_0 \cdots h_m$
15: **return** $g^{-1}A(X_1, X_2)h$ and m

Remark 2.4.5. In line 8, if $\ell = 0$, then Proposition 2.3.2 shows that at least one of the components of u contains a term of the form aX_2^m where $a \in K^\times$. Then $X_1^{-1}(ug_0)^{(1)}$ is a minimum row dependence of order $m-1$ of the matrix pencil $(g_0^{-1}\bar{A}(X_1, X_2))^{(1,1)}$, see the proof of [53, Lemma IX.6.I]. In general, we have $X_1^{-\ell}(u)^{(\ell)}$ is a minimum row dependence of order $m-\ell$ of $(\bar{A}(X_1, X_2))^{(\ell,\ell)}$. Therefore, Proposition 2.3.2 shows that at least one of the components of $u^{(\ell)}$ contains a term of the form $aX_1^\ell X_2^{m-\ell}$ where $a \in K^\times$.

Remark 2.4.6. In line 3, we have $m = 0$. Therefore u is a nonzero constant row in K^d. We can write g as a multiple of three matrices $g = \bar{g}_1\bar{g}_2\bar{g}_3$ where

$\bar{g}_1, \bar{g}_2, \bar{g}_3 \in \mathrm{GL}_d(K)$. If the first component of u is zero, then $\bar{g}_1$ swap the first component of u with any nonzero component. Otherwise $\bar{g}_1 = I_d$. If the first component of $u\bar{g}_1$ is a nonzero constant $a \neq 1$, then $\bar{g}_2$ multiply the first component by a^{-1}. Otherwise $\bar{g}_2 = I_d$. Finally, $\bar{g}_3$ casts out any nonzero component except the first component in $u\bar{g}_1\bar{g}_2$. Consider the matrix $I_d = [\mathrm{e}_1, \ldots, \mathrm{e}_d]$ where e_i's are the i column of I_d for $1 \leq i \leq d$. Those operations are given by the followings:

(i) To swap the i component with the j component in u, we multiply u by I_d after swapping e_i with e_j.

(ii) To multiply the i component in u by a constant a, we multiply u by I_d after multiplying e_i by a.

(iii) To cast out all nonzero components in u except i component, we multiply u by $[\mathrm{e}_1, \ldots, \mathrm{e}_{i-1}, (-\frac{u}{u_i})^T, \mathrm{e}_{i+1}, \ldots, \mathrm{e}_d]^T$ where u_i is the i component of u.

Similarly, we can write the operations which gives us g_ℓ in line 8 where g_ℓ makes the ℓ component in ug_ℓ contains a term $(-X_1)^\ell X_2$ and eliminates this term from the other components. In line 9, the first row in $(g_\ell^{-1}\bar{A}(X_1, X_2)h_\ell)^{(\ell,\ell)}$ does not contain a term in X_2 but it does contain a term of the form aX_1, see the proof of [53, Lemma IX.6.I]. Therefore h_ℓ and f_ℓ in line 9 and 10 can be obtained by operations similar to Gaussian elimination.

Lemma 2.4.7 ([52, Lemma IX.6.II])**.** *Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a singular matrix pencil with column dependence v of order m. Then there exist $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that*

$$gA(X_1, X_2)h^{-1} = \begin{bmatrix} L_m^T & BX_1 \\ 0 & \bar{A}(X_1, X_2) \end{bmatrix}, \tag{2.12}$$

where $B \in \mathrm{M}_{m\times(e-m-1)}(K)$ and $\bar{A}(X_1, X_2) \in \mathrm{M}_{(d-m)\times(e-m-1)}(K[X_1, X_2])$ is a matrix pencil. In particular, if $m = 0$, then we can assume $g = I_d$ so that

$$A(X_1, X_2)h^{-1} = \left[0, \bar{A}(X_1, X_2)\right].$$

Remark 2.4.8. We can use Algorithm 2.4 on the transpose of $A^T(X_1, X_2)$ to obtain a singular matrix pencil form as in (2.12).

2.5 Rational forms of singular matrix pencils

Here we recall from [53, Section IX.7] a method for writing rational forms for singular matrix pencils over arbitrary infinite fields.

Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a singular matrix pencil. If $A(X_1, X_2)$ has a minimal row dependence of order m_1, we can apply Lemma 2.4.4 on $A(X_1, X_2)$ to obtain a submatrix pencil as in (2.11), say $A_1(X_1, X_2)$. If $A_1(X_1, X_2)$ also have a minimal row dependence of order m_2, we can keep

applying Lemma 2.4.4 until we either exhaust the rows or arrive at some $A_k(X_1, X_2)$ which has independent rows (i.e. $A_k(X_1, X_2)$ has no row dependence)

$$\begin{bmatrix} L_{m_1} & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & L_{m_k} & 0 \\ * & \cdots & * & A_k(X_1, X_2) \end{bmatrix},$$

where $*$ are submatrices in X_2. Similarly, if $A_k(X_1, X_2)$ has a column dependence m_{k+1}, we can apply Lemma 2.4.7 on $A_k(X_1, X_2)$. Moreover, we can keep applying Lemma 2.4.7 until we either exhaust the columns or arrive at some non-singular matrix $\bar{A}(X_1, X_2)$ such that

$$\begin{bmatrix} L_{m_1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ * & \ddots & \ddots & & & & \vdots \\ * & * & L_{m_k} & 0 & \cdots & \cdots & 0 \\ * & * & * & L_{m_{k+1}}^T & * & * & * \\ * & * & * & 0 & \ddots & * & * \\ * & * & * & \vdots & \ddots & L_{m_\ell}^T & * \\ * & * & * & 0 & \cdots & 0 & \bar{A}(X_1, X_2) \end{bmatrix}, \tag{2.13}$$

where $*$ are submatrices in X_2. If $\bar{A}(X_1, X_2)$ exists, it can be either a non-singular polynomial matrix of degree one or a non-singular matrix pencil which can be replaced by its canonical form $\mathrm{R}_{\bar{A}}(X_1, X_2)$ using Algorithm 2.1 and the redundant $*$ can be cleared by row-column reductions, see proof of [52, Theorem IX.7.I]. In the case of $\bar{A}(X_1, X_2)$ is not regular, we replace $\bar{A}(X_1, X_2)$ with $\bar{A}((X_1, X_2)f)$ where f is defined as in Lemma 2.1.3. This step will replace $L_{m_i}(X_1, X_2)$'s and $L_{m_j}(X_1, X_2)^T$'s with $L_{m_i}((X_1, X_2)f)$'s and $L_{m_j}((X_1, X_2)f)^T$'s, for $1 \leq i \leq k < j \leq \ell$. By Lemma 2.4.3, we can retain $L_{m_i}(X_1, X_2)$'s and $L_{M_j}^T(X_1, X_2)$'s from $L_{m_i}((X_1, X_2)f)$'s and $L_{m_j}((X_1, X_2)f)^T$'s by non-singular transformations.

Theorem 2.5.1 ([53, Section IX.6 Theorem I])**.** *Let $A(X_1, X_2)$ be a singular matrix pencil. then $A(X_1, X_2)$ is weakly equivalent to a matrix pencil of the form*

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L_{m_{k+1}}^T, \ldots, L_{m_\ell}^T, \mathrm{R}_{\bar{A}}(X_1, X_2)), \tag{2.14}$$

where $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is either empty matrix or in a rational form of non-singular matrix of linear forms $\bar{A}(X_1, X_2)$.

Note that $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is an empty matrix if $\bar{A}(X_1, X_2)$ is an empty matrix. If $\bar{A}(X_1, X_2) = \bar{A}_1 X_1 + \bar{A}_2 X_2 \in \mathrm{M}_n(K[X_1, X_2])$ is not a matrix pencil (i.e.

$\bar{A}_1$ and $\bar{A}_2$ are not linearly independent), then A_1 or A_2 must be a non-singular matrix. We still can apply Algorithm 2.1 to $\bar{A}(X_1, X_2)$ to obtain $\mathrm{R}_{\bar{A}}(X_1, X_2) = \mathrm{R}X_1 + I_n X_2$ where R is a matrix in a rational canonical form.

Definition 2.5.2. Let $A(X_1, X_2)$ be a singular matrix pencil over $K[X_1, X_2]$. Then we call the form in (2.14) a **rational form of the singular matrix pencil** $A(X_1, X_2)$, denoted by $\mathrm{R}_A(X_1, X_2)$. We also call the non-singular matrix of linear forms $\bar{A}(X_1, X_2)$ the **non-singular core** of $A(X_1, X_2)$.

Next, we use the following algorithm from the proof of [52, Section IX.7 Theorem I]. For a matrix pencil $A(X_1, X_2)$, let $A(X_1, X_2)^{(i,j)}$ denote the submatrix of $A(X_1, X_2)$ after removing the first i rows and j columns.

Algorithm 2.5 Rational form of singular matrix pencil

Input: a singular matrix pencil $A(X_1, X_2)$
Output: a rational form $\mathrm{R}_A(X_1, X_2)$ of $A(X_1, X_2)$ as in (2.14)

$k := 0$
repeat
  while $A(X_1, X_2)$ is singular and $d \geq e$ **do**
    $k := k + 1$
    apply Algorithm 2.4 to $A(X_1, X_2)$ to obtain $B(X_1, X_2)$ and m_k
    $A(X_1, X_2) := B(X_1, X_2)^{(m_k+1, m_k)}$
  $\ell := 0$
  while $A(X_1, X_2)$ is singular and $d < e$ **do**
    $\ell := \ell + 1$
    apply Algorithm 2.4 to $A(X_1, X_2)^T$ to obtain $B(X_1, X_2)$ and m_ℓ
    $A(X_1, X_2) := (B(X_1, X_2)^T)^{(m_\ell, m_\ell+1)}$
until $A(X_1, X_2)$ is an empty matrix or a non-singular matrix pencil
if $A(X_1, X_2)$ is an empty matrix **then**
  return $\mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}})$
else
  use Algorithm 2.1 to obtain $\mathrm{R}_A(X_1, X_2)$
  return $\mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, \mathrm{R}_A(X_1, X_2))$

Example 2.5.3. Let $A(X_1, X_2) \in \mathrm{M}_{7\times 5}(\mathbb{Q}[X_1, X_2])$ be a matrix pencil where

$$A(X_1, X_2) = \begin{bmatrix} X_1 & X_1 + X_2 & 0 & 0 & 0 \\ X_2 & X_1 & 0 & 0 & 0 \\ 0 & 0 & X_1 & X_2 & X_2 \\ X_1 + X_2 & 0 & 0 & X_2 & X_2 \\ X_1 & 0 & X_2 & X_2 & X_2 \\ X_1 & X_2 & X_2 & 0 & 0 \\ X_1 & X_1 & X_2 & X_2 & X_2 \end{bmatrix}.$$

By using algorithm 2.2, we obtain a minimum row dependence

$$u_0 = [1, 1, 0, -1, 3, -1, -2].$$

Applying Algorithm 2.4, we obtain g_0 such that $u_0 g_0 = \omega_{(0)}$ where

$$g_0 = \begin{bmatrix} 1 & -1 & 0 & 1 & -3 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$g_0^{-1} A(X_1, X_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ X_2 & X_1 & 0 & 0 & 0 \\ 0 & 0 & X_1 & X_2 & X_2 \\ X_1 + X_2 & 0 & 0 & X_2 & X_2 \\ X_1 & 0 & X_2 & X_2 & X_2 \\ X_1 & X_1 & X_2 & 0 & 0 \\ X_1 & X_1 & X_2 & X_2 & X_2 \end{bmatrix}.$$

Hence, we have L_0 and $A_1(X_1, X_2) = (g_0^{-1} A(X_1, X_2))^{(1,0)}$. Similarly, we obtain a minimum row dependence for $A_1(X_1, X_2)$

$$u_1 = [X_2, X_2, X_1 - X_2, X_2 - X_1, 0, -X_2].$$

By Algorithm 2.4, we obtain $u_1 g_1 = \omega_{(1)}$ where

$$g_1 = \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover

$$h_1 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } f_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$f_1 g_1^{-1} A_1(X_1, X_2) h_1 = \begin{bmatrix} X_1 & 0 & 0 & 0 & 0 \\ X_2 & 0 & 0 & 0 & 0 \\ -X_2 & 0 & X_1 + X_2 & X_2 & X_2 \\ 0 & 0 & X_1 & X_2 & X_2 \\ 0 & X_2 & X_1 + X_2 & 0 & 0 \\ 0 & X_1 & X_1 + X_2 & X_2 & X_2 \end{bmatrix}.$$

Then we obtain L_1 and $A_2(X_1, X_2) = (f_1 g_1^{-1} A_1(X_1, X_2) h_1)^{(2,1)}$. By Algorithm 2.2, we obtain a minimum row dependence for $A_2(X_1, X_2)$

$$u_2 = [-X_1^2 - X_1 X_2 + X_2^2, X_1^2 + X_1 X_2, X_1 X_2, -X_2^2].$$

By Algorithm 2.4, we have $u_2 = \omega_{(2)}$ where

$$g_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover

$$h_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } f_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Then

$$f_2 g_2^{-1} A_2(X_1, X_2) h_2 = \begin{bmatrix} X_1 & 0 & 0 & 0 \\ X_2 & X_1 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ X_2 & -X_2 & X_2 & X_2 \end{bmatrix}.$$

Hence, we obtain L_2 and $A_3(X_1, X_2) = (f_2 g_2^{-1} A_2(X_1, X_2) h_2)^{(3,2)}$. By Algorithm 2.3, we obtain a minimum column dependence $A_3(X_1, X_2)\ u_3 = [1, -1]^T$. Since u_3^T is a minimum row dependence of $A_3(X_1, X_2)^T$, we use Algorithm 2.4 to obtain $u_3^T g_3 = \omega_{(0)}$ where $g_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $A_3(X_1, X_2) g_3^{-T} = [0, X_2]$. Then we obtain L_0^T and $A_4(X_1, X_2) = (A_3(X_1, X_2) g_3^{-T})^{(0,1)}$ is a non-singular matrix pencil. By Algorithm 2.1, the rational form of $A_4(X_1, X_2)$ is $\mathrm{R}_{A_4}(X_1, X_2) = [X_2]$.

Therefore $\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_0, L_1, L_2, L_0^T, \mathrm{R}_{A_4}(X_1, X_2))$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & 0 \\ X_2 & 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 \\ 0 & X_2 & X_1 & 0 & 0 \\ 0 & 0 & X_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_2 \end{bmatrix}.$$

2.6 Weak equivalence classes of matrix pencils

In this section, we recall some facts regarding rational forms of matrix pencils which allow us to write matrix pencils up to weak equivalence.

Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a matrix pencil with a rational form as in (2.14)

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L_{m_{k+1}}^T, \ldots, L_{m_{k+\ell}}^T, R_{\bar{A}}(X_1, X_2)),$$

where $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is the non-singular core. We use (2.14) as the general rational form. If $A(X_1, X_2)$ is a non-singular matrix pencil, we put $k = \ell = 0$. In this case, the non-singular core is $A(X_1, X_2)$.

Theorem 2.6.1 ([52, Sections XI.3–XI.4])**.** *For a matrix pencil $A(X_1, X_2)$, the invariant factors of the non-singular core of $A(X_1, X_2)$ are invariant under equivalent transformations of $A(X_1, X_2)$ in the field K and covariant (invariant up to change of variables) under weak equivalence.*

Theorem 2.6.1 shows that two non-singular matrix pencils are weakly equivalent if they have the same invariant factors up to change of variables. Moreover, Lemma 2.4.3 shows that two weakly equivalent singular matrix pencils have the same canonical minimal submatrices. Thus, if $\mathrm{R}_A(X_1, X_2)$ and $\bar{\mathrm{R}}_A(X_1, X_2)$ are rational forms of a matrix pencil $A(X_1, X_2)$ then $\mathrm{R}_A((X_1, X_2)f) = \bar{\mathrm{R}}_A(X_1, X_2)$ for some $f \in \mathrm{GL}_2(K)$.

Theorem 2.6.2 ([52, Theorem IX.8.II])**.** *Let $A(X_1, X_2), B(X_1, X_2)$ be two matrix pencils. Then $A(X_1, X_2)$ and $B(X_1, X_2)$ are weakly equivalent if and only if they have the same canonical minimal submatrices and same invariant factors up to change of variables.*

Let R be an integral domain with field of fractions K. Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a matrix pencil. By Corollary 1.2.3

(i) $M(A(X_1, X_2); R)$ and $M(\mathrm{R}_A(X_1, X_2); R)$ are rationally equivalent.

(ii) $\mathrm{Rel}(A(X_1, X_2); R)$ and $\mathrm{Rel}(\mathrm{R}_A(X_1, X_2); R)$ are rationally equivalent.

Therefore, we can use the general rational forms, as in (2.14), as rational equivalence classes of image and relation modules associated with matrix pencils over the integral domain R. In the later chapters, we compute ask zeta functions for these rational forms to obtain a generic formula.

Chapter 3

Compact discrete valuation rings

In this chapter, we collect some terminology and some facts found in [40, 41] regarding the so-called non-Archimedean valuations and non-Archimedean local fields. We also recall from [27] some facts about integration on non-Archimedean local fields. As in the previous chapters, K is an arbitrary field unless otherwise noted. We also denote by $\mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of integers, rational field, real numbers, and complex numbers, respectively.

3.1 Non-Archimedean valuations

Recall from [41, Section I.11] that a **local ring** is precisely a ring that has one maximal ideal. If a local ring is also a principal ideal domain, then we call it a **discrete valuation ring**.

Definition 3.1.1 ([40, Section 7.1])**.** For a field K, a **non-Archimedean absolute value** of K is a function

$$|\cdot| : K \to \mathbb{R},$$

with the following properties:

(i) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$,

(ii) $|xy| = |x||y|$, and

(iii) $|x + y| \leq \max\{|x|, |y|\}$ where $x, y \in K$.

For example, consider the p-adic absolute value $|\cdot|_p$ where p is a prime number. For $x \in \mathbb{Q}$, we can write $x = p^m \frac{a}{b}$ where $m, a, b \in \mathbb{Z}$ and p does not divide a or b. In this case, $|x|_p = p^{-m}$. For more about p-adic absolute value, see [41, Section II.2].

Remark 3.1.2 ([41, Section II.3])**.** If $|x| \neq |y|$, then $|x + y| = \max\{|x|, |y|\}$.

Define the **distance** between two points $x, y \in K$ by

$$\mathrm{d}(x, y) = |x - y|.$$

This function d is a metric. In particular, K together with this metric becomes a topological space. We say two absolute values are **equivalent** if they define the same topology on K. Then the absolute value $|\cdot|$ together with K admit a metric space, and hence in particular a topological space. The field K, together with $|\cdot|$, is called a **valued field**, denoted by $(K, |\cdot|)$.

Recall from [41, Section II.3] that let $|\cdot|$ be a non-Archimedean absolute value of K. Define a function

$$v : K \to \mathbb{R} \cup \{\infty\},$$

by $v(x) = -\log|x|$ for $x \neq 0$ and $v(0) = \infty$. This function v has the following properties for $x, y \in K$:

(i) $v(x) = \infty$ if and only if $x = 0$,

(ii) $v(xy) = v(x) + v(y)$, and

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

Then we call v a **non-Archimedean valuation** of K. A non-Archimedean valuation on K is called **discrete** if $v(K^\times) = s\mathbb{Z}$, for a real $s > 0$. Moreover, if $s = 1$, we say the valuation is **normalized**. If K is a number field, we call the valuation of an equivalence class of absolute value on K a **place**. We generally use the notation v for a normalized place.

Proposition 3.1.3 ([41, Proposition II.3.8])**.** *Let K be a field. If v is a non-Archimedean valuation then the subset*

$$\mathfrak{O} = \{x \in K : |x| \leq 1\} = \{x \in K : v(x) \geq 0\}$$

is a ring with group of units

$$\mathfrak{O}^\times = \{x \in K : |x| = 1\} = \{x \in K : v(x) = 0\},$$

and a unique maximal ideal

$$\mathfrak{P} = \{x \in K : |x| < 1\} = \{x \in K : v(x) > 0\}.$$

We call $\mathfrak{O}$ a **valuation ring**. For example, the vaulation ring $\mathfrak{O}$ could be the so-called p-adic integers $\mathbb{Z}_p$ with maximal ideal $p\mathbb{Z}_p$, see [41, Section II.1]. The field $\mathfrak{K} := \mathfrak{O}/\mathfrak{P}$ is called the **residue class field** of K. An element $\mathfrak{p} \in \mathfrak{P} \setminus \mathfrak{P}^2$ such that $v(\mathfrak{p}) = 1$ is called a **uniformizer** (or **prime element**) and every element $x \in K^\times$ admits a unique representation $x = \mathfrak{p}^m u$ where $m \in \mathbb{Z}$ and $u \in \mathfrak{O}^\times$. In this case $v(x) = m$. Moreover, $|x| = q^{-v(x)} = q^{-m}$ for a fixed real number $q > 1$. In this thesis, we normalize the absolute value $|\cdot|$ so that q is the size of the residue field $\mathfrak{K}$.

Proposition 3.1.4 ([41, Proposition II.3.9])**.** *If v is discrete valuation, then $\mathfrak{O}$ is a principal ideal domain with only one maximal ideal. Hence $\mathfrak{O}$ is a discrete valuation ring.*

Sometimes, we write $|\cdot|_v$ instead of $|\cdot|$ to denote an absolute value which represents the equivalence class v.

Definition 3.1.5 ([40, Section 7.6])**.** Let L/K be an extension of fields. For every place w of L, an absolute value $|\cdot|_w$ that represents the equivalence class w restricts to an absolute value on K that represents a place v of K. We write $w|v$ to indicate this relationship and say that w **divides** v (or w **lies over** v).

3.2 Non-Archimedean local fields

Let K be a field with an absolute value $|\cdot|$. Recall from [41, Definition II.4.1] that a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements in K is called a **Cauchy sequence** with respect to the absolute value $|\cdot|$ if, for every $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $n, m > N$. The field K is **complete** with respect to $|\cdot|$ if every Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$ in K converges to an element of $a \in K$.

$$\lim_{n\to\infty} |a_n - a| = 0.$$

Moreover, for any valued field $(K, |\cdot|)$, we obtain a complete valued field $(\mathcal{K}, |\cdot|)$ by the process of completion. For more details, see [41, Section II.4]. If K is a number field, we call its completion field the v-**adic completion** of K, denoted by K_v. In this case, we use the notation $\mathcal{O}_v$ for its valuation ring.

Definition 3.2.1 ([41, Section II.5])**.** A **non-Archimedean local field** is a field which is complete with respect to a discrete non-Archimedean valuation and has a finite residue class field.

Recall from [34, Section I.5.3] that a **directed set** is a partially ordered set $I = (I, \geq)$ such that for all $i, j \in I$ there exists $k \in I$ such that $i \geq k$ and $j \geq k$. An **inverse system** (A_i, σ_{ij}) of sets over I consists of a family of sets A_i, $i \in I$ and maps $\sigma_{ij} : A_i \to A_j$ whenever $j \leq i$. Satisfying the conditions

$$\sigma_{ii} = id_{A_i}, \text{ and } \sigma_{ij}\sigma_{jk} = \sigma_{ik} \text{ for all } i, j, k \in I \text{ with } k \leq j \leq i.$$

The **inverse limit** of the inverse system (A_i, σ_{ij}) is the set

$$\varprojlim_i A_i := \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i \mid a_i\sigma_{ij} = a_j \text{ whenever } j \leq j \right\}.$$

Proposition 3.2.2 ([41, Proposition II.4.5])**.** *Let $\mathcal{K}$ be complete with respect to a discrete valuation v. Let $\mathfrak{O}$ be the valuation ring with the maximal ideal $\mathfrak{P}$. For $n \geq 1$, we have the canonical homomorphisms $\mathfrak{O} \to \mathfrak{O}/\mathfrak{P}^n$ which gives us the isomorphism*

$$\mathfrak{O} \to \varprojlim_n \mathfrak{O}/\mathfrak{P}^n.$$

Recall from [34, Section I.3] that a topological space S is **compact** if every covering of S by open sets has a finite subcover. Moreover, S is **locally compact** if every $s \in S$ has a compact neighbourhood.

Proposition 3.2.3 ([41, Proposition II.5.1])**.** *A non-Archimedean local field $\mathcal{K}$ is locally compact, and its valuation ring $\mathfrak{O}$ is compact.*

For example. let $K = \mathbb{Q}$. Then we have a non-Archimedean local field $\mathcal{K}_v = \mathbb{Q}_p$ (p-adic number) in which case $\mathcal{O}_v = \mathbb{Z}_p$ (p-adic integers). Moreover, let $K = \mathbb{F}_q(x)$ be the rational functions with coefficients in a finite field $\mathbb{F}_q$. In this case, $\mathcal{K} = \mathbb{F}_q((x))$ (the field of formal Laurent series over $\mathbb{F}_q$) is the local field and $\mathfrak{O} = \mathbb{F}_q[[x]]$ (the ring of formal power series) its valuation ring.

Proposition 3.2.4 ([41, Proposition II.5.2])**.** *The non-Archimedean local fields are precisely the finite extensions of the fields $\mathbb{Q}_p$ and $\mathbb{F}_q((x))$.*

A finite extension of the field $\mathbb{Q}$ or $\mathbb{F}_q(x)$ is called **global field**, see [41, Chapter VI].

3.3 Hensel's Lemma

In this section, we recall Hensel's Lemma and derive other related statements. These statements will be used for later chapters. Let $\mathfrak{O}$ be a compact discrete valuation ring with a maximal ideal $\mathfrak{P}$ and a residue field $\mathfrak{K}$.

Lemma 3.3.1 (Hensel's Lemma [41, Proposition II.6.7])**.** *If a monic polynomial $f(X) \in \mathfrak{O}[X]$ admits a factorization over the residue class field $\mathfrak{K} = \mathfrak{O}/\mathfrak{P}$ as*

$$f(X) \equiv \bar{g}(X)\bar{h}(X) \bmod \mathfrak{P}$$

into relatively prime monic polynomials $\bar{g}, \bar{h} \in \mathfrak{K}[X]$, then $f(X)$ admits a factorization

$$f(X) = g(X)h(X)$$

into monic polynomials $g, h \in \mathfrak{O}[X]$ such that $g(X) \equiv \bar{g}(X) \bmod \mathfrak{P}$ and $h(X) \equiv \bar{h}(X) \bmod \mathfrak{P}$.

The following corollary is a special case of multivariable Hensel's Lemma.

Lemma 3.3.2. *Let $f \in \mathfrak{O}[X_1, \ldots, X_\ell]$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathfrak{O}^\ell$ such that $f(\alpha) \equiv 0 \bmod \mathfrak{P}$ and $\frac{\partial f(\alpha)}{\partial X_1} \not\equiv 0 \bmod \mathfrak{P}$. Then there exists a unique $x_1 \in \mathfrak{O}$ such that $x_1 \equiv \alpha_1 \bmod \mathfrak{P}$ and $f(x_1, \alpha_2, \ldots, \alpha_\ell) = 0$.*

Proof. Define $F = (f, X_2 - \alpha_2, \ldots, X_\ell - \alpha_\ell)$. By [9, Chapter III, §4.5, Corollary 2], there exists a unique $x \in \mathfrak{O}^\ell$ such that $x \equiv \alpha \bmod \mathfrak{P}$, $F(x) = 0$ and $x_k = \alpha_k$ for $2 \leq k \leq \ell$. □

Let K be a number field with ring of integers $\mathcal{O}$. Let $\mathcal{V}_K$ be the set of all non-Archimedean places of K. For $v \in \mathcal{V}_K$, let K_v be the v-adic completion of K with its valuation ring $\mathcal{O}_v$. Let $\mathfrak{P}_v$ be the maximal ideal of $\mathcal{O}_v$ and $\mathfrak{K}_v$ be the residue field.

Lemma 3.3.3. *Let $f(X)$ be a monic irreducible polynomial in $K[X]$. For almost all places $v \in \mathcal{V}_K$, if $f(\bar{\alpha}) \equiv 0 \bmod \mathfrak{P}_v$ where $\bar{\alpha} \in \mathfrak{K}_v$, there exists $\alpha \in \mathcal{O}_v$ such that for all $x \in \mathcal{O}_v$ with $x \equiv \bar{\alpha} \bmod \mathfrak{P}_v$*

$$|f(x)| = |x - \alpha|.$$

Proof. Let $n \in K[X]$ be a nonzero polynomial which divides both $f(X)$ and $f'(X)$. Note that $f'(X)$ is a nonzero polynomial. Since $\deg(n) \leq \deg(f') < \deg(f)$ and $f(X)$ is an irreducible polynomial in $K[X]$, then n must be a constant. By Bézout's identity, there exist $a, b \in K[X]$ such that $af(X) + bf'(X) = 1$. By multiplication of a suitable nonzero $N \in \mathcal{O}$, we obtain $Af(X) + Bf'(X) = N$ for some $A, B \in \mathcal{O}[X]$. For any place v with $\mathfrak{P}_v \nmid N$, we have $Af(X) + Bf'(X) = N \not\equiv 0 \bmod \mathfrak{P}_v$. Therefore $f(X)$ and $f'(X)$ have no non-unit common factor mod $\mathfrak{P}_v$. It follows that $f(X)$ has no repeated roots mod $\mathfrak{P}_v$. Let $\bar{\alpha} \in \mathfrak{K}_v$ such that $f(\bar{\alpha}) \equiv 0 \bmod \mathfrak{P}_v$. Since $f(X)$ is monic and has no repeated roots mod $\mathfrak{P}_v$, then $f(X) \equiv (X - \bar{\alpha})\bar{h}(X) \bmod \mathfrak{P}_v$ where $\bar{h}(\bar{\alpha}) \not\equiv 0 \bmod \mathfrak{P}_v$. By Hensel's Lemma 3.3.1, there exist monic polynomials $g, h \in \mathcal{O}_v[X]$ such that $f(X) = g(X)h(X)$ where $g(X) \equiv X - \bar{\alpha} \bmod \mathfrak{P}_v$ and $h(X) \equiv \bar{h}(X) \bmod \mathfrak{P}_v$. Since $g(X)$ is a monic and linear polynomial, there exists $\alpha \in \mathcal{O}_v$ such that $g(X) = X - \alpha$. Also, we have $h(\bar{\alpha}) \equiv \bar{h}(\bar{\alpha}) \not\equiv 0 \bmod \mathfrak{P}_v$. Thus for all $x \in \mathcal{O}_v$ with $x \equiv \bar{\alpha} \bmod \mathfrak{P}_v$, we have $|h(x)| = 1$ and $|f(x)| = |g(x)h(x)| = |x - \alpha|$. □

Lemma 3.3.4. *Let $f_1(X)$ and $f_2(X)$ be irreducible polynomials over $K[X]$. For almost all places $v \in \mathcal{V}_K$, if $\alpha \in \mathfrak{K}_v$ is a common root for $f_1(X)$ and $f_2(X)$ over $\mathfrak{K}$, then $f_1(X) = f_2(X)$.*

Proof. Since $f_1(X)$ and $f_2(X)$ are irreducible polynomials over $K[X]$, then either $f_1(X) = f_2(X)$ or $\gcd(f_1(X), f_2(X)) = 1$. Therefore, there exist a and $b \in K[X]$ such that $af_1(X) + bf_2(X) = 1$. For any place $v \in \mathcal{V}_K$ such that $af_1(X) + bf_2(X) \not\equiv 0 \bmod \mathfrak{P}_v$, we have $af_1(X) + bf_2(X) \equiv 1 \bmod \mathfrak{P}_v$. Since α is a common root in $f_1(X)$ and $f_2(X)$, we have $(X - \alpha)|(af_1(X) + bf_2(X))$ But $(X - \alpha) \nmid 1$. Thus $f_1(X) = f_2(X)$. □

3.4 Integration over compact discrete valuation ring

In this section, we recall the notion of integration over compact discrete valuation rings. These integrals will be used as a technique to compute the so-called ask zeta functions, see Section 4.5. Let $\mathfrak{O}$ be a compact discrete valuation ring. For $s \in \mathbb{C}$, we denote the real part of s by $\mathrm{Re}(s)$. For a non-empty subset $U \subset \mathfrak{O}$, we denote by $||U||$ the norm of U where $||U|| = \sup(|u| : u \in U)$.

Topology

Recall from [17, Section 0.7] that a **topological group** G is a group which is also a topological space such that group operations are continuous, i.e. the map

$$G \times G \to G,\ (g, h) \mapsto g^{-1}h$$

is continuous. Moreover, recall from [22, Section 2.2] that a Borel measure μ in a locally compact topological group is called a **Haar measure** if the followings hold:

(i) $\mu(U) > 0$ for every non-empty open set $U \subset G$, and

(ii) $\mu(gE) = \mu(E)$ for every $g \in G$ and every Borel set $E \subset G$.

Theorem 3.4.1 ([22, Theorem 2.10])**.** *Every locally compact topological group carries a Haar measure.*

A Haar measure in Theorem 3.4.1 is a unique measure up to multiplication by a positive real number, see [22, Theorem 2.20].

Let $\mathfrak{O}$ be a compact discrete valuation ring. Then $\mathfrak{O}$ is a topological group under addition. By Theorem 3.4.1, it admits a Haar measure μ such that $\mu(E+a) = \mu(E)$ for every Borel set $E \subset \mathfrak{O}$ and every $a \in \mathfrak{O}$. If we **normalize** μ, i.e. $\mu(\mathfrak{O}) = 1$, then the measure is unique.

Integration over $\mathfrak{O}$

For $\ell \geq 1$, let $X = (X_1, \ldots, X_\ell)$ consist of algebraically independent variables over the valuation ring $\mathfrak{O}$. Let $f_1(X), \ldots, f_m(X) \in \mathfrak{O}[X]$ be polynomials. Here, we consider the integral

$$\int_{\mathfrak{O}^\ell} \prod_{i=1}^{m} ||f_i(x)||_v^{s_i} \, \mathrm{d}\,\mu(x),$$

where $s_1, \ldots, s_m \in \mathbb{C}$ and μ is the additive normalized Haar measure on $\mathfrak{O}$. For $f(X) \in \mathfrak{O}[X]$, we obtain the special case

$$\mathsf{Z}_f(s) = \int_{\mathfrak{O}^\ell} |f(x)|_v^s \, \mathrm{d}\,\mu(x),$$

where s is a complex variable with $\mathrm{Re}(s) > 0$. This function is called **Igusa's local zeta function** associated with $f(x)$, see [30, Section 8.2]. If $\mathfrak{O} = \mathbb{Z}_p$, then we have the so-called p-adic integration.

Example 3.4.2 (Cf. [54, Example 1.22])**.** Consider $f(X) = X$. Since $\mathbb{Z}_p = \bigcup_{n=0}^{\infty} p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$, we have $|x|_p^s = p^{-ns}$ for $x \in p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$ and

$$\int_{\mathbb{Z}_p} |x|_p^s \, \mathrm{d}\,\mu(x) = \sum_{n=0}^{\infty} p^{-ns} \mu(p^n\mathbb{Z} \setminus p^{n+1}\mathbb{Z}_p).$$

Since $\mu(\mathbb{Z}_p) = 1$, we have $\mu(p^n\mathbb{Z}_p) = p^{-n}$ (see [54, Definition 1.17]). Thus

$$\begin{aligned}\int_{\mathbb{Z}_p} |x|_p^s \, \mathrm{d}\,\mu(x) &= \sum_{n=0}^{\infty} p^{-ns}(p^{-n} - p^{-(n+1)}) \\ &= (1 - p^{-1}) \sum_{n=0}^{\infty} p^{-(s+1)n} \\ &= \frac{1 - p^{-1}}{1 - p^{-s-1}}.\end{aligned}$$

For more about p-adic integration, see [38] and [54, Section 1.4]. Also, we show more applications on computing integrals over $\mathfrak{O}$ in Section 3.6.

Change of variables

Let $X = (X_1, \ldots, X_\ell)$ be algebraically independent variables over $\mathfrak{O}$. For a non-empty open subset $U \subset \mathfrak{O}^\ell$, a map $f_0 : U \to \mathfrak{O}$ is said to be an **analytic map** on U if for every point $a = (a_1, \ldots, a_\ell) \in U$, there exists a convergent power series $P_a(x) \in \mathfrak{O}[[x-a]] = \mathfrak{O}[[x_1 - a_1, \cdots, x_\ell - a_\ell]]$ such that $f(x) = P_a(x)$ for any point x in a neighbourhood of $a \in U$. If $f : U \to \mathfrak{O}^\ell$ where $f = (f_1, \ldots, f_\ell)$ and f_i's are analytic, then f is called **analytic**. Moreover, if f is invertible and both f and f^{-1} are analytic, then f is said to be **bi-analytic**, see [30, Section 2.4].

Theorem 3.4.3 ([12, Theorem 1.4.2])**.** *Let $J = (J_1, \ldots, J_\ell) : \mathfrak{O}^\ell \to \mathfrak{O}^\ell$ be an analytic map. Let U_1 be an open subset of $\mathfrak{O}^\ell$. Suppose that the determinant of the Jacobian matrix*

$$\det\left[\frac{\partial(J_1(x_0), \ldots, J_\ell(x_0))}{\partial(X_1, \ldots, X_\ell)}\right] \neq 0,$$

for all $x_0 \in U_1$. The J defines a bi-analytic map from U_1 to a set U_2. Let $f : U_2 \to \mathbb{R}$ be an integrable function. Then

$$\int_{U_2} f(y)\,\mathrm{d}\,\mu(y) = \int_{U_1} f(J(x)) \left|\det\left[\frac{\partial(J_1(x), \ldots, J_\ell(x))}{\partial(x_1, \ldots, x_\ell)}\right]\right| \mathrm{d}\,\mu(x).$$

3.5 Polyhedra and integrals

Here, we start with some terminology about polyhedra. Then, we recall a technique to compute integrals of the norms of monomial ideals by computing the generating functions associated with these monomial ideals. In this section, we denote by $\langle \cdot, \cdot \rangle$ the scalar product.

A **polyhedron** $\mathsf{P} \subset \mathbb{R}^n$ is a set defined by finitely many linear inequalities

$$\mathsf{P} = \{x \in \mathbb{R}^n : \langle \alpha_i, x \rangle \leq \beta_i, i \in I\},$$

where I is a finite set, $\alpha_i \in \mathbb{R}^n$, and β_i are real numbers. If $\alpha_i \in \mathbb{Z}^n$ and β_i are integers, then we say that P is a **rational polyhedron**. A **half-open polyhedron** is a polyhedron with some of its defining inequalities are strict. A bounded polyhedron is called a **polytope** and a polyhedron is a **cone** if $0 \in \mathsf{P}$ and for every $m \in \mathsf{P}$ and every $\lambda \geq 0$ we have $\lambda m \in \mathsf{P}$, for more details see [5, 6].

Definition 3.5.1 ([5, Definition VIII.2.1])**.** Let $\alpha_i \in \mathbb{Z}^n$. Then the set

$$\mathcal{C} = \{x \in \mathbb{R}^n : \langle \alpha_i, x \rangle \leq 0, i \in I\}$$

is called a **rational cone** where I is a finite set.

Recall from [5, Definition II.4.1] that for a vector space V, a point $s \in S \subset V$ is called an **extreme point** in S if for every two points $a, b \in S$ such that $\frac{a+b}{2} = s$, then $a = b = s$. If S is a polyhedron, then the extreme points in S are called **vertices**.

Definition 3.5.2 ([6, Definition 10.1])**.** A subset $S \subset \mathbb{R}^n$ is called a **lattice** if:

(i) S is an additive subgroup of $\mathbb{R}^n$,

(ii) S is discrete (i.e. for every bounded subset B of $\mathbb{R}^n$, the intersection $S \cap B$ is finite), and

(iii) S spans $\mathbb{R}^n$.

Recall from [4, Section 7.2] that a **lattice polytope** is a polytope with vertices in a given lattice.

Definition 3.5.3 ([5, Section VIII.1])**.** Let $\mathsf{C} \subset \mathbb{R}^n$ be a rational polyhedron and let $X = (X_1, \ldots, X_n)$ be independent variables over $\mathbb{R}$. Then the **generating function** for $\omega = (\omega_1, \ldots, \omega_n) \in \mathsf{C} \cap \mathbb{Z}^n$ is

$$[\mathsf{C}](X) = \sum_{\omega \in \mathsf{C} \cap \mathbb{Z}^n} X^{\omega},$$

where $X^{\omega} = X_1^{\omega_1} \cdots X_n^{\omega_n}$.

Recall from [6, Definition 4.1] that for $u, v \in \mathbb{R}^n$, the set $\{v + \tau u : \tau \in \mathbb{R}\}$ is called a line in $\mathbb{R}^n$.

Theorem 3.5.4 ([6, Theorem 13.8a(1)])**.** *Let $\mathsf{C} \subset \mathbb{R}^n$ be a non-empty half-open rational polyhedron which does not contain any line. Then $[\mathsf{C}](X)$ is a rational function belonging to $\mathbb{Q}(X)$.*

By the definition of a line, the assumption that C does not contain any line is satisfied when $\mathsf{P} \subset \mathbb{R}^n_{\geq 0}$.

The following proposition is used to compute integrals associated with the norms of monomial ideals. Let $\mathcal{K}$ be a non-Archimedean local field with valuation ring $\mathfrak{O}$. Let $\mathfrak{K}$ be the residue field with q denoting the size of $\mathfrak{K}$.

Proposition 3.5.5. *Let $\mathsf{C} \subset \mathbb{R}^n_{\geqslant 0}$ be a non-empty half-open rational cone. Let $\mathsf{P}_1, \ldots, \mathsf{P}_m \subset \mathbb{R}^n_{\geqslant 0}$ be non-empty lattice polytopes. Define $\mathbf{U}(\mathsf{C}) := \{x \in \mathcal{K}^n : v(x) \in \mathsf{C}\}$, where $v(x) = (v(x_1), \ldots, v(x_n))$. Then*

$$\int_{\mathbf{U}(\mathsf{C})} \prod_{j=1}^{m} ||\mathsf{P}_j(x)||^{s_j} \, \mathrm{d}\,\mu(x) = (1 - q^{-1})^n \sum_{\omega \in \mathsf{C} \cap \mathbb{Z}^n} q^{-\langle \mathbf{1}, \omega \rangle - \sum_{j=1}^{m} s_j \min(\langle \alpha, \omega \rangle : \alpha \in \mathsf{P}_j)},$$

where $\mathbf{1} = (1, \ldots, 1)$ and $s_1, \ldots, s_m \in \mathbb{C}$.

Proof. See the proof of [44, Proposition 3.9]. □

From Proposition 3.5.5, we derive the following corollary.

Corollary 3.5.6. *Let* $\mathsf{C} \subset \mathbb{R}^n_{\geqslant 0}$ *be a non-empty half-open rational cone and* $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}^n_{\geq 0}$. *Then:*

$$\int_{\mathbf{U}(\mathsf{C})} \prod_{j=1}^{m} |x^{\alpha_j}|^{s_j} \, \mathrm{d}\,\mu(x) = (1-q^{-1})^n [\mathsf{C}](q^{-1-\sum_{j=1}^m \alpha_{j1}s_j}, \ldots, q^{-1-\sum_{j=1}^m \alpha_{jn}s_j}).$$

Proof. This follow directly from Proposition 3.5.5 by setting $\mathsf{P}_j(x) = x^{\alpha_j}$ for $1 \leq j \leq m$. Then

$$\begin{aligned}\int_{\mathbf{U}(\mathsf{C})} \prod_{j=1}^{m} |x^{\alpha_j}|^{s_j} \, \mathrm{d}\,\mu(x) &= (1-q^{-1})^n \sum_{\omega \in \mathsf{C} \cap \mathbb{Z}^n} q^{-\langle 1,\omega\rangle - \langle \alpha_1,\omega\rangle s_1 - \cdots - \langle \alpha_m,\omega\rangle s_m} \\ &= (1-q^{-1})^n [\mathsf{C}](q^{-1-\sum_{j=1}^m \alpha_{j1}s_j}, \ldots, q^{-1-\sum_{j=1}^m \alpha_{jn}s_j}).\end{aligned}$$

□

In Section 3.6, we show examples about computing integrals over $\mathfrak{O}$ using 3.5.6.

3.6 Integral computations

Here, some applications on using polyhedra to compute integrals. The following lemmas are to help us compute ask zeta functions in later chapters. Let $\mathfrak{O}$ be a compact discrete valuation ring with maximal ideal $\mathfrak{P}$. Let $\mathfrak{K}$ be the residue field with size q.

Lemma 3.6.1. *Let* $A_1, \ldots, A_\ell \in \mathrm{M}_{d\times e}(\mathfrak{O})$ *and let* $X = (X_1, \ldots, X_\ell)$ *consist of algebraically independent variables over* $\mathfrak{O}$. *Define* $A(X) = A_1X_1 + \cdots + A_\ell X_\ell \in \mathrm{M}_{d\times e}(\mathfrak{O}[X])$. *Let* $x \in \mathfrak{O}^\ell$ *and* $y \in \mathfrak{O}$, *define* $I_k(x)$ *to be the ideal of* $\mathfrak{O}$ *generated by the nonzero* $k \times k$ *minors of* $A(x)$. *Then*

(i) $||I_k(x)|| \leq ||x||^k$.

(ii) $||I_k(x) \cup I_{k-1}(x)y|| = ||I_{k-1}(x)y||$ *if* $|y| \geq ||x||$.

Proof. For (i), let $A(x) = [a_{ij}(x)]_{i,j}$. Each $a_{ij}(x)$ can be written as the linear combination $\sum_{h=1}^{\ell} a_{hij}x_h$ where a_{hij} is the (i,j) entry in A_h. By Definition 3.1.1(iii), we have $|a_{ij}(x)| = |\,|\sum_{h=1}^{\ell} a_{hij}x_h|| = ||a_{1ij}x_1, \ldots, a_{\ell ij}x_\ell||$. Then $|a_{ij}(x)| \leq ||x_1, \ldots, x_\ell||$ since $|a_{hij}| \leq 1$. Hence $||I_1(x)|| = ||a_{11}(x), \ldots, a_{de}(x)|| \leq ||x||$. Suppose that (i) is true for $k = N-1$ where $N \geq 2$ and let $B(x) \in I_N(x)$. By Laplace expansion, we have $B(x) = \sum_{j=1}^{N}(-1)^{i+j}b_{ij}(x)B_{ij}(x)$ where $b_{ij}(x) \in I_1(x)$ and $B_{ij}(x) \in I_{N-1}(x)$. Therefore $||B(x)|| = ||\,|\sum_{j=1}^{N}(-1)^{i+j}b_{ij}(x)B_{ij}(x)|| \leq ||b_{i1}(x)B_{i1}(x), \ldots, b_{iN}(x)B_{iN}(x)|| \leq ||x|| \cdot ||B_{i1}(x), \ldots, B_{iN}(x)||$. Therefore $||I_N(x)|| \leq ||x|| \cdot ||I_{N-1}(x)|| \leq ||x||^N$. For (ii) similar to (i), we have $||I_k(x)|| \leq ||x|| \cdot ||I_{k-1}(x)|| \leq |y| \cdot ||I_{k-1}(x)||$. Therefore $||I_k(x) \cup I_{k-1}(x)y|| = ||I_{k-1}(x)y||$. □

The following two lemmas will be used to compute the integrals in Lemma 7.2.1, Theorem D, and Lemma 10.5.2. In the followings, we define a matrix of linear forms $A(X)$ as in Lemma 3.6.1.

Lemma 3.6.2. *Let $A(X)$ be a matrix of linear forms with $rank_{\mathcal{K}(X)}(A(X)) = r$. Let $\mathsf{C}_y = \{(v_1, \ldots, v_\ell, v_y) \in \mathbb{R}^{\ell+1}_{\geq 0} : v_y < v_1, \ldots, v_\ell\}$ be a half-open rational cone where $v_y = v(y)$ and $v_j = v(x_j)$ for $1 \leq j \leq \ell$. Define $\mathbf{U}(\mathsf{C}_y) := \{(x_1, \ldots, x_\ell, y) \in \mathfrak{O}^{\ell+1} : (v_1, \ldots, v_\ell, v_y) \in \mathsf{C}_y\}$. Let $I_k(X)$ be the ideal of $\mathfrak{O}$ generated by the nonzero $k \times k$ minors of $A(X)$ then*

$$\int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-d+r-1} \prod_{k=1}^{r} \frac{||I_{k-1}(x)||}{||I_k(x) \cup I_{k-1}(x)y||} \, \mathrm{d}\,\mu(x,y) = \frac{q^{-\ell}(1-q^{-1})}{1-q^{d-\ell}t}.$$

Proof. By Definition 3.5.3, we have the following generating function for C_y

$$\begin{aligned}
[\mathsf{C}_y](X_1, \ldots, X_\ell, Y) &= \sum_{v_1 = v_y+1}^{\infty} \cdots \sum_{v_\ell = v_y+1}^{\infty} \sum_{v_y=0}^{\infty} X_1^{v_1} \cdots X_\ell^{v_\ell} Y^{v_y} \\
&= X_1 \cdots X_\ell \sum_{v_1=0}^{\infty} X_1^{v_1} \cdots \sum_{v_\ell=0}^{\infty} X_2^{v_2} \sum_{v_y=0}^{\infty} (X_1 \ldots X_\ell Y)^{v_y} \\
&= \frac{X_1 \cdots X_\ell}{(1-X_1)\cdots(1-X_\ell)(1-X_1\cdots X_\ell Y)}. \qquad (3.1)
\end{aligned}$$

Since $|y| > ||x||$, Lemma 3.6.1(ii) shows that $||I_k(x) \cup I_{k-1}(x)y|| = ||I_{k-1}(x)y|| = ||I_{k-1}(x)|| \cdot |y|$. Hence

$$\begin{aligned}
&\int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-d+r-1} \prod_{k=1}^{r} \frac{||I_{k-1}(x)||}{||I_k(x) \cup I_{k-1}(x)y||} \, \mathrm{d}\,\mu(x,y) \\
&= \int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-d+r-1} \prod_{k=1}^{r} \frac{||I_{k-1}(x)||}{||I_{k-1}(x)|| \cdot |y|} \, \mathrm{d}\,\mu(x,y) \\
&= \int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-d-1} \, \mathrm{d}\,\mu(x,y). \qquad (3.2)
\end{aligned}$$

By Corollary 3.5.6 and (3.1), the integral in (3.2) becomes

$$\begin{aligned}
\int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-d-1} \, \mathrm{d}\,\mu(x,y) &= (1-q^{-1})^{\ell+1}[C_y](q^{-1}, \ldots, q^{-1}, q^d t) \\
&= \frac{q^{-\ell}(1-q^{-1})}{1-q^{d-\ell}t}.
\end{aligned}$$

□

Lemma 3.6.3. *Let $A(X)$ be a matrix of linear forms with $rank_{\mathcal{K}(X)}(A(X)) = r$. Let C_i be a half-open rational cone defined as follows:*

$$\mathsf{C}_i = \{(v_1, \ldots, v_\ell, v_y) \in \mathbb{R}^{\ell+1}_{\geq 0} : v_i < v_{i+1}, \ldots, v_\ell \text{ and } v_i \leq v_y, v_1, \ldots, v_{i-1}\},$$

where $v_y = v(y)$ and $v_i = v(x_i)$ for $1 \leq j \leq \ell$. Also, define $\mathbf{U}(\mathsf{C}_i) := \{(x_1, \ldots, x_\ell, y) \in \mathfrak{O}^{\ell+1} : (v_1, \ldots, v_\ell, v_y) \in \mathsf{C}_i\}$. Let $I_k(X)$ be the ideal of $\mathfrak{O}$ generated by the nonzero $k \times k$ minors of $A(X)$. For $x \in \mathfrak{O}^\ell$, if $||I_k(x)|| = |x_1|^k$ for all $1 \leq k \leq r$ then

$$\int_{\mathbf{U}(\mathsf{C}_i)} |y|^{s-d+r-1} \prod_{k=1}^{r} \frac{||I_{k-1}(x)||}{||I_k(x) \cup I_{k-1}(x)y||} \, \mathrm{d}\,\mu(x,y) = \frac{q^{-\ell+i}(1-q^{-1})^2}{(1-q^{d-r}t)(1-q^{d-\ell}t)}.$$

Proof. By Definition 3.5.3, we have the following generating function for C_i

$$\begin{aligned}[\mathsf{C}_i](X_1,\dots,X_\ell,Y) &= \sum_{v_y=v_i}^{\infty}\cdots\sum_{v_{i-1}=v_i}^{\infty}\sum_{v_i=0}^{\infty}\sum_{v_{i+1}=v_i+1}^{\infty}\cdots\sum_{v_\ell=v_i+1}^{\infty} X_1^{v_1}\cdots X_\ell^{v_\ell}Y^{v_y}\\ &= X_{i+1}\cdots X_\ell\sum_{v_i=0}^{\infty}(X_1\dots X_\ell Y)^{v_i}\sum_{v_y=0}^{\infty}Y^{v_y}\prod_{\substack{k=1\\k\neq i}}^{\ell}\sum_{v_k=0}^{\infty}X_k^{v_k}\\ &= \frac{\prod_{k=i+1}^{\ell}X_k}{\prod_{\substack{k=1\\k\neq i}}^{\ell}(1-X_k)(1-Y)(1-X_1\cdots X_\ell Y)}. \end{aligned} \tag{3.3}$$

Since $|x_i| > ||x||$, Lemma 3.6.1(i) shows that $||I_k(x)|| = |x_i|^k$. Hence

$$\begin{aligned}&\int_{\mathbf{U}(\mathsf{C}_i)}|y|^{s-d+r-1}\prod_{k=1}^{r}\frac{||I_{k-1}(x)||}{||I_k(x)\cup I_{k-1}(x)y||}\,\mathrm{d}\,\mu(x,y)\\ &= \int_{\mathbf{U}(\mathsf{C}_i)}|y|^{s-d+r-1}\prod_{k=1}^{r}\frac{|x_i|^{k-1}}{|x_i|^k}\,\mathrm{d}\,\mu(x,y)\\ &= \int_{\mathbf{U}(\mathsf{C}_i)}|y|^{s-d-1}|x_i|^{-r}\,\mathrm{d}\,\mu(x,y). \end{aligned} \tag{3.4}$$

By corollary 3.5.6 and (3.3), the integral in (3.4) becomes

$$(1-q^{-1})^{\ell+1}[C_i](q^{-1},\dots,q^{-1},q^{r-1},\dots,q^{-1},q^{d-r}t) = \frac{q^{-\ell+i}(1-q^{-1})^2}{(1-q^{d-r}t)(1-q^{d-\ell}t)}.$$

□

Chapter 4

Ask zeta functions

In this chapter, we recall from [47] the definition of ask zeta functions of modules. We also recall some facts about ask zeta functions and their abscissae of convergence. We consider a ring R to be associative, commutative, and unital.

4.1 Ask zeta functions

Let R be a ring. Let V and M be R-modules with $|V|, |M| < \infty$. Let $M \xrightarrow{\theta} \mathrm{Hom}(V, W)$ be a module representation. Then the **average size of the kernel** of M is

$$\mathrm{ask}(\theta) := \frac{1}{|M|} \sum_{a \in M} |\ker(a\theta)|.$$

Lemma 4.1.1 ([47, Lemma 2.1])**.** *Let $M \xrightarrow{\theta} \mathrm{Hom}(V, W)$ be a module representation. Let $M\theta$ be the image of M under θ then* $\mathrm{ask}(\theta) = \sum_{x \in V} |xM\theta|^{-1}$.

Lemma 4.1.1 shows that if we have two module representations $M \xrightarrow{\theta} \mathrm{Hom}(V, W)$ and $\bar{M} \xrightarrow{\bar{\theta}} \mathrm{Hom}(V, W)$ where $xM\theta = x\bar{M}\bar{\theta}$ for all $x \in V \setminus \{0\}$. Then $\mathrm{ask}(\theta) = \mathrm{ask}(\bar{\theta})$. Next, we will be using the following notations to define local ask zeta functions.

Notation 4.1.2. *Let $\mathfrak{O}$ be a compact discrete valuation ring with field of fractions $\mathcal{K}$ and maximal ideal $\mathfrak{P}$. Let $\mathfrak{K} := \mathfrak{O}/\mathfrak{P}$ denote the residue field, and let q and p denote the size and the characteristic of $\mathfrak{K}$, respectively. We write $\mathfrak{P}^n = \mathfrak{P} \cdots \mathfrak{P}$ for the nth ideal power of $\mathfrak{P}$ and $\mathfrak{O}^n = \mathfrak{O} \times \cdots \times \mathfrak{O}$ for the nth Cartesian power of $\mathfrak{O}$. For $\mathfrak{O}$-modules V and W, we write $V_n = V \otimes \mathfrak{O}/\mathfrak{P}^n$ and $W_n = W \otimes \mathfrak{O}/\mathfrak{P}^n$, respectively.*

Definition 4.1.3 (Cf. [47, Definition 1.3])**.** Let V and M be finitely generated $\mathfrak{O}$-modules. Let $M \xrightarrow{\theta} \mathrm{Hom}(V, W)$ be a module representation. Let θ_n denote the map $M \otimes \mathfrak{O}/\mathfrak{P}^n \to \mathrm{Hom}(V_n, W_n)$. Then the **ask zeta function** of θ is defined by

$$\mathsf{Z}_\theta^{\mathsf{ask}}(t) = \sum_{n=0}^{\infty} \mathrm{ask}(\theta_n) t^n.$$

In this thesis, we focus on modules of matrices $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$. Hence we consider $V = \mathfrak{O}^d$ and $W = \mathfrak{O}^e$. For a module of matrices $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$, let θ be the inclusion map $M \to \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$. Then we write $\mathsf{Z}^{\mathsf{ask}}_{\theta}(t) = \mathsf{Z}^{\mathsf{ask}}_{M}(t)$.

Example 4.1.4 ([47, Section 1])**.** Let $\{0_{d\times e}\} \xrightarrow{\theta} \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$ be the inclusion map. Then $|\ker(0_{d\times e}\theta_n)| = q^{dn}$ and $\mathrm{ask}(\theta_n) = q^{dn}$. Hence

$$\mathsf{Z}^{\mathsf{ask}}_{\{0_{d\times e}\}}(t) = \sum_{n=0}^{\infty} q^{dn}t^n = \frac{1}{1-q^dt}.$$

Corollary 4.1.5. *Let $M, \bar{M} \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be two submodules. Let $xM = x\bar{M}$ for every $x \in \mathfrak{O}^d \setminus \{0\}$. Then $\mathsf{Z}^{\mathsf{ask}}_{M}(t) = \mathsf{Z}^{\mathsf{ask}}_{\bar{M}}(t)$.*

Proof. Let $M \xrightarrow{\theta} \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$ and $M \xrightarrow{\bar{\theta}} \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$ be the inclusion maps. Since $xM\theta = x\bar{M}\bar{\theta}$, then $xM\theta_n = x\bar{M}\bar{\theta}_n$ for all n. By Lemma 4.1.1, we have $\mathrm{ask}(\theta_n) = \mathrm{ask}(\bar{\theta}_n)$. Therefore $\mathsf{Z}^{\mathsf{ask}}_{M}(t) = \mathsf{Z}^{\mathsf{ask}}_{\bar{M}}(t)$. □

Theorem 4.1.6 ([47, Theorem 4.10])**.** *Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule. Then there exist $f(t) \in \mathbb{Z}[t]$, nonzero $(a_1, b_1), \ldots, (a_m, b_m) \in \mathbb{Z} \times \mathbb{N}_0$, and $n \in \mathbb{N}_0$ such that*

$$\mathsf{Z}^{\mathsf{ask}}_{M}(t) = \frac{f(t)}{q^n(1-q^{a_1}t^{b1})\cdots(1-q^{a_m}t^{b_m})} \in \mathbb{Q}[t]. \tag{4.1}$$

4.2 Global and local ask zeta functions

Let R be a ring which contains only finitely many ideals $\mathfrak{a}$ of given finite norms $|R/\mathfrak{a}|$. Let M, V and W be finitely generated R-modules. Let θ be a module representation $M \to \mathrm{Hom}(V, W)$. Given $\mathfrak{a} \lhd R$, let $V_{\mathfrak{a}} = V \otimes R/\mathfrak{a}$ and $W_{\mathfrak{a}} = W \otimes R/\mathfrak{a}$. Let $\theta_{\mathfrak{a}}$ denote the map $M \otimes R/\mathfrak{a} \to \mathrm{Hom}(V_{\mathfrak{a}}, W_{\mathfrak{a}})$. Define a formal Dirichlet series

$$\zeta^{\mathsf{ask}}_{\theta}(s) = \sum_{\mathfrak{a}} \mathrm{ask}(\theta_{\mathfrak{a}}) \cdot |R/\mathfrak{a}|^{-s}, \tag{4.2}$$

where $s \in \mathbb{C}$. We also define the **abscissa of convergence**, denoted by α_θ, of the Dirichlet series $\zeta^{\mathsf{ask}}_{\theta}(s)$ as follows:

$$\alpha_\theta = \inf\{\alpha \in \mathbb{R} : \zeta^{\mathsf{ask}}_{\theta}(s) \text{ converges for every } s \text{ for which } \mathrm{Re}(s) > \alpha\}.$$

Next, we will be using the following notations to define the local and global ask zeta functions, see [47, Definition 3.1].

Notation 4.2.1. *Let K be a number field with ring of integers $\mathcal{O}$. Let $\mathcal{V}_K$ be the set of non-Archimedean places of K. For $v \in \mathcal{V}_K$, let K_v be the v-adic completion of K and $\mathcal{O}_v$ be its valuation ring with a maximal ideal $\mathfrak{P}_v$. Let $\mathfrak{K}_v$ denote the residue field with size q_v.*

For example, if $K = \mathbb{Q}$, non-Archimedean places of $\mathbb{Q}$ are the prime numbers (see the proof of [41, Proposition II.3.7]). For every p prime, we have $K_v = \mathbb{Q}_p$ the p-adic numbers and $\mathcal{O}_v = \mathbb{Z}_p$ the p-adic integers, see Section 3.2.

In Notation 4.2.1, we consider K_v and $\mathcal{O}_v$ to be finite extensions of $\mathbb{Q}_p$ and $\mathbb{Z}_p$, respectively.

Given $M \subset \mathrm{M}_{d\times e}(R)$, if θ is the inclusion map $M \to \mathrm{Hom}(R^d, R^e)$, we also write $\zeta^{\mathsf{ask}}_\theta(s) = \zeta^{\mathsf{ask}}_M(s)$ and $\alpha_\theta = \alpha_M$.

Global setting: Put $R = \mathcal{O}$ in (4.2). Then $\zeta^{\mathsf{ask}}_M(s)$ defines the **global zeta function** for $M \subset \mathrm{M}_{d\times e}(\mathcal{O})$. In this case, the abscissa of convergence α_M of $\zeta^{\mathsf{ask}}_M(s)$ is called the **global abscissa of convergence**.

Local setting: For $M_v \subset \mathrm{M}_{d\times e}(\mathcal{O}_v)$, the formal Dirichlet series $\zeta^{\mathsf{ask}}_{M_v}$ defines the **local zeta function** where $\zeta^{\mathsf{ask}}_{M_v}(s) = \mathsf{Z}^{\mathsf{ask}}_{M_v}(q_v^{-s})$. Then the abscissa of convergence of $\zeta^{\mathsf{ask}}_{M_v}(s)$, denoted by α_{M_v}, is called the **local abscissa of convergence**.

Proposition 4.2.2 ([47, Proposition 3.4(ii)])**.** *Let $M \subset \mathrm{M}_{d\times e}(\mathcal{O})$ be submodule. For $v \in \mathcal{V}_K$, let M_v be $\mathcal{O}_v$-submodule generated by M. Then $\zeta^{\mathsf{ask}}_M(s) = \prod_{v\in\mathcal{V}_K} \zeta^{\mathsf{ask}}_{M_v}(s)$.*

Recal from [47, Section 3.2] that the local abscissa of convergence is the largest real pole of $\mathsf{Z}^{\mathsf{ask}}_{M_v}(q_v^{-s})$.

Remark 4.2.3. It is well-known to experts in the area that removing finitely many Euler factors from a global zeta function does not affect the global abscissa of convergence. This analytic property was established by du Sautoy and Grunewald [19] for Euler products of cone integrals. Dung and Voll [21, Section 4.1] proved analogous results for representation zeta functions of nilpotent groups. The type of integrals which expresses their local zeta functions are essentially the same type of integrals that expresses local ask zeta functions of modules of matrices as we shall recall in Section 4.5.

The following proposition is analogous to Remark 4.2.3 for ask zeta functions.

Proposition 4.2.4. *Let M and M_v be defined as in Proposition 4.2.2. Let $S \subset \mathcal{V}_K$ be a finite set of places. Then the global abscissa of convergence is the abscissa of convergence of $\prod_{v\notin S} \mathsf{Z}^{\mathsf{ask}}_{M_v}(q_v^{-s})$.*

Proposition 4.2.4 allows us to obtain the global abscissa of convergence from generic formulas of ask zeta functions.

Theorem 4.2.5 ([47, Theorem 4.18])**.** *Let $M_v \subset \mathrm{M}_{d\times e}(\mathcal{O}_v)$ be a submodule. Let $\mathsf{Z}^{\mathsf{ask}}_{M_v}(t) = W(q_v, t)$. Then for almost all places $v \in \mathcal{V}_K$*

$$W(q_v, t) = \frac{W(q_v^{-1}, t^{-1})}{-q_v^d t}.$$

4.3 Ask and module representations

In this section, we recall from [49] ask zeta functions of isotopic module representations and Knuth duals. Here, we use Notation 4.1.2.

In the following Remark, the first part was noted in [49, Section 3.1] and the second part is an immediate consequence of the first part.

Remark 4.3.1. Let $M_1 \xrightarrow{\theta} \mathrm{Hom}(V_1, W_1)$ and $M_2 \xrightarrow{\bar{\theta}} \mathrm{Hom}(V_2, W_2)$ be isotopic module representations where M_i, V_i, and W_i be finitely generated $\mathfrak{O}$-modules. Then $\mathrm{ask}(\theta_n) = \mathrm{ask}(\bar{\theta}_n)$, and $\mathsf{Z}^{\mathsf{ask}}_{\theta}(t) = \mathsf{Z}^{\mathsf{ask}}_{\bar{\theta}}(t)$.

In the following theorem, for a ring R, let V and M be R-modules with $|V|, |M| < \infty$. Let $M \xrightarrow{\theta} \mathrm{Hom}(V, W)$ be a module representation.

Theorem 4.3.2 ([49, Theorem 5.2])**.** *Let $\theta^{\circ}, \theta^{\bullet}$ and $\theta^{\vee}$ be Knuth duals of the module representation θ. Then:*
(i) $\mathrm{ask}(\theta^{\circ}) = \frac{|M|}{|V|}\mathrm{ask}(\theta)$*, (ii)* $\mathrm{ask}(\theta^{\bullet}) = \mathrm{ask}(\theta)$*, and (iii)* $\mathrm{ask}(\theta^{\vee}) = \frac{|W|}{|V|}\mathrm{ask}(\theta)$.

In the following corollary, let $X = (X_1, \ldots, X_\ell)$ consist of algebraically independent variables over $\mathfrak{O}$. Let $A(X) \in \mathrm{M}_{d\times e}(\mathfrak{O}[X])$ be a matrix of linear forms.

Proposition 4.3.3 (Cf. [49, Corollary 5.6])**.** *Let $A^{\circ}(X), A^{\bullet}(X)$ and $A^{\vee}(X)$ be Knuth duals of $A(X)$. Then:*

$$\mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathfrak{O})}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(A^{\circ}(X);\mathfrak{O})}(q_v^{d-\ell}t) = \mathsf{Z}^{\mathsf{ask}}_{M(A^{\bullet}(X);\mathfrak{O})}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(A^{\vee}(X);\mathfrak{O})}(q_v^{d-e}t).$$

4.4 Weak equivalence and ask zeta functions

Here, we show that modules of weakly equivalent matrices of linear forms have the same ask zeta functions.

In the following corollary, we use Notation 4.1.2.

Corollary 4.4.1. *Let $A(X), B(X) \in \mathrm{M}_{d\times e}(\mathfrak{O}[X])$ be two matrices of linear forms. If $A(X)$ and $B(X)$ are weakly equivalent over $\mathfrak{O}$, then:*
(i) $\mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathfrak{O})}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(B(X);\mathfrak{O})}(t)$*, and (ii)* $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X);\mathfrak{O})}(t) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(B(X);\mathfrak{O})}(t)$.

Proof. By Corollary 1.2.3, we have $M(A(X);\mathfrak{O})$ is equivalent to $M(B(X);\mathfrak{O})$ and $\mathrm{Rel}(A(X);\mathfrak{O})$ is equivalent to $\mathrm{Rel}(B(X);\mathfrak{O})$. For (i), define θ_1 and θ_2 to be the inclusion maps such that $M(A(X);\mathfrak{O}) \xrightarrow{\theta_1} \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$ and $M(B(X);\mathfrak{O}) \xrightarrow{\theta_2} \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$. By Lemma 1.3.4, we have θ_1 and θ_2 are isotopic. By Remark 4.3.1, we have $\mathsf{Z}^{\mathsf{ask}}_{\theta_1}(t) = \mathsf{Z}^{\mathsf{ask}}_{\theta_2}(t)$. Similarly, we define $\tilde{\theta}_1$ and $\tilde{\theta}_2$ to be the inclusion maps such that $\mathrm{Rel}(A(X);\mathfrak{O}) \xrightarrow{\tilde{\theta}_1} \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$ and $\mathrm{Rel}(B(X);\mathfrak{O}) \xrightarrow{\tilde{\theta}_2} \mathrm{Hom}(\mathfrak{O}^d, \mathfrak{O}^e)$. Then $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are isotopic. Thus $\mathsf{Z}^{\mathsf{ask}}_{\tilde{\theta}_1}(t) = \mathsf{Z}^{\mathsf{ask}}_{\tilde{\theta}_2}(t)$. □

Next, we use Notation 4.2.1. We will show that if $A(X)$ and $B(X)$ are equivalent matrices over a number field K, then the modules associated with $A(X)$ and $B(X)$ are equivalent over $\mathcal{O}_v$ for almost all places $v \in \mathcal{V}_K$.

Lemma 4.4.2. *If $g \in \mathrm{GL}_d(K)$, then $g \in \mathrm{GL}_d(\mathcal{O}_v)$ for almost all places $v \in \mathcal{V}_k$.*

Proof. First, we want to show that if $x \in K^\times$, then $x \in \mathcal{O}_v$ for almost all places $v \in \mathcal{V}_K$. If $K = \mathbb{Q}$, we can write $x = \frac{a}{b}$ where $a, b \in \mathbb{Z}^\times$ and $\gcd(a, b) = 1$. Then $x \in \mathbb{Z}_p$ if and only if $0 \leq |x|_p \leq 1$. If $b = 1$ then clearly $x \in \mathbb{Z}_p$. For $b \neq 1$, we can write $b = \pm p_1^{s_1} \cdots p_n^{s_n}$ where $s_i > 0$ and $p_i \neq p_j$ for $i \neq j$. Define $S[x] := \{p_1, \ldots, p_n\}$. For any prime $p \notin S[x]$, then b is a unit in $\mathbb{Z}_p$. Hence $x \in \mathbb{Z}_p$. In general, let $x \in K^\times$ and let $f = a_0 + a_1 X + \ldots + a_{n-1} X^{n-1} + X^n \in \mathbb{Q}(X)$ be a minimal polynomial of x over $\mathbb{Q}$ where $X \in \mathcal{O}$. Then $a_i \in \mathbb{Z}_p$ for almost all primes p where $0 \leq i \leq n-1$. Hence $|x|_v \leq 1$ for $v|p$ and $x \in \mathcal{O}_v$. For $g_{ij} \in K$ be an entry in g, we have $g_{ij} \in \mathcal{O}_v$ for almost all places $v \in \mathcal{V}_K$ and $g \in \mathrm{M}_d(\mathcal{O}_v)$. Since $g \in \mathrm{GL}_d(K)$, we have $\det(g) \in K^\times$. Then $\det(g)$ and $\det(g)^{-1} \in \mathcal{O}_v$. Thus $\det(g) \in \mathcal{O}_v^\times$. Therefore $g \in \mathrm{GL}_d(\mathcal{O}_v)$. □

In the following, we will show that the modules associated with these matrices of linear forms have the same ask zeta function for almost all places $v \in \mathcal{V}_K$.

Corollary 4.4.3. *Let $A(X), B(X) \in \mathrm{M}_{d\times e}(K[X])$ be two matrices of linear forms. If $A(X)$ and $B(X)$ are weakly equivalent over K, then for almost all places $v \in \mathcal{V}_K$:*
(i) $\mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(B(X);\mathcal{O}_v)}(t)$, and (ii) $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(B(X);\mathcal{O}_v)}(t)$.

Proof. Since $A(X)$ and $B(X)$ are equivalent over K, there exist $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that $gA(X)h = B(X)$. By Lemma 4.4.2, we have $g \in \mathrm{GL}(\mathcal{O}_v)$ and $h \in \mathrm{GL}_e(\mathcal{O}_v)$ for almost all places $v \in \mathcal{V}_K$. Hence, the proof follows directly from Corollary 4.4.1. □

Note that the argument in the proof of Lemma 4.4.2 can be also used to show that if $A(X)$ and $B(X)$ are equivalent over K_v, then they are equivalent over $\mathcal{O}_v$ for almost all places $v \in \mathcal{V}_K$. Hence the proof of the following remark is the same in Corollary 4.4.3.

Remark 4.4.4. Let $A(X), B(X) \in \mathrm{M}_{d\times e}(\mathcal{O}_v[X])$ be two matrices of linear forms. If $A(X)$ and $B(X)$ are weakly equivalent over K_v, then for almost all places $v \in \mathcal{V}_K$:
(i) $\mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(B(X);\mathcal{O}_v)}(t)$, and (ii) $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(B(X);\mathcal{O}_v)}(t)$.

4.5 Computing ask zeta functions

Here, we recall from [47] methods to compute the ask zeta function of an image module of a matrix of linear forms over a compact discrete valuation ring $\mathfrak{O}$. In this section, we use Notation 4.1.2.

In the followings, let $A(X) = \sum_{i=1}^{\ell} A_i X_i \in \mathrm{M}_{d\times e}(\mathfrak{O}[X])$ be a matrix of linear forms.

Proposition 4.5.1 (Cf. [47, Corollary 4.9]). *Let* $\mathrm{rank}_{\mathcal{K}(X)}(A(X)) = r$. *Then*

$$(1-q^{-1})\cdot \mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathfrak{O})}(q^{-s}) = \int\limits_{\mathfrak{O}^{\ell+1}} |y|^{s+r-d-1}\cdot \prod_{i=1}^{r} \frac{||f_{i-1}(x)||}{||f_i(x)\cup f_{i-1}(x)y||}\,\mathrm{d}\mu(x,y),$$

where $f_i(X)$ *is the set of nonzero* $i\times i$ *minors of* $A(X)$.

In some cases, computing the image modules of Knuth duals of $A(X)$ can be less complicated than computing the image module of $A(X)$. The following corollary is used to compute the image module of $A(X)$ by integrating the set of minors of the image module of $A^{\circ}(X)$.

Proposition 4.5.2 (Cf. [47, Corollary 4.13]). *Let* $\mathrm{rank}_{\mathcal{K}(X)}(A^{\circ}(X)) = \bar{r}$. *Then*

$$(1-q^{-1})\cdot \mathsf{Z}^{\mathsf{ask}}_{M(A^{\circ}(X);\mathfrak{O})}(q^{-s}) = \int\limits_{\mathfrak{O}^{d+1}} |y|^{s+\bar{r}-d-1}\prod_{i=1}^{\bar{r}} \frac{||g_{i-1}(x)||}{||g_i(x)\cup g_{i-1}(x)y||}\,\mathrm{d}\mu(x,y),$$

where $g_i(X)$ *is the set of nonzero* $i\times i$ *minors of* $A^{\circ}(X)$.

Proposition 4.5.3 ([47, Proposition 1.5]). $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{M}_{d\times e}(\mathcal{O}_v)}(t) = \frac{1-q^{-e}t}{(1-t)(1-q^{d-e}t)}$.

Chapter 5

Orbit and conjugacy class zeta functions

In this chapter, we recall other related zeta functions, conjugacy class and orbit zeta functions. We also prove Theorem A which is analogue to [47, Proposition 8.13] for conjugacy class zeta functions. Theorem A allows us to convert conjugacy class zeta functions of saturable Lie groups into ask zeta functions of Lie algebras. Here, we use Notation 4.1.2. Let $\mathfrak{gl}_d(\mathfrak{O})$ denote the set of $d \times d$ Lie algebra matrices over a ring $\mathfrak{O}$. Let $\mathfrak{O}$ be the valuation ring of a local field $\mathcal{K} \supset \mathbb{Q}_p$, and let $\mathfrak{e}(\mathcal{K}/\mathbb{Q}_p)$ denote the ramification index of $\mathcal{K}/\mathbb{Q}_p$.

Theorem A. *Let $\mathfrak{g} \subseteq \mathfrak{gl}_d(\mathfrak{O})$ be a Lie subalgebra, where $\mathfrak{g}$ is an isolated submodule of $\mathfrak{gl}_d(\mathfrak{O})$. Let $G^m = \exp(\mathfrak{P}^m\mathfrak{g})$ where $m > \frac{\mathfrak{e}(\mathcal{K}/\mathbb{Q}_p)}{p-1}$ and $r = \operatorname{rank}\mathfrak{g}$. Then:*

(i) $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}}}(t) = \sum_{n=0}^{m} t^n + \sum_{n=m+1}^{2m-1} q^{r(n-m)}t^n + q^{rm}t^{2m}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{g}}}(t)$.

(ii) $\mathsf{Z}^{\mathsf{cc}}_{G^m}(t) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}}}(t)$.

5.1 Lie correspondence

In this section, we recall the correspondence between nilpotent Lie algebras and nilpotent finite p-groups. We also recall the notion of saturability (or p-saturable). The structure of saturability on Lie algebras allows us to use the correspondence between saturable p-groups and Lie algebras. Moreover, saturability allows us to use the ingredients in Section 5.3 for proving Theorem A.

Let R be a ring. Recall from [34, Section I.2.3] that a **Lie algebra** $\mathfrak{g}$ is an R-module equipped with a bilinear skew-symmetric map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and satisfies the Jacobi identity:

$$[x,x] = 0 \text{ and } [[x,y],z] + [[y,z],x] + [[z,x],y] = 0,$$

for all $x, y, z \in \mathfrak{g}$. Moreover, if R is a principal ideal domain and $\mathfrak{g}$ is a free R-module of finite rank, then we say $\mathfrak{g}$ is a **Lie lattice**.

For an indeterminate X, define

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \text{ and } \log(1+X) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{X^n}{n}. \tag{5.1}$$

Then the two formal series in (5.1) can translate between a nilpotent finite p-group G of nilpotency less than p and a nilpotent Lie ring $\mathfrak{g}$ of nilpotency class less than p whose additive group is a finite p-group via

$$\exp : \mathfrak{g} \to G \text{ and } \log : G \to \mathfrak{g}.$$

This correspondence is called **Lazard's correspondence**. We call G the **Lie group associated with** $\mathfrak{g}$ and we write $G = \exp(\mathfrak{g})$, see [34, Section I.2.4].

Definition 5.1.1 ([34, Section I.5.6])**.** A **pro-p group** is a topological group which is isomorphic to the inverse limit of finite groups of p-power order.

Let G be a group and $X, Y \leq G$. For $x \in X$ and $y \in Y$, define $[x,y] = x^{-1}y^{-1}xy$. We call $[x,y]$ the **commutator** of x with y. In the following definition, we write $[X,Y]$ to denote the subgroup of G which is generated by $[x,y]$ for all $x \in X$ and $y \in Y$.

Definition 5.1.2 ([34, Section I.8.3])**.** Let G be a pro-p group and let N be a closed normal subgroup of G. A **potent filtration** of N in G is a descending series $N_i, i \in \mathbb{N}$ of closed normal subgroups of G such that (i) $N_1 = N$, (ii) $\bigcap \{N_i | i \in \mathbb{N}\} = 1$, (iii) $[N_i, G] \subseteq N_{i+1}$ and $[N_{i,(p-1)}\, G] := [\cdots [[N_i, \underbrace{G], G] \cdots, G}_{p-1}] \subseteq N_{i+1}^p$ for all $i \in \mathbb{N}$.

In [37], Lazard developed the theory of saturable (p-saturable) groups. However, we will be using the description of González-Sánchez [24] of saturable pro-p groups in terms of potent filtrations.

Definition 5.1.3 ([24, Theorem A])**.** A torsion-free finitely generated pro-p group is called **saturable** if it admits a potent filtration.

For example, Lazard [37] proved that the pro-p Sylows of $\mathrm{GL}_d(\mathbb{Z}_p)$ and $\mathrm{SL}_d(\mathbb{Z}_p)$ with $n \leq p-2$ are saturable.

Let $\mathcal{L}/\mathcal{K}/\mathbb{Q}_p$ be finite field extensions. Let $\mathfrak{O}_{\mathcal{K}}$ and $\mathfrak{O}_{\mathcal{L}}$ denote the valuation rings of $\mathcal{K}$ and $\mathcal{L}$ with maximal ideals $\mathfrak{P}_{\mathcal{K}}$ and $\mathfrak{P}_{\mathcal{L}}$, respectively. Let $\mathfrak{p}_{\mathcal{K}}$ be the uniformizer of $\mathcal{K}$. Then $\mathfrak{p}_{\mathcal{K}}\, \mathfrak{O}_{\mathcal{L}} = \mathfrak{P}_{\mathcal{L}}^{\mathfrak{e}}$ for some positive integer $\mathfrak{e}$. The integer $\mathfrak{e}$ is called the **ramification index** for $\mathcal{L}$ over $\mathcal{K}$, denoted by $\mathfrak{e}(\mathcal{L}/\mathcal{K})$. If $\mathfrak{e}(\mathcal{L}/\mathcal{K}) = 1$, we say that $\mathcal{L}/\mathcal{K}$ is **unramified**, see [41, Sections I.8 & II.6]. In the following, recall Notation 4.1.2 and let $\mathbb{Q}_p \subset \mathcal{K}$.

Proposition 5.1.4 ([1, Proposition 2.3])**.** *Let $\mathfrak{g}$ be a Lie $\mathfrak{O}$-algebra whose underlying $\mathfrak{O}$-module is free of finite rank. Then $\mathfrak{P}^m\mathfrak{g}$ is saturable when $m > \frac{\mathfrak{e}(\mathcal{K}/\mathbb{Q}_p)}{p-1}$.*

For example, consider $\mathfrak{gl}_d(\mathbb{Z}_p)$ with $p > 2$. Then Proposition 5.1.4 shows that $p^m\mathfrak{gl}_d(\mathbb{Z}_p)$ is saturable for all $m \geq 1$.

Write $\mathfrak{gl}_d^m(\mathfrak{O}) := \{x \in \mathfrak{gl}_d(\mathfrak{O}) : x \equiv 0 \bmod \mathfrak{P}^m\}$ and $\mathrm{GL}_d^m(\mathfrak{O}) := \{x \in \mathrm{GL}_d(\mathfrak{O}) : x \equiv 1 \bmod \mathfrak{P}^m\}$. When $m > \frac{\mathfrak{e}(\mathcal{K}/\mathbb{Q}_p)}{p-1}$, then the two formal series in (5.1) induce the bijection maps:

$$\exp : \mathfrak{gl}_d^m(\mathfrak{O}) \to \mathrm{GL}_d^m(\mathfrak{O}) \text{ and } \log : \mathrm{GL}_d^m(\mathfrak{O}) \to \mathfrak{gl}_d^m(\mathfrak{O})$$

between a saturable Lie lattice $\mathfrak{g} \subset \mathfrak{gl}_d^m(\mathfrak{O})$ and a saturable pro-p group $G \subset \mathrm{GL}_d^m(\mathfrak{O})$, see [32, Lemma B.1]. We write $G := \exp(\mathfrak{g})$.

5.2 Orbit-counting and conjugacy class zeta functions

In this section, we recall other types of zeta functions, orbit-counting zeta functions and conjugacy class zeta functions. We also recall some cases where we can convert those zeta functions into ask zeta functions. In this section, we use Notation 4.1.2.

Definition 5.2.1. Let $G \leq \mathrm{GL}_d(\mathfrak{O})$ be a subgroup.

(i) ([47, Definition 1.2]) The **orbit-counting zeta function** of G is

$$\mathsf{Z}_G^{\mathsf{oc}}(t) = \sum_{n=0}^{\infty} |(\mathfrak{O}/\mathfrak{P}^n)^d/G|t^n.$$

(ii) ([18, Definition 1.1]) The **conjugacy class zeta function** of G is

$$\mathsf{Z}_G^{\mathsf{cc}}(t) = \sum_{n=0}^{\infty} k(G_n)t^n,$$

where G_n is the image of G under the map $\mathrm{GL}_d(\mathfrak{O}) \to \mathrm{GL}_d(\mathfrak{O}_n)$ and $k(G_n)$ denotes the number of the conjugacy classes of G_n.

Note that if $\mathfrak{g} \subset \mathfrak{gl}_d(\mathfrak{O})$ is an isolated submodule (i.e. $\mathfrak{gl}_d(\mathfrak{O})/\mathfrak{g}$ is torsion-free), then we may naturally identify $\mathfrak{g} \otimes \mathfrak{O}_n$ with the image of $\mathfrak{g}$ under the map $\mathfrak{gl}_d(\mathfrak{O}) \to \mathfrak{gl}_d(\mathfrak{O}_n)$, see [47, Section 4.1]. In the following Theorem, let $\mathrm{ad}_{\mathfrak{g}}$ denote the adjoint representation where $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $\mathrm{ad}_{\mathfrak{g}}(a) : x \mapsto [x, a]$ for $a \in \mathfrak{g}$. Also, let $\mathfrak{n}_d(\mathfrak{O}) \subset \mathfrak{gl}_d(\mathfrak{O})$ denote the Lie algebra of strictly upper triangular $d \times d$ matrices.

Theorem 5.2.2 ([47, Theorem 1.7])**.** *Let $\mathfrak{g} \subseteq \mathfrak{n}_d(\mathfrak{O})$ be a Lie subalgebra and $G := \exp(\mathfrak{g})$. Suppose that $p \geq d$ and $\mathfrak{g}$ is an isolated submodule of $\mathfrak{n}_d(\mathfrak{O})$. Then $\mathsf{Z}_G^{\mathsf{oc}}(t) = \mathsf{Z}_{\mathfrak{g}}^{\mathsf{ask}}(t)$, and $\mathsf{Z}_G^{\mathsf{cc}}(t) = \mathsf{Z}_{\mathrm{ad}_{\mathfrak{g}}}^{\mathsf{ask}}(t)$.*

Theorem 5.2.2 relies on Lazard's correspondence to convert orbit-counting and conjugacy class zeta functions of nilpotent groups associated with nilpotent Lie algebra to ask zeta functions of these nilpotent Lie algebra when they have nilpotency $d \leq p$.

Proposition 5.2.3 ([47, Proposition 8.13])**.** *Let $\mathfrak{O}$ be the valuation ring of a local field $\mathcal{K} \supset \mathbb{Q}_p$. Let $\mathfrak{g} \subseteq \mathfrak{gl}_d(\mathfrak{O})$ be a saturable subalgebra and define $G^m := \exp(\mathfrak{P}^m\mathfrak{g})$. If $m > \max(\mathfrak{e}(\mathcal{K}/\mathbb{Q}_p) - 1, \frac{\mathfrak{e}(\mathcal{K}/\mathbb{Q}_p)}{p-1})$, then $\mathsf{Z}_{G^m}^{\mathsf{oc}}(T) = \mathsf{Z}_{\mathfrak{P}^m\mathfrak{g}}^{\mathsf{ask}}(T)$.*

5.3 Preliminaries

This section contains the ingredients for proving Theorem A. For a group G, we denote by $\mathcal{C}_G(a)$ the centralizer of $a \in G$. We also use the notation $[X,Y]_{\text{Group}} = X^{-1}Y^{-1}XY$ for $X, Y \in G$. For a Lie algebra $\mathfrak{g}$, we use $[x,y]_{\text{Lie}} = xy - yx$ for $x, y \in \mathfrak{g}$.

Lemma 5.3.1. *For finite group G, the number of the conjugacy classes of G is*

$$k(G) = \frac{1}{|G|} \sum_{a \in G} |\mathcal{C}_G(a)|.$$

Proof. Consider the conjugacy action $f : G \times G \to G$ defined as $a \mapsto x^{-1}ax$, for all $x, a \in G$. The orbits of this action are the conjugacy classes (i.e. $k(G) = |O_f(G)|$, where $O_f(G)$ is the set of orbits in G under the group action f). Let $\text{stab}_f(a)$ denotes the stabilizer of $a \in G$ under the group action f. Then $\text{stab}_f(a) = \mathcal{C}_G(a)$. By the orbit-counting theorem

$$k(G) = |O_f(G)| = \frac{1}{|G|} \sum_{a \in G} |\text{stab}_f(a)| = \frac{1}{|G|} \sum_{a \in G} |\mathcal{C}_G(a)|.$$

□

Lemma 5.3.2. *Let $\theta : A \to B$ be a module epimorphism and let $|A|, |B| < \infty$. If $\beta \leq B$, then $|\alpha| = |\ker\theta| \cdot |\beta|$ where $\alpha := \beta\theta^{-1}$.*

Proof. Since $0 \in \beta$, we have $\ker(\theta) \leq \alpha$. By the 1st isomorphism theorem, we have $\alpha/\ker(\theta) \cong \beta$. Hence $|\alpha|/|\ker(\theta)| = |\beta|$. □

Recall from [34, Section I.7.8] that the Hausdorff formula gives an expression for

$$\Phi(X,Y) := \log(\exp(X)\exp(Y)) \in \mathbb{Q}[[X,Y]],$$

where X, Y are non-commuting indeterminates. A computation of the first three terms shows that

$$\Phi(X,Y) = X + Y + \frac{[X,Y]_{\text{Group}}}{2} + \frac{[X,[X,Y]_{\text{Group}}]_{\text{Group}} + Y, [Y,X]_{\text{Group}}}{12} + \cdots.$$

Corollary 5.3.3. *For some $m \geq 0$, let $\mathfrak{P}^m\mathfrak{g}$ be a saturable Lie algebra and let $G^m = \exp(\mathfrak{P}^m\mathfrak{g})$ be the Lie group associated with $\mathfrak{P}^m\mathfrak{g}$. Then $x, y \in \mathfrak{P}^m\mathfrak{g}$ commute if and only if $X, Y \in G^m$ commute where $X = \exp(x)$ and $Y = \exp(y)$.*

Proof. Consider the mapping $\exp : \mathfrak{P}^m\mathfrak{g} \to G^m$ and let $\text{ad}_{\mathfrak{P}^m\mathfrak{g}} : \mathfrak{P}^m\mathfrak{g} \to \mathfrak{gl}(\mathfrak{P}^m\mathfrak{g})$ denote the adjoint representation given by $\text{ad}_{\mathfrak{P}^m\mathfrak{g}}(y) : x \mapsto [x,y]_{\text{Lie}}$ for $y \in \mathfrak{P}^m\mathfrak{g}$. If $x \in \mathcal{C}_{\mathfrak{P}^m\mathfrak{g}}(y)$, then $[x,y]_{\text{Lie}} = 0$. By Hausdorff formula

$$\exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x,y]_{\text{Lie}} + \cdots).$$

Thus

$$XY = \exp(x)\exp(y) = \exp(x+y) = \exp(y)\exp(x) = YX.$$

Now consider the mapping $\log : G^m \to \mathfrak{P}^m\mathfrak{g}$. By [24, Theorem 4.2]

$$[x,y]_{Lie} = \log\big(\lim_{n\to\infty}[X^{p^n}, Y^{p^n}]^{p^{-2n}}_{\text{Group}}\big).$$

If $X \in \mathcal{C}_G(Y)$, then $[X,Y]_{\text{Group}} = 1$. By [24, Proposition 2.2]

$$\lim_{n\to\infty}[X^{p^n}, Y^{p^n}]^{p^{-2n}}_{\text{Group}} = 1.$$

Hence $[x,y]_{Lie} = 0$. □

Lemma 5.3.4 ([47, Lemma 8.2])**.** *Let $\mathfrak{g}$ be a p-saturable Lie $\mathbb{Z}_p$-algebra and let $\mathfrak{h}$ be a p-saturable subalgebra of finite additive index. Let G and H be the saturable pro-p groups associated with $\mathfrak{g}$ and $\mathfrak{h}$ via the Hausdorff series, respectively. Then $|G:H| = |\mathfrak{g}:\mathfrak{h}|$.*

5.4 Proof of Theorem A

Proof of Theorem A(i). Let $\mathfrak{g}_n$ denote the image of $\mathfrak{g}$ under the natural map $\mathfrak{gl}_d(\mathfrak{O}) \to \mathfrak{gl}_d(\mathfrak{O}_n)$. Let $\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n} : \mathfrak{P}^m\mathfrak{g}_n \to \mathfrak{gl}(\mathfrak{P}^m g_n)$ denote the adjoint representation given by $\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a) : \mathfrak{P}^m x \mapsto [\mathfrak{P}^m x, \mathfrak{P}^m a]_{\text{Lie}}$ for $a \in \mathfrak{g}_n$. By Definition 4.1.3

$$\begin{aligned}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}}}(t) &= \sum_{n=0}^{\infty}\mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n})t^n \\ &= \sum_{n=0}^{\infty}\frac{1}{|\mathfrak{P}^m\mathfrak{g}_n|}\sum_{\mathfrak{P}^m a\in\mathfrak{P}^m\mathfrak{g}_n}|\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a))|t^n.\end{aligned}$$

Note that if $m \geq n$, then $\mathfrak{P}^m\mathfrak{g}_n = \mathbf{0}_d \bmod \mathfrak{P}^n$. Therefore $|\mathfrak{P}^m\mathfrak{g}_n| = 1$ and $|\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a))| = 1$ for every $\mathfrak{P}^m a \in \mathfrak{P}^m\mathfrak{g}_n$. Hence

$$\mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}}) = 1. \tag{5.2}$$

If $m < n \leq 2m$, then

$$\begin{aligned}\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a)) &= \{\mathfrak{P}^m x \in \mathfrak{P}^m\mathfrak{g}_n : [\mathfrak{P}^m x, \mathfrak{P}^m a]_{\text{Lie}} \equiv 0 \bmod \mathfrak{P}^n\} \\ &= \{\mathfrak{P}^m x \in \mathfrak{P}^m\mathfrak{g}_n : \mathfrak{P}^{2m}[x,a]_{\text{Lie}} \equiv 0 \bmod \mathfrak{P}^n\} \\ &= \{\mathfrak{P}^m\mathfrak{g}_n\}.\end{aligned}$$

Therefore $|\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a))| = |\mathfrak{P}^m\mathfrak{g}_n| = q^{r(n-m)}$. Hence

$$\mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}}) = \frac{1}{|\mathfrak{P}^m\mathfrak{g}_n|}\sum_{\mathfrak{P}^m a\in\mathfrak{P}^m\mathfrak{g}_n}|\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a))| = |\mathfrak{P}^m\mathfrak{g}_n| = q^{r(n-m)}. \tag{5.3}$$

If $2m < n$, then

$$\begin{aligned}\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a)) &= \{\mathfrak{P}^m x \in \mathfrak{P}^m\mathfrak{g}_n : \ [\mathfrak{P}^m x, \mathfrak{P}^m a]_{\mathrm{Lie}} \equiv 0 \bmod \mathfrak{P}^n\} \\ &= \{\mathfrak{P}^m x \in \mathfrak{P}^m\mathfrak{g}_n : \ \mathfrak{P}^{2m}[x,a]_{\mathrm{Lie}} \equiv 0 \bmod \mathfrak{P}^n\} \\ &= \{x \in \mathfrak{g}_{n-2m} : \ [x,a]_{\mathrm{Lie}} \equiv 0 \bmod \mathfrak{P}^{n-2m}\}.\end{aligned}$$

Consider the module epimorphism $\theta : \mathfrak{P}^m\mathfrak{g}_n \to \mathfrak{g}_{n-2m}$. Then $\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a))$ is the preimage of $\ker(\mathrm{ad}_{\mathfrak{g}_{n-2m}}(a))$ under θ and by Lemma 5.3.2

$$\begin{aligned}|\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(\mathfrak{P}^m a))| &= |\ker\theta| \cdot |\ker(\mathrm{ad}_{\mathfrak{g}_{n-2m}}(a))| \\ &= |\mathfrak{g}_m| \cdot |\ker(\mathrm{ad}_{\mathfrak{g}_{n-2m}}(a))| \\ &= q^{rm} \cdot |\ker(\mathrm{ad}_{\mathfrak{g}_{n-2m}}(a))|.\end{aligned}$$

Therefore

$$\mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}) = q^{rm} \cdot \mathrm{ask}(\mathrm{ad}_{\mathfrak{g}_{n-2m}}). \tag{5.4}$$

Since $\mathfrak{g}$ is an isolated submodule of $\mathfrak{gl}_d(\mathfrak{O})$, we obtain the following by combining (5.2), (5.3), and (5.4)

$$\begin{aligned}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}}(t) &= \sum_{n=0}^{\infty} \mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}})t^n \\ &= \sum_{n=0}^{m} t^n + \sum_{n=m+1}^{2m-1} q^{r(n-m)}t^n + q^{rm}t^{2m}\sum_{n=2m}^{\infty} \mathrm{ask}(\mathrm{ad}_{\mathfrak{g}_{n-2m}})t^{n-2m} \\ &= \sum_{n=0}^{m} t^n + \sum_{n=m+1}^{2m-1} q^{r(n-m)}t^n + q^{rm}t^{2m}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{g}}}(t).\end{aligned}$$

□

Proof of Theorem A(ii). Let G_n^m denote the image of G^m in $\mathrm{GL}_d^m(\mathfrak{O}_n)$. Let $k(G_n^m)$ be the number of conjugacy classes of G_n^m. By Lemma 5.3.1

$$k(G_n^m) = \frac{1}{|G_n^m|}\sum_{A \in G_n^m} |\mathcal{C}_{G_n^m}(A)|,$$

where $\mathcal{C}_{G_n^m}(A)$ is the centralizer of A in G_n^m. Let $A = \exp(a)$ for some $a \in \mathfrak{P}^m\mathfrak{g}_n$. Now consider the mapping $\log : G_n^m \to \mathfrak{P}^m\mathfrak{g}_n$. By Lemma 5.3.3, we have correspondence between $\mathcal{C}_{G_n^m}(A)$ and $\mathcal{C}_{\mathfrak{P}^m\mathfrak{g}_n}(a)$. Consider the adjoint representation $\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n} : \mathfrak{P}^m\mathfrak{g}_n \to \mathfrak{gl}(\mathfrak{P}^m\mathfrak{g}_n)$ given by $\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(a) : x \mapsto [x,a]$ for $a \in \mathfrak{P}^m\mathfrak{g}_n$. Note that $\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(a)) = \mathcal{C}_{\mathfrak{P}^m\mathfrak{g}_n}(a)$. By Lemma 5.3.4

$$\begin{aligned}k(G_n^m) &= \frac{1}{|G_n^m|}\sum_{A \in G_n^m} |\mathcal{C}_{G_n^m}(A)| = \frac{1}{|\mathfrak{P}^m\mathfrak{g}_n|}\sum_{a \in \mathfrak{P}^m\mathfrak{g}_n} |\mathcal{C}_{\mathfrak{P}^m\mathfrak{g}_n}(a)| \\ &= \frac{1}{|\mathfrak{P}^m\mathfrak{g}_n|}\sum_{a \in \mathfrak{P}^m\mathfrak{g}_n} |\ker(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}(a))| = \mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n}).\end{aligned}$$

Since $\mathfrak{g}$ is an isolated submodule of $\mathfrak{gl}_d(\mathfrak{O})$, we have

$$\begin{aligned}\mathsf{Z}^{\mathsf{cc}}_{G^m}(t) &= \sum_{n=0}^{\infty} k(G^m_n)t^n = \sum_{n=0}^{\infty} \mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n})t^n \\ &= \mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}}}(t).\end{aligned}$$

□

Remark 5.4.1. (i) If $\mathfrak{g}$ is an abelian Lie algebra, then we have $|\mathfrak{P}^m\mathfrak{g}_n| = \mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n})$. Hence $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}}}(t) = \sum_{n=0}^{m-1} t^n + t^m\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{g}}}(t)$.

(ii) If G is an abelian group, then the Lie algebra $\mathfrak{g}$ which is associated with the Lie group G is also an abelian Lie algebra. Therefore, $|\mathfrak{P}^m\mathfrak{g}_n| = \mathrm{ask}(\mathrm{ad}_{\mathfrak{P}^m\mathfrak{g}_n})$ and we obtain $\mathsf{Z}^{\mathsf{cc}}_{G^m}(t) = \sum_{n=0}^{m-1} t^n + t^m\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{g}}}(t)$.

(iii) Theorem A does not generally hold if $\mathfrak{g}$ is not an isolated submodule of $\mathfrak{gl}_d(\mathfrak{O})$ (Cf. [47, Remark 8.18]).

Example 5.4.2. Let $\mathfrak{g} = \mathfrak{gl}_1(\mathbb{Z}_p)$ with $p > 2$, and define $G^m := \exp(p^m\mathfrak{g})$ where $m \in \mathbb{Z}_{\geq 0}$. Then

$$k(G^m_n) = \frac{1}{|G^m_n|}\sum_{a\in G^m_n} |C_{G^m_n}(a)|.$$

Since G^m_n is an abelian group, we have $|C_{G^m_n}(a)| = |G^m_n|$ for every $a \in G^m_n$. Note that G^m_n is trivial for $m \geq n$. Thus $|G^m_n| = 1$. However, if $m < n$, then $|G^m_n| = p^{n-m}$. Therefore, we have

$$\begin{aligned}\mathsf{Z}^{\mathsf{cc}}_{G^m}(t) &= \sum_{n=0}^{\infty} k(G^m_n)t^n = \sum_{n=0}^{\infty}\frac{1}{|G^m_n|}\sum_{a\in G^m_n}|C_{G^m_n}(a)|t^n \\ &= 1 + t + t^2 + \cdots + t^{m-1} + t^m\sum_{n=m}^{\infty} p^{n-m}t^{n-m} \\ &= 1 + t + t^2 + \cdots + t^{m-1} + \frac{t^m}{1-pt}.\end{aligned}$$

By comparing our answer with $L_{1,1}$ in [47, Section 9 Table 1], we obtain

$$\mathsf{Z}^{\mathsf{cc}}_{G^m}(t) = 1 + t + t^2 + \cdots + t^{m-1} + t^m\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}_{\mathfrak{g}}}(t).$$

Chapter 6

Pfaffians and ask zeta functions for antisymmetric matrices

In this section, we write an explicit formula to compute ask zeta functions of modules of antisymmetric matrices using the sets of all nonzero principal Pfaffians. Moreover, we prove Theorem B which is analogue to [47, Theorem 7.1] for the image modules of antisymmetric matrices of linear forms with smooth Pfaffian hypersurfaces. Also, a similar formula was given in [55, Theorem 3] for the so-called normal zeta function of class 2-nilpotent groups with smooth Pfaffian hypersurfaces. Using Notation 4.1.2, let $\mathsf{p} \in \mathfrak{P} \setminus \mathfrak{P}^2$ be the uniformizer. For an antisymmetric matrix A, let $\mathrm{Pf}(A)$ denote the Pfaffian of A where $(\mathrm{Pf}(A))^2 = \det(A)$. Moreover, let $\mathbb{P}^\ell(\mathfrak{K})$ denote the rational points of the projective ℓ-space over $\mathfrak{K}$. For a background on rational points, see [25, Chapter 5].

Theorem B. *Let $A_1, \dots, A_\ell \in \mathrm{M}_{2d}(\mathfrak{O})$ be antisymmetric matrices where $\ell \geq 1$. Let $X = (X_1, \dots, X_\ell)$ consist of algebraically independent variables over $\mathfrak{O}$. Write $A(X) = A_1X_1 + \cdots + A_\ell X_\ell \in \mathrm{M}_{2d}(\mathfrak{O}[X])$ and let $F(X) := \mathrm{Pf}(A(X))$. Suppose that the following smoothness condition is satisfied:*

$$\text{For all } \bar{x}_0 \in \mathfrak{K},\ \text{if } F(\bar{x}_0) = \frac{\partial F(\bar{x}_0)}{\partial X_1} = \cdots = \frac{\partial F(\bar{x}_0)}{\partial X_\ell} = 0,\ \text{then } \bar{x}_0 = 0.$$

Let $H := \{[x_1 : \cdots : x_\ell] \in \mathbb{P}^{\ell-1}(\mathfrak{K}) : F(x_1, \dots, x_\ell) = 0\}$. If q is odd, then

$$\mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathfrak{O})}(t) = \frac{1 - q^{-\ell}t}{(1-t)(1-q^{2d-\ell}t)} + \#H\frac{(q-1)^2(q+1)q^{-\ell}t}{(1-t)(1-qt)(1-q^{2d-\ell}t)}.$$

6.1 Formula for computing ask zeta functions of antisymmetric matrices

In [1, Section 3], Avni et al. introduced a new method to compute the so-called representation zeta functions associated with groups over $\mathfrak{O}$ in the form of antisymmetric matrices. Here, we use a similar approach to give a new method to compute ask zeta functions associated with antisymmetric matrices over $\mathfrak{O}$.

Since the valuation ring $\mathfrak{O}$ is a principal ideal domain, we can obtain the Smith normal form for any antisymmetric matrix $A \in \mathrm{M}_{2d}(\mathfrak{O})$ with $r = \frac{1}{2}\operatorname{rank}_{\mathcal{K}}(A)$ such that

$$gAh = \begin{bmatrix} \operatorname{diag}(\mathfrak{p}^{\lambda_1}, \ldots, \mathfrak{p}^{\lambda_{2r}}) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\lambda_1 \leq \cdots \leq \lambda_{2r}$ and $g, h \in \mathrm{GL}_{2d}(\mathfrak{O})$.

Theorem 6.1.1 (Cf. [39, Theorem 4.2])**.** *Let $A \in \mathrm{M}_{2d}(\mathfrak{O})$ be an antisymmetric matrix with $r = \frac{1}{2}rank_{\mathcal{K}}(A)$. Let $D = \operatorname{diag}(\mathfrak{p}^{\lambda_1}, \ldots, \mathfrak{p}^{\lambda_{2r}}, 0, \ldots, 0)$ be the Smith normal form of A. Then $\lambda_{2i-1} = \lambda_{2i}$ for all $i = 1, \ldots, r$.*

For an antisymmetric matrix $A \in \mathrm{M}_{2d}(\mathfrak{O})$, the smith normal form in Theorem 6.1.1 gives rise to "smith-like" form called the **antisymmetric canonical form**. We define this canonical form in the following corollary.

Proposition 6.1.2 (Cf. [39, Corollary 4.3])**.** *Let $A \in \mathrm{M}_{2d}(\mathfrak{O})$ be an antisymmetric matrix with smith normal form $D = \operatorname{diag}(\mathfrak{p}^{\lambda_1}, \mathfrak{p}^{\lambda_1}, \ldots, \mathfrak{p}^{\lambda_r}, \mathfrak{p}^{\lambda_r}, 0, \ldots, 0)$. Then there exist $g, h \in \mathrm{GL}_{2d}(\mathfrak{O})$ such that*

$$gAh = \begin{bmatrix} 0 & \mathfrak{p}^{\lambda_1} \\ -\mathfrak{p}^{\lambda_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \mathfrak{p}^{\lambda_r} \\ -\mathfrak{p}^{\lambda_r} & 0 \end{bmatrix} \oplus 0_{2(d-r)}. \tag{6.1}$$

Corollary 6.1.3. *Let $A(X) \in \mathrm{M}_{2d}(\mathfrak{O}[X])$ be an antisymmetric matrix of linear forms with $r = \frac{1}{2}\operatorname{rank}_{\mathcal{K}(X)}(A(X))$. For $x \in \mathfrak{O}$, define $f_i(x)$ to be the set of all the nonzero $i \times i$ minors of $A(x)$. Then*

$$\prod_{i=1}^{2r} \frac{||f_{i-1}(x)||}{||f_i(x) \cup f_{i-1}(x)y||} = \prod_{i=1}^{r} \frac{||f_{2(i-1)}(x)||}{||f_{2i}(x) \cup f_{2(i-1)}(x)y^2||}.$$

Proof. Let $g, h \in \mathrm{GL}_{2d}(\mathfrak{O})$ be defined as in (6.1). Define $\bar{f}_i(x)$ to be the set of nonzero $i \times i$ minors of $gA(x)h$. Recall from [10, Section 1.B] that the ideals generated by the nonzero minors are invariant under linear transformations. Therefore, we can replace $||f_i(x)||$ by $||\bar{f}_i(x)||$. Since $\lambda_1 \leq \cdots \leq \lambda_r$, we have $||\bar{f}_{2i}(x)|| = ||\bar{f}_{2i-1}(x)\mathfrak{p}^{\lambda_i}|| = ||\bar{f}_{2(i-1)}(x)\mathfrak{p}^{2\lambda_i}||$. Hence

$$\begin{aligned}
\prod_{i=1}^{2r} \frac{||\bar{f}_{i-1}(x)||}{||\bar{f}_i(x) \cup \bar{f}_{i-1}(x)y||} &= \prod_{i=1}^{r} \frac{||\bar{f}_{2(i-1)}(x)|| \cdot ||\bar{f}_{2i-1}(x)||}{||\bar{f}_{2i-1}(x) \cup \bar{f}_{2(i-1)}(x)y|| \cdot ||\bar{f}_{2i}(x) \cup \bar{f}_{2i-1}(x)y||} \\
&= \prod_{i=1}^{r} \frac{||\bar{f}_{2(i-1)}(x)||}{||\bar{f}_{2i-1}(x) \cup \bar{f}_{2(i-1)}(x)y|| \cdot ||\mathfrak{p}^{\lambda_i} \cup y||} \\
&= \prod_{i=1}^{r} \frac{||\bar{f}_{2(i-1)}(x)||}{||\bar{f}_{2i}(x) \cup \bar{f}_{2(i-1)}(x)\mathfrak{p}^{\lambda_i}y \cup \bar{f}_{2(i-1)}(x)y^2||} \\
&= \prod_{i=1}^{r} \frac{||\bar{f}_{2(i-1)}(x)||}{||\bar{f}_{2i}(x) \cup \bar{f}_{2(i-1)}(x)y^2||}.
\end{aligned}$$

□

For a commutative ring R whose additive group is torsion-free, let $A \in \mathrm{M}_{2d}(R)$ be an antisymmetric matrix (i.e. $A^T = -A$). Recall from [3, Section 3.2] that the **Pfaffian** of A, denoted by $\mathrm{Pf}(A)$, is defined by $(\mathrm{Pf}(A))^2 = \det(A)$. A **principal minor** is the determinant of a square submatrix obtained by removing the same number of rows and columns with the same indices. In an antisymmetric matrix, the square root of a principal minor of even order is called a **principal Pfaffian**.

In the following corollary, we write an explicit formula to compute ask zeta functions of modules associated with antisymmetric matrices of linear forms.

Corollary 6.1.4. *Let $A(X) \in \mathrm{M}_{2d}(\mathfrak{O}[X])$ be an antisymmetric matrix of linear forms with $r := \frac{1}{2}\operatorname{rank}_{\mathcal{K}(X)}(A(X))$. Write $M := M(A(X); \mathfrak{O})$. Then*

$$(1-q^{-1}) \cdot \mathsf{Z}^{\mathsf{ask}}_M(q^{-s}) = \int_{\mathfrak{O}^{\ell+1}} |y|^{s+2r-2d-1} \cdot \prod_{i=1}^{r} \frac{||F_{2(i-1)}(x)||^2}{||F_{2i}(x) \cup F_{2(i-1)}(x)y||^2} \,\mathrm{d}\mu(x,y),$$

where $F_{2i}(X)$ is the set of nonzero $2i \times 2i$ principal Pfaffians of $A(X)$.

Proof. By Proposition 4.5.1 and Corollary 6.1.3

$$(1-q^{-1}) \cdot \mathsf{Z}^{\mathsf{ask}}_M(q^{-s}) = \int_{\mathfrak{O}^{\ell+1}} |y|^{s+2r-2d-1} \cdot \prod_{i=1}^{r} \frac{||f_{2(i-1)}(x)||}{||f_{2i}(x) \cup f_{2(i-1)}(x)y^2||} \,\mathrm{d}\mu(x,y),$$

where $f_{2i}(X)$ is the set of all nonzero $2i \times 2i$ minors of $A(X)$. Let $\bar{f}_{2i}(X)$ be the set of all nonzero $2i \times 2i$ principal minors of $A(X)$. By [1, Remark 3.6], we replace the set of minors $f_{2i}(X)$ by the set of principal minors $\bar{f}_{2i}(X)$. Since $||F_{2i}(X)||^2 = ||\bar{f}_{2i}(X)||$, we have

$$\begin{aligned}(1-q^{-1}) \cdot \mathsf{Z}^{\mathsf{ask}}_M(q^{-s}) &= \int_{\mathfrak{O}^{\ell+1}} |y|^{s+2r-2d-1} \cdot \prod_{i=1}^{r} \frac{||\bar{f}_{2(i-1)}(x)||}{||\bar{f}_{2i}(x) \cup \bar{f}_{2(i-1)}(x)y^2||} \,\mathrm{d}\mu(x,y) \\ &= \int_{\mathfrak{O}^{\ell+1}} |y|^{s+2r-2d-1} \cdot \prod_{i=1}^{r} \frac{||F_{2(i-1)}(x)||^2}{||F_{2i}(x) \cup F_{2(i-1)}(x)y||^2} \,\mathrm{d}\mu(x,y).\end{aligned}$$

□

Example 6.1.5. Let $A(X) = \begin{bmatrix} 0 & X & 0 & 0 \\ -X & 0 & X & 0 \\ 0 & -X & 0 & X \\ 0 & 0 & -X & 0 \end{bmatrix}$. For $i = 1, 2$, we have $F_{2i}(X) = \{\pm X^i\}$. Using Corollary 6.1.4

$$(1-q^{-1})\mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathfrak{O})} = \int_{\mathfrak{O}^2} |y|^{s-1} \frac{|x|^2}{||x,y||^2 \cdot ||x^2,xy||^2} \,\mathrm{d}\mu(x,y). \tag{6.2}$$

Define $\mathbf{U}(\cdot) = \{(x,y) \in \mathfrak{O}^2 : (v_x, x_y) \in \cdot\}$. Then we divide the domain of the integral in (6.2) into $\mathbf{U}(\mathsf{C}_x)$ and $\mathbf{U}(\mathsf{C}_y)$ where

$$\mathsf{C}_x = \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : v_x \leq v_y\} \text{ and } \mathsf{C}_y = \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : v_x > v_y\},$$

where v_x and v_y denote the valuation of x and y, respectively. By Lemma 3.6.2 and Lemma 3.6.3 (put $\ell = 1$), we have

$$\begin{aligned} \mathsf{Z}^{\mathsf{ask}}_{M(A(X);\mathfrak{O})} &= \frac{q^{-1}}{1-q^3t} + \frac{(1-q^{-1})}{(1-t)(1-q^3t)} \\ &= \frac{1-q^{-1}t}{(1-t)(1-q^3t)}. \end{aligned}$$

Let $M \subset \mathrm{M}_d(\mathbb{Z})$ be a submodule consisting of antisymmetric matrices. A unipotent group scheme, denoted by G_M, can be constructed from M, see [50, Section 1.2]. This group scheme is called the **Baer group scheme associated with** M.

Proposition 6.1.6 (Cf. [50, Proposition 1.1])**.** *Let $M \subset \mathrm{M}_d(\mathbb{Z})$ be a submodule of antisymmetric matrices of $\mathbb{Z}$-rank ℓ. Let G_M be the Baer group scheme associated with M. Then $\zeta^{\mathsf{cc}}_{G_M}(s) = \zeta^{\mathsf{ask}}_M(s-\ell)$.*

Note that $\zeta^{\mathsf{cc}}_{G_M}(s)$ in Proposition 6.1.6 is defined in [50, Section 1.3]. Proposition 6.1.6 shows that conjugacy classes zeta functions of Baer group scheme associated with modules of antisymmetric matrices are instances of ask zeta functions of those modules of antisymmetric matrices. Hence, the method in Corollary 6.1.3 can also be used to obtain the conjugacy classes zeta functions of the Bear group scheme associated with modules of antisymmetric matrices.

6.2 Preliminaries

This section contains the ingredients for proving Theorem B. Let R be a commutative ring whose additive group is torsion-free.

Recall from [3, §3.2 Definition 1] that for an antisymmetric matrix $A \in \mathrm{M}_{2d}(R)$, we call the matrix $\mathrm{Padj}(A) = [a_{ij}]$ the **Pfaffian-adjugate matrix** of A where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ (-1)^{i+j}\mathrm{Pf}(A^{i,j}), & \text{if } i < j, \\ (-1)^{i+j+1}\mathrm{Pf}(A^{i,j}), & \text{if } i > j. \end{cases}$$

Note that $A^{i,j}$ denotes the $(2d-2) \times (2d-2)$ submatrix of A obtained by removing the i and j rows and columns.

Next, we recall some facts about Padj(A).

Proposition 6.2.1 ([3, §3.2 Corollary 1])**.** *Let $A \in \mathrm{M}_{2d}(R)$ be an antisymmetric matrix. Then*

$$A \cdot \mathrm{Padj}(A) = \mathrm{Pf}(A) \cdot I_{2d}.$$

In particular, if A is invertible, then $A^{-1} = \frac{1}{\mathrm{Pf}(A)}\mathrm{Padj(A)}$.

In the following, we assume 2 is a unit in R.

Lemma 6.2.2 ([3, §3.2 Lemma 1])**.** *Let $A(t) \in \mathrm{M}_{2d}(R[t])$ be an antisymmetric matrix. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathrm{Pf}(A(t)) = \frac{1}{2}\mathrm{tr}(A'(t)\cdot \mathrm{Padj}(A(t)),$$

where $A'(t) = \frac{d}{dt}A(t)$.

Recall Notation 4.1.2. In the following lemma, let q be an odd number. Let $I_{2i}(X)$ be the ideal of $\mathfrak{O}[X]$ generated by the set of all nonzero $2i \times 2i$ Pfaffians of $a(X)$.

Lemma 6.2.3. *Let $F(X)$ be defined as in Theorem B. Assume the smoothness condition in Theorem B is satisfied. Then:*

(i) $\frac{\partial F(X)}{\partial X_k} \in I_{2d-2}(X)$*, for all* $k = 1, \ldots, \ell$.

(ii) Let $\bar{x} \in \mathfrak{K}^\ell \setminus \{0\}$*. Then* $\mathrm{rank}_{\mathfrak{K}}(A(\bar{x})) \in \{2d-2, 2d\}$.

Proof. For (i), Lemma 6.2.2 gives us the Pfaffian version of the Jacobi's formula

$$\frac{\partial}{\partial X_k}\mathrm{Pf}(A(X)) = \frac{1}{2}\sum_{i=1}^{2d}\sum_{j=1}^{2d}\frac{\partial a_{ij}(X)}{\partial X_k}\cdot \mathrm{Pf}(A^{i,j}(X)),$$

where $\mathrm{Pf}(A^{i,j}(X)) \in I_{2d-2}(X)$ and $a_{ij}(X)$ is the (i,j) entry in $A(X)$. Hence $\frac{\partial F(X)}{\partial X_k} = \frac{\partial \mathrm{Pf}(A(X))}{\partial X_k} \in I_{2d-2}(X)$. For (ii), suppose that $\mathrm{rank}_{\mathfrak{K}}(A(\bar{x})) < 2d-2$ for some $\bar{x} \in \mathfrak{K}^\ell$. Then $F(\bar{x}) = 0$ and $I_{2d-2}(\bar{x}) = 0$. However, part (i) and the smoothness condition in Theorem B imply that $\bar{x} = 0$. □

6.3 Proof of Theorem B

In this section, we will prove Theorem B by using a similar approach to the proof of [47, Theorem 7.1]. We also give an example that can be solved by using Theorem B instead of [47, Theorem 7.1].

Proof of Theorem B. By Lemma 6.2.3(ii), we have $\mathrm{rank}_{\mathfrak{K}}(A(\bar{x})) \geq 2d-2$ for all $\bar{x} \in \mathfrak{K}^\ell \setminus \{0\}$. Hence $||I_{2i}(x)|| = 1$ for $0 \leq i \leq d-1$. Thus

$$\prod_{i=1}^{d}\frac{||I_{2i-2}(x)||^2}{||I_{2i}(x) \cup yI_{2i-2}(x)||^2} = ||F(x), y||^{-2}.$$

Write $t = q^{-s}$ and $M := M(A(X); \mathfrak{O})$. By [47, Proposition 4.17(ii)], we obtain

$$(1-q^{2d-\ell}t)\mathsf{Z}_M^{\mathsf{ask}}(t) = 1 + \frac{1}{1-q^{-1}}\int\limits_{(\mathfrak{O}^\ell \setminus \mathfrak{P}\mathfrak{O}^\ell)\times\mathfrak{P}} |y|^{s-1}||F(x), y||^{-2}\,\mathrm{d}\,\mu(x,y). \quad (6.3)$$

We decompose the domain of the integral in (6.3) into sets of the form $(\alpha + \mathfrak{P}\mathfrak{O}^\ell) \times \mathfrak{P}$ where $\alpha \in \mathfrak{O}^\ell \setminus \mathfrak{P}\mathfrak{O}^\ell$. If $F(\alpha) \not\equiv 0 \bmod \mathfrak{P}$, then $||F(x)|| = 1$. Put $\mathsf{p}x' = x - \alpha$ and $\mathsf{p}\bar{y} = y$ then we map $\mathfrak{O}^{\ell+1} \to (\alpha + \mathfrak{P}\,\mathfrak{O}^\ell) \times \mathfrak{P}$ via $J_1(x', \bar{y}) =$

$(\mathsf{p}x' + \alpha, \mathsf{p}\bar{y})$ where the determinant of the Jacobian matrix $|\frac{\partial J_1(x',\bar{y})}{\partial(x',\bar{y})}| = |\mathsf{p}|^{\ell+1}$. By Theorem 3.4.3, the integral in (6.3) becomes

$$\begin{aligned}\int\limits_{(\alpha+\mathfrak{P}\mathfrak{O}^\ell)\times\mathfrak{P}} |y|^{s-1}||F(x),y||^{-2}\,\mathrm{d}\,\mu(x,y) &= \int\limits_{(\alpha+\mathfrak{P}\mathfrak{O}^\ell)\times\mathfrak{P}} |y|^{s-1}\,\mathrm{d}\,\mu(x,y)\\ &= \int\limits_{\mathfrak{O}^{\ell+1}} |\bar{y}|^{s-1}|\mathsf{p}|^{s+\ell}\,\mathrm{d}\,\mu(x',\bar{y})\\ &= \frac{q^{-\ell}t(1-q^{-1})}{1-t}.\end{aligned}$$

Write $A(q,t) = \frac{q^{-\ell}t}{1-t}$. Suppose now that $F(\alpha) \equiv 0 \mod \mathfrak{P}$. By the smoothness condition and Lemma 3.3.2, we can transform the map induced by $F(\alpha)$ on $\alpha + \mathfrak{P}\mathfrak{O}^\ell$ into the map induced by X_1 on $\mathfrak{P}\mathfrak{O}^\ell$ via $J_2(x_1,\ldots,x_\ell) = (F(x), x_2 - \alpha_2, \ldots, x_\ell - \alpha_\ell)$. By [30, Corollary 2.1.1], the map $J_2 : \alpha + \mathfrak{P}\,\mathfrak{O}^\ell \to \mathfrak{P}\,\mathfrak{O}^\ell$ is a bi-analytic map. Put $\mathsf{p}\bar{x} = x$. Then we also map $\mathfrak{O}^{\ell+1} \to \mathfrak{P}\mathfrak{O}^\ell \times \mathfrak{P}$ via $J_3(\bar{x}_1,\bar{y}) = (\mathsf{p}\bar{x}_1, \mathsf{p}\bar{y})$ where the determinant of the Jacobian matrix $|\frac{\partial J_3(\bar{x}_1,\bar{y})}{\partial(\bar{x}_1,\bar{y})}| = |\mathsf{p}|^{\ell+1}$. By the special case of [47, Lemma 5.8] (put $r = 1, a_0 = s-1$ and $a_1 = -2$), we obtain

$$\begin{aligned}\int\limits_{(x_0+\mathfrak{P}\mathfrak{O}^\ell)\times\mathfrak{P}} |y|^{s-1}||F(x),y||^{-2}\,\mathrm{d}\,\mu(x,y) &= \int\limits_{\mathfrak{P}\mathfrak{O}^\ell\times\mathfrak{P}} |y|^{s-1}||x_1,y||^{-2}\,\mathrm{d}\,\mu(x,y)\\ &= \int\limits_{\mathfrak{O}^{\ell+1}} |\mathsf{p}\bar{y}|^{s-1}||\mathsf{p}\bar{x}_1,\mathsf{p}\bar{y}||^{-2}|\mathsf{p}|^{\ell+1}\,\mathrm{d}\,\mu(\bar{x},\bar{y})\\ &= q^{2-\ell}t\int\limits_{\mathfrak{O}^2} |\bar{y}|^{s-1}||\bar{x}_1,\bar{y}||^{-2}\,\mathrm{d}\,\mu(\bar{x}_1,\bar{y})\\ &= \frac{q^{2-\ell}t(1-q^{-1})(1-q^{-1}t)}{(1-t)(1-qt)}.\end{aligned}$$

Write $B(q,t) = \frac{q^{2-\ell}t(1-q^{-1}t)}{(1-t)(1-qt)}$. From the proof of [47, Theorem 7.1], the condition $F(\alpha) \equiv 0 \mod \mathfrak{P}$ is equivalent to the image of α in $\mathbb{P}^{\ell-1}(\mathfrak{K})$ being contained in H. Thus (6.3) becomes

$$\begin{aligned}(1-q^{2d-\ell}t)\mathsf{Z}^{\mathsf{ask}}_M(t) &= 1 + (q-1)\left(\#\mathbb{P}^{\ell-1}(\mathfrak{K}) - \#H\right)A(q,t)\\ &\quad + \#H(q-1)B(q,t)\\ &= 1 + (q^\ell-1)\frac{q^{-\ell}t}{1-t} + \#H(q-1)\frac{q^{-\ell}t(q^2-1)}{(1-t)(1-qt)}\\ &= \frac{1-q^{-\ell}t}{1-t} + \#H(q-1)(q^2-1)\frac{q^{-\ell}t}{(1-t)(1-qt)}.\end{aligned}$$

□

Note that any polynomial $F(X)$ that satisfies the smoothness condition in Theorem B and has determinantal representation (i.e. there exists a matrix of linear forms $A(X)$ such that $\det(A(X)) = F(X)$) also has Pfaffian representation

(i.e. there exists a matrix of linear forms $B(X)$ such that $\mathrm{Pf}(B(X)) = F(X)$). We can simply show that by writing

$$B(X) = \begin{bmatrix} 0 & A(X) \\ -A(X)^T & 0 \end{bmatrix}. \tag{6.4}$$

However, the opposite is not always true as we shall show in the next example.

Example 6.3.1. Let $F(X, Y, W, Z) = X^2 + Y^2 + WZ$ where X, Y, W, Z are algebraically independent variables over $\mathfrak{O}$. Then $F(X, Y, W, Z)$ satisfies the smoothness condition in Theorem B. Suppose for the sake of contradiction that $A(X, Y, W, Z)$ is a matrix of linear forms such that $\det(A(X, Y, W, Z)) = F(X, Y, W, Z)$. Then $A(X, Y, W, Z) \in \mathrm{M}_2(\mathfrak{O}[X, Y, W, Z])$. Write

$$A(X, Y, W, Z) = \begin{bmatrix} a_1X + a_2Y + a_3W + a_4Z & b_1X + b_2Y + b_3W + b_4Z \\ c_1X + c_2Y + c_3W + c_4Z & d_1X + d_2Y + d_3W + d_4Z \end{bmatrix}.$$

Since $\det(A(X, Y, W, Z) = F(X, Y, W, Z)$, we obtain the linear equations system:

$$\begin{aligned} &a_3d_3 = b_3c_3,\ a_4b_4 = b_4c_4,\ b_1c_2 + b_2c_1 = a_1d_2 + a_2d_1, \\ &b_1c_3 + b_3c_1 = a_1d_3 + a_3d_1,\ b_1c_4 + b_4c_1 = a_1d_4 + a_4d_1, \\ &b_2c_3 + b_3c_2 = a_2d_3 + a_3d_2,\ b_2c_4 + b_4c_2 = a_2d_4 + a_4d_2, \\ &a_1d_1 - b_1c_1 = 1, a_2d_2 - b_2c_2 = 1, b_3c_4 + b_4c_3 - a_3d_4 - a_4d_3 = 1. \end{aligned}$$

Computing by Sage [51], the linear equations system has no solution. Therefore, the polynomial $F(X, Y, W, Z)$ has no determinantal representation. However, let $B(X, Y, W, Z) \in \mathrm{M}_4(\mathfrak{O}[X, Y, W, Z])$ be an antisymmetric matrix of linear forms defined as follows:

$$B(X, Y, W, Z) = \begin{bmatrix} 0 & X & Y & Z \\ -X & 0 & W & Y \\ -Y & -W & 0 & X \\ -Z & -Y & -X & 0 \end{bmatrix}.$$

Then $\mathrm{Pf}(B(X, Y, W, Z)) = F(X, Y, W, Z)$. Computing by Zeta [48], we obtain $\#H := \{[x : y : w : z] \in \mathbb{P}^2(\mathfrak{K}) : x^2 + y^2 + wz = 0\} = q^2 + N(q)q + 1$ for almost all p where

$$N(q) = \begin{cases} 1, & \text{if } 2 \mid q, \\ 0, & \text{if } q \equiv 3 \bmod 4, \\ 2, & \text{if } q \equiv 1 \bmod 4. \end{cases}$$

By Theorem B

$$\mathsf{Z}^{\mathsf{ask}}_{M(B(X);\mathfrak{O})}(t) = \frac{(1 - q^{-4}t)(1 - qt) + (q^2 + N(q)q + 1)(q - 1)^2(q + 1)q^{-4}t}{(1 - t)^2(1 - qt)}.$$

Remark 6.3.2. When an antisymmetric matrix of linear forms $B(X)$ defined as in (6.4), the ask zeta function of the image module associated with $B(X)$ can be obtained by using [49, Theorem 7.9 & Remark 7.10].

Chapter 7

Ask zeta functions of image modules of single matrices

In this chapter, we give a generic and explicit formula for local ask zeta functions of image modules associated with single matrices (modules generated by a constant matrix). A related work by Rossmann [45] studies a type of zeta functions enumerating submodules invariant under a constant matrix. We also give a formula for global ask zeta functions of image modules of single matrices. Moreover, we study the analytic properties of these zeta functions and their Knuth duals. Theorem C gives an explicit computation for the local ask zeta function of the image module associated with a given matrix. We use Notation 4.1.2 for Theorem C. For a variable X over $\mathfrak{O}$, we use the notation $M(AX;\mathfrak{O})$ for the image module of a matrix A over a compact discrete valuation ring $\mathfrak{O}$.

Theorem C. *Let $A \in \mathrm{M}_{d\times e}(\mathfrak{O})$ with $r = \mathrm{rank}_{\mathcal{K}}(A)$. There exist $g \in \mathrm{GL}_d(\mathfrak{O})$ and $h \in \mathrm{GL}_e(\mathfrak{O})$ such that $gAh = \begin{bmatrix} \mathrm{diag}(\mathsf{p}^{\lambda_1},\dots,\mathsf{p}^{\lambda_r}) & 0 \\ 0 & 0 \end{bmatrix}$ where $0 \le \lambda_1 \le \cdots \le \lambda_r$. Define $\sigma_i := \lambda_1 + \cdots + \lambda_i$. Then*

$$\mathsf{Z}^{ask}_{M(AX;\mathfrak{O})}(t) = \sum_{i=0}^{r} Q_i(q,t),$$

where $Q_0(q,t) = \frac{(1-q^{d-1}t)\sum_{n=0}^{\lambda_1-1}(q^dt)^n + q^{d\lambda_1-1}t^{\lambda_1}}{1-q^{d-1}t}$,
$Q_i(q,t) = \frac{q^{\sigma_i+(d-i)\lambda_i}t^{\lambda_i}(1-q^{-1})\sum_{n=0}^{\lambda_{i+1}-\lambda_i-1}(q^{d-i}t)^n}{(1-q^{d-1}t)}$ *for* $1 \le i \le r-1$, *and*
$Q_r(q,t) = \frac{(1-q^{-1})q^{\sigma_r+(d-r)\lambda_r}t^{\lambda_r}}{(1-q^{d-r}t)(1-q^{d-1}t)}$.

7.1 Proof of Theorem C

Here, we use Notation 4.1.2. In the following lemma, we compute generating functions which we will use later to compute (7.2) in the proof of Theorem C.

Lemma 7.1.1. *Let* C_a, C_b *and* $\mathsf{C}_{ab} \subset \mathbb{R}^2_{\geqslant 0}$ *be rational polyhedra such that*

$$\begin{aligned}\mathsf{C}_a &= \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : a + v_x \leq v_y\},\\ \mathsf{C}_b &= \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : v_y < b + v_x\}, \text{ and}\\ \mathsf{C}_{ab} &= \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : a + v_x \leq v_y < b + v_x\},\end{aligned}$$

where $a, b \in \mathbb{R}_{\geqslant 0}$. *Then the generating functions of* C_a, C_b *and* C_{ab} *in* X *and* Y *are the followings:*

$$\begin{aligned}[\mathsf{C}_a](X,Y) &= \frac{Y^a}{(1-Y)(1-XY)},\\ [\mathsf{C}_b](X,Y) &= \frac{1}{(1-X)(1-Y)} - \frac{Y^b}{(1-Y)(1-XY)}, \text{ and}\\ [\mathsf{C}_{ab}](X,Y) &= \frac{Y^a - Y^b}{(1-Y)(1-XY)}.\end{aligned}$$

Proof. By Definition 3.5.3, we have

$$\begin{aligned}[\mathsf{C}_a](X,Y) &= \sum_{v_x=0}^{\infty} \sum_{v_y=v_x+a}^{\infty} X^{v_x} Y^{v_y} = Y^a \sum_{v_x=0}^{\infty} X^{v_x} Y^{v_x} \sum_{v_y=0}^{\infty} Y^{v_y}\\ &= Y^a \left(\frac{1}{1-XY}\right)\left(\frac{1}{1-Y}\right) = \frac{Y^a}{(1-Y)(1-XY)}.\end{aligned}$$

Similarly

$$\begin{aligned}[\mathsf{C}_b](X,Y) &= \sum_{v_x=0}^{\infty} \sum_{v_y=0}^{v_x+b-1} X^{v_x} Y^{v_y} = \sum_{v_x=0}^{\infty} X^{v_x} \left[\frac{1-Y^{v_x+b}}{1-Y}\right]\\ &= \frac{1}{1-Y} \sum_{v_x=0}^{\infty} X^{v_x} - \frac{Y^b}{1-Y} \sum_{v_x=0}^{\infty} (XY)^{v_x}\\ &= \frac{1}{(1-X)(1-Y)} - \frac{Y^b}{(1-Y)(1-XY)}.\end{aligned}$$

Also

$$\begin{aligned}[\mathsf{C}_{ab}](X,Y) &= \sum_{v_x=0}^{\infty} \sum_{v_y=v_x+a}^{v_x+b-1} X^{v_x} Y^{v_y} = \sum_{v_x=0}^{\infty} X^{v_x} Y^{v_x} \left[\frac{Y^a - Y^b}{1-Y}\right]\\ &= \frac{1}{1-XY} \left[\frac{Y^a - Y^b}{1-Y}\right] = \frac{Y^a - Y^b}{(1-Y)(1-XY)}.\end{aligned}$$

□

Let $A \in \mathrm{M}_{d\times e}(\mathfrak{O})$ with $r = \mathrm{rank}_{\mathcal{K}}(A)$. Since $\mathfrak{O}$ is a principal ideal domain, we can obtain the Smith normal form for A such that

$$gAh = \begin{bmatrix} \mathrm{diag}(\mathsf{p}^{\lambda_1}, \ldots, \mathsf{p}^{\lambda_r}) & 0 \\ 0 & 0 \end{bmatrix}, \tag{7.1}$$

where $\lambda_1 \leq \cdots \leq \lambda_r$, for some $g \in \mathrm{GL}_d(\mathfrak{O})$ and $h \in \mathrm{GL}_e(\mathfrak{O})$. Let X be algebraically independent variable over $\mathfrak{O}$. Our goal is to compute $M(gAh;\mathfrak{O})$ instead of $M(A;\mathfrak{O})$ since the local ask zeta function of the former can be computed using Proposition 4.5.1. Moreover, Corollary 4.4.1 shows that $\mathsf{Z}^{\mathsf{ask}}_{M(AX;\mathfrak{O})}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(gAhX;\mathfrak{O})}(t)$.

Proof of Theorem C. Consider gAh as in (7.1). The norm of the set of all nonzero $i \times i$ minors of $gAhx$ is

$$||f_i(x)|| = |\sup\{|gAhx|_{i\times i} : x \in \mathfrak{O}\}| = |\mathsf{p}^{\sigma_i}x^i|.$$

where $\sigma_i = \lambda_1 + \cdots + \lambda_i$ for $i \leq r$ since $|\mathsf{p}^{\lambda_1}| \geq \ldots \geq |\mathsf{p}^{\lambda_r}|$. Define $M := M(gAhX;\mathfrak{O})$. By Proposition 4.5.1

$$\mathsf{Z}^{\mathsf{ask}}_M(q^{-s}) = \frac{1}{1-q^{-1}} \int_{\mathfrak{O}^2} |y|^{s-d+r-1} \prod_{i=0}^{r-1} \frac{|\mathsf{p}^{\sigma_i}x^i|}{||\mathsf{p}^{\sigma_{i+1}}x^{i+1}, \mathsf{p}^{\sigma_i}x^iy||} \,\mathrm{d}\,\mu(x,y).$$

We construct the polyhedra $\mathsf{C}_y, \mathsf{C}_1, \ldots, \mathsf{C}_r$ such that

$$\begin{aligned}
\mathsf{C}_y &= \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : v_y < \lambda_1 + v_x\},\\
\mathsf{C}_1 &= \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : \lambda_1 + v_x \leq v_y < \lambda_2 + v_x\},\\
&\vdots\\
\mathsf{C}_i &= \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : \lambda_i + v_x \leq v_y < \lambda_{i+1} + v_x\},\\
&\vdots\\
\mathsf{C}_r &= \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : \lambda_r + v_x \leq v_y\},
\end{aligned}$$

where v_x and v_y denote the valuation of x and y, respectively. Define $\mathbf{U}(\cdot) := \{(x,y) \in \mathfrak{O}^2 : (v_x, v_y) \in \cdot\}$. Then

$$\begin{aligned}
\mathsf{Z}^{\mathsf{ask}}_M(q^{-s}) = \frac{1}{1-q^{-1}} \Bigg[&\int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-d-1} \,\mathrm{d}\,\mu(x,y) + \int_{\mathbf{U}(\mathsf{C}_1)} |y|^{s-d}|\mathsf{p}^{\lambda_1}x|^{-1} \,\mathrm{d}\,\mu(x,y)\\
&+ \cdots + \int_{\mathbf{U}(\mathsf{C}_r)} |y|^{s-d+r-1}|\mathsf{p}^{\sigma_r}x^r|^{-1} \,\mathrm{d}\,\mu(x,y)\Bigg].
\end{aligned}$$

Now put $t = q^{-s}$. By Corollary 3.5.6

$$\begin{aligned}
\mathsf{Z}^{\mathsf{ask}}_M(t) = (1-q^{-1}) \Big[&[\mathsf{C}_y](q^{-1}, q^d t) + q^{\lambda_1}[\mathsf{C}_1](1, q^{d-1}t)\\
&+ \cdots + q^{\sigma_r}[\mathsf{C}_r](q^{r-1}, q^{d-r}t)\Big]. \qquad (7.2)
\end{aligned}$$

By Lemma 7.1.1

$$\begin{aligned}
[\mathsf{C}_y](q^{-1}, q^d t) &= \frac{1}{(1-q^{-1})(1-q^d t)} - \frac{q^{d\lambda_1} t^{\lambda_1}}{(1-q^d t)(1-q^{d-1}t)} \\
&= \frac{(1-q^{d-1}t) - (q^{d\lambda_1}t^{\lambda_1} - q^{d\lambda_1-1}t^{\lambda_1})}{(1-q^{-1})(1-q^d T)(1-q^{d-1}t)} \\
&= \frac{(1-q^{d\lambda_1}t^{\lambda_1}) - q^{d-1}t(1-q^{d(\lambda_1-1)}t^{\lambda_1-1}))}{(1-q^{-1})(1-q^d t)(1-q^{d-1}t)} \\
&= \frac{(1-q^{d-1}t)\sum_{n=0}^{\lambda_1-1}(q^d t)^n + q^{d\lambda_1-1}t^{\lambda_1}}{(1-q^{-1})(1-q^{d-1}t)} = \frac{Q_0(q,t)}{1-q^{-1}}. \qquad (7.3)
\end{aligned}$$

For $1 \leq i < r$, we have

$$\begin{aligned}
[\mathsf{C}_i](q^{i-1}, q^{d-i}t) &= \frac{q^{(d-i)\lambda_i}t^{\lambda_i} - q^{(d-i)\lambda_{i+1}}t^{\lambda_{i+1}}}{(1-q^{d-i}t)(1-q^{d-1}t)} \\
&= \frac{q^{(d-i)\lambda_i}t^{\lambda_i}(1-q^{(d-i)(\lambda_{i+1}-\lambda_i)}t^{\lambda_{i+1}-\lambda_i})}{(1-q^{d-i}t)(1-q^{d-1}t)} \\
&= \frac{q^{(d-i)\lambda_i}t^{\lambda_i}\sum_{n=0}^{\lambda_{i+1}-\lambda_i-1}(q^{d-i}t)^n}{1-q^{d-1}t}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=1}^{r-1} q^{\sigma_i}[\mathsf{C}_i](q^{i-1}, q^{d-i}t) &= \frac{\sum_{i=1}^{r-1} q^{\sigma_i+(d-i)\lambda_i}t^{\lambda_i}\sum_{n=0}^{\lambda_{i+1}-\lambda_i-1}(q^{d-i}t)^n}{1-q^{d-1}t} \\
&= \frac{\sum_{i=1}^{r-1} Q_i(q,t)}{1-q^{-1}}. \qquad (7.4)
\end{aligned}$$

Moreover

$$q^{\sigma_r}[\mathsf{C}_r](q^{r-1}, q^{d-r}t) = \frac{q^{\sigma_r+(d-r)\lambda_r}t^{\lambda_r}}{(1-q^{d-r}t)(1-q^{d-1}t)} = \frac{Q_r(q,t)}{1-q^{-1}}. \qquad (7.5)$$

By inserting (7.3), (7.4), and (7.5) in (7.2), we obtain

$$\mathsf{Z}_M^{\mathrm{ask}}(t) = \sum_{i=0}^{r} Q_i(q,t).$$

□

Example 7.1.2. Let $A = \begin{bmatrix} \mathsf{p}^2 & 0 \\ 0 & \mathsf{p}^3 \end{bmatrix}$. By Theorem C

$$Q_0(q,t) = \frac{1-qt+q^2t}{1-qt},\ Q_1(q,t) = \frac{q^4t^2-q^3t^2}{1-qt},\ Q_2(q,t) = \frac{q^5t^3-q^4t^3}{(1-t)(1-qt)}.$$

Hence

$$\mathsf{Z}_{M(AX;\mathfrak{O})}^{\mathsf{ask}}(t) = \frac{(q^2-2q+1)q^3t^3 + (q^3-q^2-q+1)qt^2 + (q^2-q-1)t + 1}{(1-t)(1-qt)}.$$

Corollary 7.1.3. *Let $A \in \mathrm{M}_{d\times e}(\mathfrak{O})$ with $r = \mathrm{rank}_{\mathcal{K}}(A)$. Then the local abscissa of convergence of $\mathsf{Z}_{M(AX;\mathfrak{O})}^{\mathsf{ask}}(q^{-s})$ is $\alpha_{M(AX;\mathfrak{O})} = \max(d-1, d-r)$.*

Proof. The proof follows directly from the largest real pole of the local ask zeta function of $M(AX;\mathfrak{O})$ in Theorem C. □

7.2 Global ask zeta functions

In this section, we give a generic computation for local ask zeta functions of image modules associated with single matrices. We also compute global ask zeta functions of image modules of single matrices. Here, we use Notation 4.2.1.

For a matrix $A \in \mathrm{M}_{d\times e}(K)$ with $r = \mathrm{rank}_K(A)$, there exist $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that

$$gAh = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.6}$$

By Corollary 4.4.3, we have $\mathsf{Z}^{\mathsf{ask}}_{M(AX;\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(gAhX;\mathcal{O}_v)}(t)$ for almost all places $v \in \mathcal{V}_K$. In the following lemma, we replace $M(AX;\mathcal{O}_v)$ with $M(gAXh;\mathcal{O}_v)$ since the latter is easier to compute using Proposition 4.5.1.

Lemma 7.2.1. *Let $A \in \mathrm{M}_{d\times e}(K)$ with $r = \mathrm{rank}_K(A)$. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{M(AX;\mathcal{O}_v)}(t) = \frac{1 - q_v^{d-r-1}t}{(1 - q_v^{d-1}t)(1 - q_v^{d-r}t)}. \tag{7.7}$$

Proof. Consider gAh in (7.6). Since all nonzero $i \times i$ minors are 1 or 0 for $i \leq r$, the norm of the set of all nonzero $i \times i$ minors of $gAhX$ is

$$||f_i(x)|| = |\sup\{|gAhx|_{i\times i} : x \in \mathcal{O}_v\}| = |x|^i,$$

for $i \leq r$. Define $M_v := M(gAhX;\mathcal{O}_v)$. By Proposition 4.5.1

$$\mathsf{Z}^{\mathsf{ask}}_{M_v}(q_v^{-s}) = \frac{1}{1 - q_v^{-1}} \int\limits_{\mathcal{O}_v^2} |y|^{s-d+r-1} \prod_{i=0}^{r-1} \frac{|x^i|}{||x^{i+1}, x^i y||} \,\mathrm{d}\,\mu(x,y).$$

Then we construct two cones C_x and C_y such that

$$\mathsf{C}_x = \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : v_x \leq v_y\} \text{ and } \mathsf{C}_y = \{(v_x, v_y) \in \mathbb{R}^2_{\geqslant 0} : v_x > v_y\},$$

where v_x and v_y denote the valuation of x and y, respectively. Define $\mathbf{U}(\cdot) = \{(x,y) \in \mathcal{O}_v^2 : (v_x, x_y) \in \cdot\}$. Then

$$\mathsf{Z}^{\mathsf{ask}}_{M_v}(q_v^{-s}) = \frac{1}{1 - q_v^{-1}} \left(\int\limits_{\mathbf{U}(\mathsf{C}_x)} |y|^{s-d+r-1}|x|^{-r} \,\mathrm{d}\,\mu(x,y) + \int\limits_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-d-1} \,\mathrm{d}\,\mu(x,y) \right).$$

Put $t = q_v^{-s}$. By Lemma 3.6.2 and 3.6.3 (put $\ell = i = 1$)

$$\begin{aligned} \mathsf{Z}^{\mathsf{ask}}_{M_v}(t) &= (1 - q_v^{-1}) \left(\frac{1}{(1 - q_v^{d-r}t)(1 - q_v^{d-1}t)} + \frac{q_v^{-1}}{(1 - q_v^{-1})(1 - q_v^{d-1}t)} \right) \\ &= \frac{1 - q_v^{d-r-1}t}{(1 - q_v^{d-1}t)(1 - q_v^{d-r}t)}. \end{aligned}$$

□

Note that the formula in Lemma 7.2.1 can be also derived from [47, Proposition 6.4] when $\ell = \dim_K(M_v \otimes K) = 1$. In the following, recall Notation 4.1.2.

Remark 7.2.2. Let $A \in \mathrm{M}_{d\times e}(\mathfrak{O})$ with $r = \mathrm{rank}_{\mathcal{K}}(A)$. Define $\lambda_1, \ldots, \lambda_r$ to be as in Theorem C. If $\lambda_1 = \cdots = \lambda_r = m \geq 0$, then $\sigma_i = im$ and $W_i(q,t) = 0$ for $1 \leq i < r$. Then

$$\mathsf{Z}^{\mathsf{ask}}_{M(AX;\mathfrak{O})}(t) = \sum_{n=0}^{m-1}(q^d t)^n + q^{dm}t^m \frac{1-q^{d-r-1}t}{(1-q^{d-r}t)(1-q^{d-1}t)}.$$

Define $\tilde{A} = \mathrm{diag}(I_r, 0) \in \mathrm{M}_{d\times e}(\mathfrak{O})$. From the proof of Lemma 7.2.1

$$\mathsf{Z}^{\mathsf{ask}}_{M(AX;\mathfrak{O})}(t) = \sum_{n=0}^{m-1}(q^d t)^n + q^{dm}t^m \mathsf{Z}_{M(\tilde{A}X;\mathfrak{O})}(t).$$

Let $\bar{Q}(q_v, q_v^{-s})$ be the rational function in (7.7). For each place $v \in \mathcal{V}_K$, we regard $\tilde{Q}_v(q_v, q_v^{-s})$ to be the sum of rational functions in Theorem C for $A_v \in \mathrm{M}_{d\times e}(\mathcal{O}_v)$. Next, we give a formula for global ask zeta functions of image modules associated with single matrices.

Corollary 7.2.3. *Let $A \in \mathrm{M}_{d\times e}(\mathcal{O})$. Then*

$$\zeta^{\mathsf{ask}}_{M(AX;\mathcal{O})}(s) = \prod_{v\in\mathcal{V}_K\setminus S}\bar{Q}(q_v, q_v^{-s}) \cdot \prod_{v\in S}\tilde{Q}_v(q_v, q_v^{-s}).$$

where $S \subset \mathcal{V}_K$ is a finite set of places.

Proof. By Proposition 4.2.2, the global ask zeta function

$$\zeta^{\mathsf{ask}}_{M(AX;\mathcal{O})}(s) = \prod_{v\in\mathcal{V}_K}\zeta^{\mathsf{ask}}_{M(AX;\mathcal{O}_v)}(s).$$

By Lemma 7.2.1

$$\zeta^{\mathsf{ask}}_{M(AX;\mathcal{O}_v)}(s) = \mathsf{Z}^{\mathsf{ask}}_{M(AX;\mathcal{O}_v)}(q_v^{-s}) = \bar{Q}(q_v, q_v^{-s})$$

for almost all places $v \in \mathcal{V}_K$. Let S be the excluded finite set of places. By Theorem C

$$\zeta^{\mathsf{ask}}_{M(AX;\mathcal{O})}(s) = \prod_{v\in\mathcal{V}_K\setminus S}\bar{Q}(q_v, q_v^{-s}) \cdot \prod_{v\in S}\tilde{Q}_v(q_v, q_v^{-s}).$$

□

Corollary 7.2.4. *Let $A \in \mathrm{M}_{d\times e}(\mathcal{O})$ with $r = \mathrm{rank}_K(A)$. Then the global abscissa of convergence of $\zeta^{\mathsf{ask}}_{M(AX;\mathcal{O})}(s)$ is $\alpha_{M(AX;\mathcal{O})} = \max(d, d+1-r)$.*

Proof. Let S be the excluded places from the generic formula in Lemma 7.2.1. By Proposition 4.2.4 and Lemma 7.2.1, the global abscissa of convergence is the abscissa of convergence of $\prod_{v\in\mathcal{V}_K\setminus S}\bar{Q}(q_v, q_v^{-s})$. □

7.3 Ask zeta functions of Knuth duals

Here, we also give formulas for local and global ask zeta functions of image modules associated with Knuth duals of a single matrix.

In the following two corollaries, we use Notation 4.1.2. Let $A \in \mathrm{M}_{d\times e}(\mathfrak{O})$ with $r = \mathrm{rank}_{\mathcal{K}}(A)$. Write $A(X_1) = AX_1$. Recall the notations in Section 1.4.

Corollary 7.3.1. *Let $A^{\circ}(X_1,\dots,X_d), A^{\bullet}(X_1,\dots,X_e)$, and $A^{\vee}(X_1)$ be Knuth duals of $A(X_1)$. Then:*

(i) $A^{\circ}(X_1,\dots,X_d) \approx [\mathsf{p}^{\lambda_1}X_1,\dots,\mathsf{p}^{\lambda_r}X_r,0,\dots,0] \in \mathrm{M}_{1\times e}(\mathfrak{O}[X_1,\dots,X_d])$.

(ii) $A^{\bullet}(X_1,\dots,X_e) \approx [\mathsf{p}^{\lambda_1}X_1,\dots,\mathsf{p}^{\lambda_r}X_r,0,\dots,0]^T \in \mathrm{M}_{d\times 1}(\mathfrak{O}[X_1,\dots,X_e])$.

(iii) $A^{\vee}(X_1) = A^TX_1$.

Proof. This follows directly by applying Definition 1.4.1 on $gAhX_1$ in (7.1). □

In the following corollary, we give explicit formulas for the local ask zeta functions of the image modules associated with Knuth duals of $A(X_1)$.

Corollary 7.3.2. *Let $A^{\circ}(X_1,\dots,X_d), A^{\bullet}(X_1,\dots,X_e)$, and $A^{\vee}(X_1)$ be Knuth duals of $A(X_1)$. Then the local ask zeta functions of the duals are:*

(i) $\mathsf{Z}^{ask}_{M(A^{\circ}(X_1,\dots,X_d);\mathfrak{O})}(t) = \sum_{i=0}^{r} Q_i(q,q^{1-d}t)$,

(ii) $\mathsf{Z}^{ask}_{M(A^{\bullet}(X_1,\dots,X_e);\mathfrak{O})}(t) = \sum_{i=0}^{r} Q_i(q,t)$, *and*

(iii) $\mathsf{Z}^{ask}_{M(A^{\vee}(X_1);\mathfrak{O})}(t) = \sum_{i=0}^{r} Q_i(q,q^{e-d}t)$,

where Q_i defined as in Theorem C.

Proof. This follows directly from Proposition 4.3.3. □

Recall Notation 4.2.1. Let $A \in \mathrm{M}_{d\times e}(\mathcal{O})$ with $r = \mathrm{rank}_K(A)$. Let $\tilde{Q}_v$ and $\bar{Q}$ be defined as in Section 7.2. In the following corollaries, let $A(X_1) = AX_1$.

Corollary 7.3.3. *Let $A^{\circ}(X_1,\dots,X_d), A^{\bullet}(X_1,\dots,X_e)$, and $A^{\vee}(X_1)$ be Knuth duals of $A(X_1)$. For a finite set of places S, the global ask zeta functions of the duals are:*

(i) $\zeta^{\mathsf{ask}}_{M(A^{\circ}(X_1,\dots,X_d);\mathcal{O}_v)}(s) = \prod_{v\in\mathcal{V}_K\setminus S} \bar{Q}(q_v,q_v^{1-d-s}) \cdot \prod_{v\in S} \tilde{Q}_v(q_v,q_v^{1-d-s})$,

(ii) $\zeta^{\mathsf{ask}}_{M(A^{\bullet}(X_1,\dots,X_e);\mathcal{O}_v)}(s) = \prod_{v\in\mathcal{V}_K\setminus S} \bar{Q}(q_v,q_v^{-s}) \cdot \prod_{v\in S} \tilde{Q}_v(q_v,q_v^{-s})$, *and*

(iii) $\zeta^{\mathsf{ask}}_{M(A^{\vee}(X_1);\mathcal{O}_v)}(s) = \prod_{v\in\mathcal{V}_K\setminus S} \bar{Q}(q_v,q_v^{e-d-s}) \cdot \prod_{v\in S} \tilde{Q}_v(q_v,q_v^{e-d-s})$.

Proof. This follows directly by applying Proposition 4.3.3 to Corollary 7.2.3. □

Corollary 7.3.4. *Let $\alpha_{M(A(X_1);\mathcal{O})}$ and $\alpha_{M(A(X_1);\mathcal{O}_v)}$ be the global and local abscissa of the image module of $A(X_1)$. Then the global and the local abscissa of Knuth duals are:*

(i) $\alpha_{M(A^{\circ}(X_1,\ldots,X_d);\mathcal{O})} = \max(2-r,1)$ *and* $\alpha_{M(A^{\circ}(X_1,\ldots,X_d);\mathcal{O}_v)} = \max(1-r,0)$.

(ii) $\alpha_{M(A^{\bullet}(X_1,\ldots,X_e);\mathcal{O})} = \max(d, d-r+1)$ *and* $\alpha_{M(A^{\bullet}(X_1,\ldots,X_e);\mathcal{O}_v)} = \max(d-1, d-r)$.

(iii) $\alpha_{M(A^{\vee}(X_1);\mathcal{O})} = \max(e, e-r+1)$ *and* $\alpha_{M(A^{\vee}(X_1);\mathcal{O}_v)} = \max(e-1, e-r)$.

Proof. Similar to the proof of Corollary 7.2.4, Corollary 7.3.3 and Proposition 4.2.4 show that the global abscissae of convergence in (i), (ii), and (iii) are the abscissae of convergence of $\prod_{v\in\mathcal{V}_K\setminus S} \bar{Q}(q_v, q_v^{1-d-s})$, $\prod_{v\in\mathcal{V}_K\setminus S} \bar{Q}(q_v, q_v^{-s})$, and $\prod_{v\in\mathcal{V}_K\setminus S} \bar{Q}(q_v, q_v^{e-d-s})$, respectively. Moreover, the local abscissae of (i), (ii), and (iii) are the largest real poles of (i), (ii), and (iii) in Corollary 7.3.2, respectively. □

Chapter 8

Ask zeta functions of image modules of matrices pencil

In this chapter, we write a generic formula for ask zeta functions of image modules associated with matrix pencils, see Theorem D. A similar work can be found for normal zeta functions of class 2-nilpotent groups, see [56, Theorem 2]. Here, we use Notation 4.2.1.

Theorem D. *For $A_1, A_2 \in \mathrm{M}_{d\times e}(K)$, let $A(X_1, X_2) = A_1X_1 + A_2X_2$ be a matrix pencil with a rational form*

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_\ell}, \mathrm{R}_{\bar{A}}(X_1, X_2)),$$

where $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is a rational form of the non-singular core of $A(X_1, X_2)$ (If $A(X_1, X_2)$ is a non-singular matrix pencil, we put $k = \ell = 0$ and $\mathrm{R}_A(X_1, X_2) = \mathrm{R}_{\bar{A}}(X_1, X_2)$). Then

$$\mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1 - q_v^{k-2}t}{(1 - q_v^k t)(1 - q_v^{d-2}t)} + \sum_{i=1}^{h} N_i(\mathfrak{K}_v)W_i(q_v, t),$$

where $N_i(\mathfrak{K}_v) = \#\{\bar{\alpha} \in \mathfrak{K}_v : g_i(\bar{\alpha}) = 0\}$, g_i is a univariate polynomial, and $W_i(q_v, t)$ is a rational function. The univariate polynomials g_i's and the rational functions $W_i(q_v, t)$'s can both be read off from the matrix core $\mathrm{R}_{\bar{A}}(X_1, X_2)$ (see Definition 2.5.2).

8.1 Preliminaries

In this section, we provide some computations on the non-singular core which we will use to prove Theorem D. Recall that A is called a **diagonal block matrix** if $A = \mathrm{diag}(A_1, \ldots, A_w)$ where each A_j is a matrix for $1 \leq j \leq w$. We also call each A_j a **diagonal block** of A.

Lemma 8.1.1. *Define $[n] = \{1, \ldots, n\}$. Let $J_1, \ldots, J_w$ be disjoint subsets of $[n]$ such that $J_1 \cup \cdots \cup J_w = [n]$. Let $A = \mathrm{diag}(A_1, \ldots, A_w)$ be a square diagonal block matrix of size n over an arbitrary ring R where each A_ℓ is a diagonal*

block of A with nonzero determinant and formed by J_ℓ rows and columns for $1 \leq \ell \leq w$. For $J_R, J_C \subset [n]$ with $|J_R| = |J_C|$, define B to be a square submatrix of A formed by J_R rows and J_C columns. Define B_ℓ to be a submatrix of B formed by $J_R \cap J_\ell$ rows and $J_C \cap J_\ell$ columns. Then

(i) If $|J_R \cap J_\ell| \neq |J_C \cap J_\ell|$ for some ℓ, then $\det(B) = 0$.

(ii) if B is a nonzero determinant submatrix with size $n - w + r$ where $1 \leq r \leq w$, then it contains at least r diagonal blocks of A.

Proof. For (i), if B is contained in one of the diagonal blocks of A, say A_s, then $B = B_s$ where B_ℓ is an empty matrix for each $\ell \neq s$. Since B is a square submatrix, then B_s is also a square submatrix. Hence $|J_R \cap J_\ell| = |J_C \cap J_\ell|$ for all ℓ. If $B \neq B_\ell$ for any ℓ. Write $B = B_1 \oplus \cdots \oplus B_w$. Let B_s be a submatrix such that $|J_R \cap J_s| > |J_C \cap J_s|$ for some $1 \leq s \leq w$. Let $\mathcal{B}_s$ be a square submatrix of B with size $|J_R \cap J_s|$ which contains the submatrix B_s. Then $\mathcal{B}_s$ has $|J_R \cap J_s| - |J_C \cap J_s|$ columns of zeros. Hence $\det(\mathcal{B}_s) = 0$. By induction on k, suppose that any square submatrix of B which contains B_s with size $|J_R \cap J_s| + k$ is zero determinant. Let $\mathcal{B}$ be a square submatrix of B which contains B_s with size $|J_R \cap J_s| + k + 1$. By Laplace expansion, $\det(\mathcal{B}) = \sum_{j=1}^{|J_R \cap J_s|+k+1} b_{ij} \det(\mathcal{B}_{ij})$ where b_{ij}'s are entries in $\mathcal{B}$ and $\mathcal{B}_{ij}$'s are submatrices of $\mathcal{B}$ obtained by removing i^{th} row and j^{th} column. Fix $i \notin J_R \cap J_s$, then $b_{ij} = 0$ for all $j \in J_C \cap J_s$ since B_ℓ are diagonal block for all ℓ. Hence $\det(\mathcal{B}) = \sum_{j \notin J_C \cap J_s} b_{ij} \det(\mathcal{B}_{ij})$. However $\mathcal{B}_{ij}$'s are square submatrices of B with size $|J_R \cap J_s| + k$ and contains B_s for all $j \notin J_C \cap J_s$. Therefore $\det(\mathcal{B}) = 0$ since $\det(\mathcal{B}_{ij}) = 0$ for all $j \notin J_C \cap J_s$. Thus, all square submatrices of B which contain B_s has a zero determinant. Similarly, if B_s is a square submatrix of A with $|J_R \cap J_s| < |J_C \cap J_s|$ for some s, then all square submatrices of B which contain B_s has a zero determinant. Thus $\det(B) = 0$ if $|J_R \cap J_\ell| \neq |J_C \cap J_\ell|$ for some ℓ. For (ii), if $\det(B) \neq 0$. By (i), each B_ℓ is either an empty or square submatrix of B for all ℓ. Let B be size $n - w + r$. Let S be another submatrix of A obtained by removing a column and a row from $w - r$ diagonal blocks of A. Then S is submatrix with size $n - w + r$ and contains the least possible diagonal blocks of A. However, since S contains r diagonal blocks of A, then B must contain at least r blocks. □

Notation 8.1.2. *For an infinite field K, let $\mathrm{R} \in \mathrm{M}_n(K)$ be a matrix in a rational canonical form with the companion matrices $C_1, \ldots, C_w$ the diagonal blocks of R. Let $n_1, \ldots, n_w$ be the sizes of the companion matrices, respectively. For $1 \leq j \leq w$, suppose that each C_j is a companion matrix for a monic polynomial which can be written as $F_j(X) = c_{j0} + c_{j1}X + \cdots + c_{jn_j-1}X^{n_j-1} + X^{n_j} = \prod_{i=1}^{h} g_i^{e_{ij}}(X)$ where $g_i(X)$'s are different monic irreducible polynomials over K and $e_{ij} \geq 0$. Since $F_1(x) \mid F_2(x) \mid \cdots \mid F_w(x)$, then $e_{i1} \leq e_{i2} \leq \cdots \leq e_{iw}$ and we write $e_{ij} = 0$ for all $g_i(x) \nmid F_j(x)$. Write $0 < d_{i1} < \cdots < d_{ir_i}$ where $d_{i1} = \min\{e_{ij} : e_{ij} > 0\}, d_{i2} = \min\{e_{ij} : e_{ij} > d_{i1}\}, \ldots, d_{ir_i} = \min\{e_{ij} : e_{ij} > d_{ir_i-1}\}$. For $1 \leq j \leq r_i$, let a_{ij} be the number of $g_i(x)$'s with the exponent d_{ij} in $F_j(x)$.*

In the followings, we use Notation 8.1.2. Also, define $\tilde{\mathrm{R}}_{\bar{A}}(X_1, X_2) := -\mathrm{R}X_2 + I_n X_1$.

Lemma 8.1.3. *Let $A_j(X_1, X_2)$ be a diagonal block of $\tilde{\mathrm{R}}_{\bar{A}}(X_1, X_2)$ such that $A_j(X_1, X_2) = -C_jX_2 + I_{n_j}X_1$ for $1 \leq j \leq w$. Then $\det(A_j(X_1, X_2)) = X_2^{n_j}F_j(X_1X_2^{-1}) = X_1^{n_j}F_j^*(X_1^{-1}X_2)$ where F_j^* is the reciprocal polynomial of F_j.*

Proof. By (2.1), we have $\det(A_j(X_1, X_2)) = \det(-C_jX_2 + I_{n_j}X_1) = c_{j0}X_2^{n_j} + c_{j1}X_2^{n_j-1}X_1 + \cdots + c_{jn_j-1}X_2X_1^{n_j-1} + X_1^{n_j}$ where $c_{j1}, \ldots, c_{jn_j-1} \in K$. Hence

$$\begin{aligned}\det(A_j(X_1, X_2)) &= X_2^{n_j}\left[c_{j0} + c_{j1}\left(\frac{X_1}{X_2}\right) + \cdots + c_{jn_j-1}\left(\frac{X_1}{X_2}\right)^{n_j-1} + \left(\frac{X_1}{X_2}\right)^{n_j}\right] \\ &= X_2^{n_j}F_j(X_1X_2^{-1}).\end{aligned}$$

Similarly

$$\begin{aligned}\det(A_j(X_1, X_2)) &= X_1^{n_j}\left[c_{j0}\left(\frac{X_2}{X_1}\right)^{n_j} + c_{j1}\left(\frac{X_2}{X_1}\right)^{n_j-1} + \cdots + c_{jn_j-1}\left(\frac{X_2}{X_1}\right) + 1\right] \\ &= X_1^{n_j}F_j^*(X_1^{-1}X_2).\end{aligned}$$

□

Let $\mathfrak{O}$ be a compact discrete valuation ring.

Corollary 8.1.4. *Let $A_j(X_1, X_2) = -C_jX_2 + I_{n_j}X_1$ be a diagonal block of $\tilde{\mathrm{R}}_{\bar{A}}(X_1, X_2)$. Define $M := \{\mathrm{diag}(A_1(x_1, x_2), \ldots, A_w(x_1, x_2)) : x_1, x_2 \in \mathfrak{O}\}$. Let $f_k(x_1, x_2)$ be the set of all nonzero $k \times k$ minors of M. Then:*

(i) If $|x_1| \leq |x_2|$, then $||f_k(x_1, x_2)|| = |x_2|^k$ for $1 \leq k \leq n - w$ and $||f_{n-w+w_0}(x_1, x_2)|| = |x_2|^{n-w+w_0}\prod_{j=1}^{w_0}|F_j(x_1x_2^{-1})|$ where $1 \leq w_0 \leq w$.

(ii) If $|x_1| > |x_2|$, then $||f_k(x_1, x_2)|| = |x_1|^k$ for $1 \leq k \leq n$.

Proof. For (i), we have $|x_1| \leq |x_2|$. By Lemma 3.6.1(i), we have $||f_k(x_1, x_2)|| \leq |x_2|^k$. For $1 \leq k \leq n_j - 1$, there exists a minor equals to X_2^k in the set of all nonzero $k \times k$ minors of $A_j(X_1, X_2)$. Thus there exists a minor equals to x_2^k in the set of $k \times k$ minors of M for $1 \leq k \leq n - w$. Hence $||f_k(x_1, x_2)|| = |x_2|^k$ for $0 \leq k \leq n-w$. By Lemma 8.1.1(ii), a nonzero determinant submatrix of the size $n-w+w_0$ must include at least w_0 blocks of M for $1 \leq w_0 \leq w$. By Lemma 8.1.3, we have $\det(A_j(X_1, X_2)) = X_2^{n_j}F_j(X_1X_2^{-1})$ and $F_1(X_1X_2^{-1})|\cdots|F_w(X_1X_2^{-1})$. Therefore, we have $||f_{n-w+w_0}(x_1, x_2)|| = |x_2|^{n-w+w_0}\prod_{j=1}^{w_0}|F_j(x_1x_2^{-1})|$. Similarly for (ii), we have $||f_k(x_1, x_2)|| = |x_1|^k$ for $0 \leq k \leq n - w$. Moreover, we have $||f_{n-w+w_0}(x_1, x_2)|| = |x_1|^{n-w+w_0}\prod_{j=1}^{w_0}|F_j^*(x_1^{-1}x_2)|$ for $1 \leq w_0 \leq w$. Since $|x_1| > |x_2|$, we have

$$|F_j^*(x_1^{-1}x_2)| = |c_{j0}\left(\frac{x_2}{x_1}\right)^{n_j} + \cdots + c_{jn_j-1}\left(\frac{x_2}{x_1}\right) + 1| = 1.$$

Thus $||f_k(x_1, x_2)|| = |x_1|^k$ for $1 \leq k \leq n$. □

Corollary 8.1.5. *Let $\bar{\alpha} \in \mathfrak{K}_v$ such that $F_j(\bar{\alpha}) = 0$ where $F_j(X)$ defined as in Notation 8.1.2. Then for almost all places $v \in \mathcal{V}_K$, there exists $\alpha \in \mathcal{O}_v$ such that for all $x \in \mathcal{O}_v$ with $x \equiv \bar{\alpha} \bmod \mathfrak{P}_v$*

$$|F_j(x)| = |x - \alpha|^{e_{ij}},$$

where $1 \leq i \leq h$.

Proof. By Notation 8.1.2, we have $g_i(x)$'s are different monic irreducible polynomials. By Lemma 3.3.4, each $g_i(x)$ has a different root over $\mathfrak{K}_v$ for almost all places $v \in \mathcal{V}_K$. Since $F_j(x) = \prod_{i=1}^{h} g_i^{e_{ij}}(x)$, there exist $g_m(x)$ such that $g_m(\bar{\alpha}) = 0$ and $g_i(\bar{\alpha}) \neq 0$ for all $i \neq m$. Write $F(x) = g_m^{e_{mj}}(x)\bar{F}_j(x)$ then $\bar{F}_j(\bar{\alpha}) \neq 0$. For all $x \in \mathcal{O}_v$ with $x \equiv \bar{\alpha} \bmod \mathfrak{P}_v$, we have $|\bar{F}_j(x)| = 1$ and $|F_j(x)| = |g_m(x)|^{e_{mj}}$. By Lemma 3.3.3, we have $|g_m(x)| = |x - \alpha|$. Thus $|F_j(x)| = |x - \alpha|^{e_{mj}}$. □

8.2 Technical lemmas

In this section, we compute the integral in (8.9) to prove Theorem D. We will use Notation 4.1.2.

Lemma 8.2.1. *Let $d_1, d_2 \in \mathbb{Z}$ with $d_1 < d_2$. Define the half-open polyhedra C_y, C_1 and C_2 as follows:*

$$\begin{aligned}
\mathsf{C}_y &= \{(v_x, v_y) \in \mathbb{R}^2_{\geq 0} : v_y < d_1 \cdot (v_x + 1)\},\\
\mathsf{C}_1 &= \{(v_x, v_y) \in \mathbb{R}^2_{\geq 0} : d_1 \cdot (v_x + 1) \leq v_y < d_2 \cdot (v_x + 1)\}, \text{ and}\\
\mathsf{C}_2 &= \{(v_x, v_y) \in \mathbb{R}^2_{\geq 0} : d_2 \cdot (v_x + 1) \leq v_y\}.
\end{aligned}$$

Then the generating functions for C_y, C_1 and C_2 in X and Y are

$$\begin{aligned}
[\mathsf{C}_y](X,Y) &= \frac{1 - Y^{d_1}}{(1-X)(1-Y)(1-XY^{d_1})},\\
[\mathsf{C}_1](X,Y) &= \frac{Y^{d_1} - X^{d_2}}{(1-Y)(1-XY^{d_1})(1-XY^{d_2})}, \text{ and}\\
[\mathsf{C}_2](X,Y) &= \frac{Y^{d_2}}{(1-Y)(1-XY^{d_2})}.
\end{aligned}$$

Proof. By Definition 3.5.3

$$\begin{aligned}
[\mathsf{C}_y](X,Y) &= \sum_{v_x=0}^{\infty} \sum_{v_y=0}^{d_1\cdot(v_x+1)-1} X^{v_x}Y^{v_y} = \sum_{v_x=0}^{\infty} X^{v_x}\left(\frac{1 - Y^{d_1\cdot(v_x+1)}}{1-Y}\right)\\
&= \frac{1}{1-Y}\left(\sum_{v_x=0}^{\infty} X^{v_x} - Y^{d_1}\sum_{v_x=0}^{\infty}(XY^{d_1})^{v_x}\right)\\
&= \frac{1}{1-Y}\left(\frac{1}{1-X} - \frac{Y^{d_1}}{1-XY^{d_1}}\right) = \frac{1 - Y^{d_1}}{(1-Y)(1-X)(1-XY^{d_1})}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\mathsf{C}_1](X,Y) &= \sum_{v_x=0}^{\infty} \sum_{v_y=d_1\cdot(v_x+1)}^{d_2\cdot(v_x+1)-1} X^{v_x}Y^{v_y} = \sum_{v_x=0}^{\infty} X^{v_x}\left(\frac{Y^{d_1\cdot(v_x+1)} - Y^{d_2\cdot(v_x+1)}}{1-Y}\right)\\
&= \frac{1}{1-Y}\left(Y^{d_1}\sum_{v_x=0}^{\infty}(XY^{d_1})^{v_x} - Y^{d_2}\sum_{v_x=0}^{\infty}(XY^{d_2})^{v_x}\right)\\
&= \frac{1}{1-Y}\left(\frac{Y^{d_1}}{1-XY^{d_1}} - \frac{Y^{d_2}}{1-XY^{d_2}}\right)
\end{aligned}$$

$$= \frac{Y^{d_1} - Y^{d_2}}{(1-Y)(1-XY^{d_1})(1-XY^{d_2})}.$$

Also,

$$[\mathsf{C}_2](X,Y) = \sum_{v_x=0}^{\infty} \sum_{v_y=d_2\cdot(v_x+1)}^{\infty} X^{v_x}Y^{v_y} = Y^{d_2} \sum_{v_x=0}^{\infty} (XY^{d_2})^{v_x} \sum_{v_y=0}^{\infty} Y^{v_y}$$
$$= \frac{Y^{d_2}}{(1-Y)(1-XY^{d_2})}.$$

□

Lemma 8.2.2. *Let x and $y \in \mathfrak{O}$. let $k, a_j, d_j \in \mathbb{Z}$ where $1 \leq j \leq r$ and $0 < d_1 < \ldots < d_r$. Define $\pi(j) = a_1d_1 + \cdots + a_jd_j$ and $\pi(0) = 0$. Also, define $\gamma(j) = k + a_j + \cdots + a_r$. Then*

$$\int_{\mathfrak{O}^2} |y|^{s-k-1} \prod_{j=1}^{r} ||(\mathfrak{p}x)^{d_j}, y||^{-a_j} \, \mathrm{d}\,\mu(x,y) = (1-q^{-1}) \left[B_y + (1-q^{-1}) \sum_{j=1}^{r} B_j \right],$$

where

$$B_y = \frac{1 - q^{\gamma(1)d_1}t^{d_1}}{(1-q^{\gamma(1)}t)(1-q^{\gamma(1)d_1-1}t^{d_1})},$$
$$B_j = \frac{q^{\pi(j-1)+\gamma(j)d_j}t^{d_j} - q^{\pi(j)+\gamma(j+1)d_{j+1}}t^{d_{j+1}}}{(1-q^{\gamma(j+1)}t)(1-q^{\pi(j-1)+\gamma(j)d_j-1}t^{d_j})(1-q^{\pi(j)+\gamma(j+1)d_{j+1}-1}t^{d_{j+1}})},$$
for $1 \leq j < r$, *and*
$$B_r = \frac{q^{\pi(r)+kd_r}t^{d_r}}{(1-q^kt)(1-q^{\pi(r)+kd_r-1}t^{d_r})}.$$

Proof. Consider the half-open polyhedra $\mathsf{C}_y, \mathsf{C}_1, \ldots, \mathsf{C}_r$ where

$$\mathsf{C}_y = \{(v_x, v_y) \in \mathbb{R}^2_{\geq 0} : v_y < d_1 \cdot (v_x + 1)\},$$
$$\mathsf{C}_1 = \{(v_x, v_y) \in \mathbb{R}^2_{\geq 0} : d_1 \cdot (v_x + 1) \leq v_y < d_2 \cdot (v_x + 1)\},$$
$$\vdots$$
$$\mathsf{C}_{r-1} = \{(v_x, v_y) \in \mathbb{R}^2_{\geq 0} : d_{r-1} \cdot (v_x + 1) \leq v_y < d_r \cdot (v_x + 1)\},$$
$$\mathsf{C}_r = \{(v_x, v_y) \in \mathbb{R}^2_{\geq 0} : d_r \cdot (v_x + 1) \leq v_y\},$$

where $v_y = v(y)$ and $v_x = v(x)$. Define $\mathbf{U}(\cdot) = \{(x,y) \in \mathfrak{O} : (v_x, v_y) \in \cdot\}$. Then divide the domain of the integral into $U(\mathsf{C}_y), U(\mathsf{C}_1), \ldots, U(\mathsf{C}_r)$. We obtain the

following

$$\begin{aligned}\int_{\mathfrak{O}^2} |y|^{s-k-1} \prod_{j=1}^{r} ||(\mathsf{p}x)^{d_j}, y||^{-a_j} \,\mathrm{d}\,\mu(x,y) = &\int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-\gamma(1)-1} \,\mathrm{d}\,\mu(x,y) \\ &+ q^{\pi(1)} \int_{\mathbf{U}(\mathsf{C}_1)} |y|^{s-\gamma(2)-1}|x|^{-\pi(1)} \,\mathrm{d}\,\mu(x,y) \\ &\vdots \\ &+ q^{\pi(r)} \int_{\mathbf{U}(\mathsf{C}_r)} |y|^{s-k-1}|x|^{-\pi(r)} \,\mathrm{d}\,\mu(x,y).\end{aligned} \tag{8.1}$$

Put $t = q^{-s}$. By Corollary 3.5.6, the integral in (8.1) becomes

$$(1-q^{-1})^2 \left[[\mathsf{C}_y](q^{-1}, q^{\gamma(1)}t) + \sum_{j=1}^{r} q^{\pi(j)} [\mathsf{C}_j](q^{\pi(j)-1}, q^{\gamma(j+1)}t) \right]. \tag{8.2}$$

By Lemma 8.2.1, the generating functions for these polyhedra are:

$$\begin{aligned}[\mathsf{C}_y](X,Y) &= \frac{1-Y^{d_1}}{(1-X)(1-Y)(1-XY^{d_1})}, \\ [\mathsf{C}_j](X,Y) &= \frac{(Y)^{d_j} - (Y)^{d_{j+1}}}{(1-Y)(1-XY^{d_j})(1-XY^{d_{j+1}})}, \text{ for } 1 \le j < r, \text{ and} \\ [\mathsf{C}_r](X,Y) &= \frac{Y^{d_r}}{(1-Y)(1-XY^{d_r})}.\end{aligned}$$

Define $B_y = (1-q^{-1})[\mathsf{C}_y](q^{-1}, q^{\gamma(1)}t)$, $B_j = q^{\pi(j)}[\mathsf{C}_j](q^{\pi(j)-1}, q^{\gamma(j+1)}t)$ for $1 \le j \le r$. Then

$$\begin{aligned}B_y &= \frac{1-q^{\gamma(1)d_1}t^{d_1}}{(1-q^{\gamma(1)}t)(1-q^{\gamma(1)d_1-1}t^{d_1})}, \\ B_j &= \frac{q^{\pi(j-1)+\gamma(j)d_j}t^{d_j} - q^{\pi(j)+\gamma(j+1)d_{j+1}}t^{d_{j+1}}}{(1-q^{\gamma(j+1)}t)(1-q^{\pi(j-1)+\gamma(j)d_j-1}t^{d_j})(1-q^{\pi(j)+\gamma(j+1)d_{j+1}-1}t^{d_{j+1}})}, \\ &\quad \text{for } 1 \le j < r, \text{ and} \\ B_r &= \frac{q^{\pi(r)+kd_r-1}t^{d_r}}{(1-q^k t)(1-q^{\pi(r)+kd_r-1}t^{d_r})}.\end{aligned}$$

Note that $\pi(j) + \gamma(j+1)d_j = \pi(j-1) + \gamma(j)d_j$. Then (8.2) becomes

$$(1-q^{-1}) \left[B_y + (1-q^{-1}) \sum_{j=1}^{r} B_j \right].$$

□

8.3 Proof of Theorem D

In this section, we prove Theorem D. We will use Notation 4.2.1 and Notation 8.1.2.

Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(\mathcal{O}_v)$ be a matrix pencil with a rational form

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_\ell}, \mathrm{R}_{\bar{A}}(X_1, X_2)).$$

By Corollary 4.4.3, we have $\mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(\mathrm{R}_A(X_1,X_2);\mathcal{O}_v)}(t)$ for almost all places $v \in \mathcal{V}_k$. Define $\tilde{\mathrm{R}}_A(X_1, X_2) = \mathrm{R}_A(-X_2, X_1)$. By Corollary 1.1.4, we have $M(\tilde{\mathrm{R}}_A(X_1, X_2); \mathcal{O}_v) = M(\mathrm{R}_A(X_1, X_2); \mathcal{O}_v)$. In proof of Theorem D, we write a formula for ask zeta function of $M(\tilde{\mathrm{R}}_A(X_1, X_2); \mathcal{O}_v)$ instead of $M(A(X_1, X_2); \mathcal{O}_v)$ since both have the same ask zeta function for almost all places $v \in \mathcal{V}_K$.

Proof of Theorem D. Let $r = m_1+\cdots+m_k+m_{k+1}+\cdots+m_{k+\ell}$ and let n be the size of the non-singular core. Since $L_{m_i} \in \mathrm{M}_{(m_i+1)m_i}(K[X_1, X_2])$ for $1 \le i \le k+\ell$, we have $d = n+r+k$ and $e = n+r+\ell$ where $\mathrm{rank}_{K(X_1,X_2)}(\tilde{\mathrm{R}}_A(X_1, X_2)) = n+r$. If $A(X_1, X_2)$ is a non-singular matrix pencil, then $\ell = k = r = 0$ and $d = e = n$. Define $M_v := M(\tilde{\mathrm{R}}_A(X_1, X_2); \mathcal{O}_v)$. By Proposition 4.5.1

$$\mathsf{Z}^{\mathsf{ask}}_{M_v}(q_v^{-s}) = (1-q_v^{-1})^{-1} \int\limits_{\mathcal{O}_v^3} |y|^{s-k-1} \prod_{j=1}^{n+r} \frac{||f_{j-1}(x_1, x_2)||}{||f_j(x_1, x_2) \cup f_{j-1}(x_1, x_2)y||} \,\mathrm{d}\,\mu(x_1, x_2, y) \tag{8.3}$$

where $f_j(x_1, x_2)$ is the set of all nonzero $j \times j$ minors of M_v. Consider the following half-open cones

$$\begin{aligned}
\mathsf{C}_y &= \{(v_1, v_2, v_y) \in \mathbb{R}^3_{\ge 0} : v_y < v_1 \text{ and } v_y < v_2\},\\
\mathsf{C}_1 &= \{(v_1, v_2, v_y) \in \mathbb{R}^3_{\ge 0} : v_1 < v_2 \text{ and } v_1 \le v_y\},\\
\mathsf{C}_2 &= \{(v_1, v_2, v_y) \in \mathbb{R}^3_{\ge 0} : v_2 \le v_1 \text{ and } v_2 \le v_y\},
\end{aligned}$$

where $v_1 = v(x_1), v_2 = v(x_2)$ and $v_y = v(y)$. Define $\mathbf{U}(\cdot) := \{(x_1, x_2, y) \in \mathcal{O}_v^3 : (v_1, v_2, v_y) \in \cdot\}$. Divide the domain of the integral in (8.3) into $\mathbf{U}(\mathsf{C}_y), \mathbf{U}(\mathsf{C}_1)$ and $\mathbf{U}(\mathsf{C}_2)$. In $\mathbf{U}(\mathsf{C}_y)$, we have $|y| > ||x_1, x_2||$. By Lemma 3.6.2 (put $\ell = 2$)

$$\int\limits_{\mathbf{U}(\mathsf{C}_y)} |y|^{s-k-1} \prod_{j=1}^{n+r} \frac{||f_{j-1}(x_1, x_2)||}{||f_{j-1}(x_1, x_2)|| \cdot |y|} \,\mathrm{d}\,\mu(x_1, x_2, y) = \frac{q_v^{-2}(1-q^{-1})}{1-q_v^{d-2}t}. \tag{8.4}$$

where $t = q^{-s}$. In $U(C_1)$, we have $|x_1| \ge ||x_2, y||$ and $|x_1| > |x_2|$. Definition 2.4.2 shows that each L_{m_i} contains a minor equal to $X_1^{m_i}$ for $1 \le i \le k+\ell$. Hence, we have $\mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}})$ contains a minor X_1^r. By Corollary 8.1.4(ii), we have $||f_j(x_1, x_2)|| = |x_1|^j$ for $1 \le j \le n+r$. By Lemma 3.6.3 (put $\ell = 2$ and $i = 1$)

$$\int\limits_{\mathbf{U}(\mathsf{C}_1)} |y|^{s-k-1} \prod_{j=1}^{n+r} \frac{|x_1|^{j-1}}{|x_1|^j} \,\mathrm{d}\,\mu(x_1, x_2, y) = \frac{q_v^{-1}(1-q_v^{-1})^2}{(1-q_v^k t)(1-q_v^{d-2}t)}. \tag{8.5}$$

In $U(C_2)$, we have $|x_2| \geq ||x_1, y||$. Put $x_1 = x_1'x_2$ and $y = x_2y'$. Then we map $J : \mathcal{O}_v^3 \longrightarrow U(C_2)$ via $J(x_1', x_2, y') \mapsto (x_1'x_2, x_2, x_2y')$ where the determinant of Jacobian matrix $\det\left[\frac{\partial J(x_1', x_2, y')}{\partial(x_1', x_2, y')}\right] = x_2^2$. Similarly, Definition 2.4.2 shows that each L_{m_i} contains a minor equal to $X_2^{m_i}$ for $1 \leq i \leq k+\ell$. Hence $\mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L_{m_{k+1}}^T, \ldots, L_{m_{k+\ell}}^T)$ contains a minor X_2^r. By Corollary 8.1.4(i), we have $||f_j(x_1'x_2, x_2)|| = |x_2^j|$ for $1 \leq j \leq n+r-w$ and $||f_{n+r-w+w_0}(x_1'x_2, x_2)|| = |x_2|^{n+r-w+w_0} \prod_{j=1}^{w_0} |F_j(x_1')|$ for $1 \leq w_0 \leq w$. By Theorem 3.4.3

$$\begin{aligned}
&\int_{\mathbf{U}(\mathsf{C}_2)} |y|^{s-k-1} \prod_{j=1}^{n+r} \frac{||f_{j-1}(x_1, x_2)||}{||f_j(x_1, x_2) \cup f_j(x_1, x_2)y||} \,\mathrm{d}\,\mu(x_1, x_2, y) \\
&= \int_{\mathcal{O}_v^3} |x_2y'|^{s-k-1}|x_2|^2 \prod_{j=1}^{n+r} \frac{||f_{j-1}(x_1'x_2, x_2)||}{||f_j(x_1'x_2, x_2) \cup f_{j-1}(x_1'x_2, x_2)x_2y'||} \,\mathrm{d}\,\mu(x_1', x_2, y') \\
&= \int_{\mathcal{O}_v^3} |y'|^{s-k-1}|x_2|^{s-d+1} \prod_{j=1}^{w} \frac{|\prod_{j_0=1}^{j-1} F_{j_0}(x_1')|}{||\prod_{j_0=1}^{j} F_{j_0}(x_1'), \prod_{j_0=1}^{j-1} F_{j_0}(x_1')y'||} \,\mathrm{d}\,\mu(x_1', x_2, y') \\
&= \frac{1-q_v^{-1}}{1-q_v^{d-2}t} \int_{\mathcal{O}_v^2} |y'|^{s-k-1} \prod_{j=1}^{w} ||F_j(x_1'), y'||^{-1} \,\mathrm{d}\,\mu(x_1', y'). \qquad (8.6)
\end{aligned}$$

We can decompose the domain of x_1' in the integral (8.6) into sets of the form $\alpha + \mathfrak{P}_v$ where $\alpha \in \mathcal{O}_v$. If $F_j(\alpha) \not\equiv 0 \bmod \mathfrak{P}_v$ for all j, then $|F_j(x_1')| = 1$ for all $x_1' \in \alpha + \mathfrak{P}_v$. Put $\mathsf{p}_v x = x_1' - \alpha$ then we map $\mathcal{O}_v \longrightarrow \alpha + \mathfrak{P}_v$ via $J_0(x) \mapsto \mathsf{p}_v x + \alpha$ where $|\frac{\mathrm{d}\,J_0(x)}{\mathrm{d}\,x}| = |\mathsf{p}_v|$. Therefore, the integral in (8.6) becomes

$$\begin{aligned}
\int_{\alpha+\mathfrak{P}_v \times \mathcal{O}_v} |y'|^{s-k-1} \prod_{j=1}^{w} ||F_j(x_1'), y'||^{-1} \,\mathrm{d}\,\mu(x_1', y') &= \int_{\mathcal{O}_v^2} |y'|^{s-k-1}|\mathsf{p}_v| \,\mathrm{d}\,\mu(x, y') \\
&= \frac{q_v^{-1}(1-q_v^{-1})}{(1-q_v^k t)}. \qquad (8.7)
\end{aligned}$$

Let $\bar{\alpha} \in \mathfrak{K}_v$ such that $F_j(\bar{\alpha}) = 0$. Then there exists g_i such that $g_i(\bar{\alpha}) = 0$ for $1 \leq i \leq h$. By Corollary 8.1.5, there exist $\alpha \in \mathcal{O}_v$ such that for all $x_1' \in \mathcal{O}_v$ with $x_1' \equiv \bar{\alpha} \bmod \mathfrak{P}_v$

$$|F_j(x_1')| = |x_1' - \alpha|^{e_{ij}},$$

Hence, if $F_j(\alpha) \equiv 0 \bmod \mathfrak{P}_v$ for any j, then the integral in (8.6) becomes

$$\int_{\alpha+\mathfrak{P}_v \times \mathcal{O}_v} |y'|^{s-k-1} \prod_{j=1}^{w} ||(x_1' - \alpha)^{e_{ij}}, y'||^{-1} \,\mathrm{d}\,\mu(x_1', y'). \qquad (8.8)$$

Put $\mathsf{p}_v x = x_1' - \alpha$ and consider $0 < d_{i1} < \cdots < d_{ir_i}$ and $a_{i1}, \ldots, a_{ir_i}$ as in Notation 8.1.2. Then (8.8) becomes

$$q_v^{-1} \int_{\mathcal{O}_v^2} |y'|^{s-k-1} \prod_{j=1}^{r_i} ||(\mathsf{p}_v x)^{d_{ij}}, y'||^{-a_{ij}} \,\mathrm{d}\,\mu(x, y'). \qquad (8.9)$$

Define $\gamma_i(j) = k + a_{ij} + a_{ij+1} + \cdots + a_{ir_i}$ and $\pi(j) = a_1d_1 + \cdots + a_jd_j$. By Lemma 8.2.2, then (8.9) becomes

$$(1 - q_v^{-1}) \left[B_y(i) + (1 - q_v^{-1}) \sum_{j=1}^{r_i} B_j(i) \right], \tag{8.10}$$

where

$$B_y(i) = \frac{q^{-1}(1 - q_v^{\gamma_i(1)d_{i1}} t^{d_{i1}})}{(1 - q_v^{\gamma_i(1)} t)(1 - q_v^{\gamma_i(1)d_{i1}-1} t^{d_{i1}})},$$

$$B_j(i) = \frac{q_v^{\pi_i(j-1)+\gamma_i(j)d_{ij}-1} t^{d_{ij}} - q_v^{\pi_i(j)+\gamma_i(j+1)d_{ij+1}-1} t^{d_{ij+1}}}{(1 - q_v^{\gamma_i(j+1)} t)(1 - q_v^{\pi_i(j-1)+\gamma_i(j)d_{ij}-1} t^{d_{ij}})(1 - q_v^{\pi_i(j)+\gamma_i(j+1)d_{ij+1}-1} t^{d_{ij+1}})},$$

for $1 \leq j < r_i$, and

$$B_{r_i}(i) = \frac{q_v^{\pi_i(r_i)+kd_{ir_i}-1} t^{d_{ir_i}}}{(1 - q_v^k t)(1 - q_v^{\pi_i(r_i)+kd_{ir_i}-1} t^{d_{ir_i}})}.$$

Now define $N_i(\mathfrak{K}_v) = \#\{\bar{\alpha} \in \mathfrak{K}_v : g_i(\bar{\alpha}) = 0\}$. For $1 \leq i \leq h$, define $L(q_v, t)$ and $H_i(q_v, t)$ to be the rational functions in (8.7) and (8.10), respectively. Then (8.6) becomes

$$\frac{1 - q_v^{-1}}{1 - q_v^{d-2}t} \left[\left(q_v - \sum_{i=1}^{h} N_i(\mathfrak{K}_v) \right) L(q_v, t) + \sum_{i=1}^{h} N_i(\mathfrak{K}_v) H_i(q_v, t) \right]. \tag{8.11}$$

Define the rational function $W_i(q_v, t)$ as follows

$$W_i(q_v, t) = \frac{(1 - q_v^{-1})(1 - q_v^k t) \left[B_y(i) + (1 - q_v^{-1}) \sum_{j=1}^{r_i} B_j(i) \right] - q_v^{-1}(1 - q_v^{-1})}{(1 - q_v^k t)(1 - q_v^{d-2} t)}.$$

Then (8.11) becomes

$$\frac{(1 - q_v^{-1})^2}{(1 - q_v^k t)(1 - q_v^{d-2} t)} + (1 - q_v^{-1}) \sum_{i=1}^{h} N_i(\mathfrak{K}_v) W_i(q_v, t). \tag{8.12}$$

By (8.4), (8.5) and (8.12), we obtain

$$\mathsf{Z}^{\mathsf{ask}}_{M_v}(t) = \frac{1 - q_v^{k-2} t}{(1 - q_v^k t)(1 - q_v^{d-2} t)} + \sum_{i=1}^{h} N_i(\mathfrak{K}_v) W_i(q_v, t).$$

□

Remark 8.3.1. Define $\pi_i(j) = a_{i1}d_{i1} + \cdots + a_{ij}d_{ij}$ and $\pi_i(0) = 0$. Also, define $\gamma_i(j) = k + a_{ij} + \cdots + a_{ir_i}$. The rational function $W_i(q_v, t)$ in Theorem D is defined by:

$$W_i(q_v, t) := \frac{(1 - q_v^{-1})(1 - q_v^k t) \left[B_y(i) + (1 - q_v^{-1}) \sum_{j=1}^{r_i} B_j(i) \right] - q_v^{-1}(1 - q_v^{-1})}{(1 - q_v^k t)(1 - q_v^{d-2} t)},$$

where

$$B_y(i) = \frac{q_v^{-1}\sum_{m=0}^{d_{i1}-1}(q_v^{\gamma_i(1)}t)^m}{(1-q_v^{\gamma_i(1)d_{i1}-1}t^{d_{i1}})},$$

$$B_j(i) = \frac{q_v^{\pi_i(j-1)+\gamma_i(j)d_{ij}-1}t^{d_{ij}}\sum_{m=0}^{d_{ij+1}-d_{ij}-1}(q_v^{\gamma_i(j+1)}t)^m}{(1-q_v^{\pi_i(j-1)+\gamma_i(j)d_{ij}-1}t^{d_{ij}})(1-q_v^{\pi_i(j)+\gamma_i(j+1)d_{ij+1}-1}t^{d_{ij+1}})},$$

for $1 \leq j < r_i$, and

$$B_{r_i}(i) = \frac{q_v^{\pi_i(r_i)+kd_{ir_i}-1}t^{d_{ir_i}}}{(1-q_v^k t)(1-q_v^{\pi_i(r_i)+kd_{ir_i}-1}t^{d_{ir_i}})}.$$

Example 8.3.2. Let $A(X_1, X_2)$ be a non-singular matrix pencil with a rational form $\mathrm{R}(X_1, X_2) = \mathrm{R}X_1 + I_6X_2$, where

$$\mathrm{R} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -2 & 0 \end{bmatrix}.$$

Using Notation 8.1.2, we have $F_1(X) = X^2 + 1$ and $F_2(X) = (X^2 + 1)^2$. Hence $h = 1$ where $g_1(X) = X^2 + 1$. By Remark 8.3.1

$$B_y(1) = \frac{q_v^{-1}}{1-q_v t},\ B_1(1) = \frac{q_v t}{(1-q_v t)(1-q_v^2t^2)},\ \text{and}\ B_2(1) = \frac{q_v^2t^2}{1-q_v^2t^2}.$$

From Remark 8.3.1, we can obtain the rational function $W_1(q_v, t)$. If $q_v \equiv 3 \mod 4$, then $g_1(X) = X^2 + 1$ has no solutions over $\mathfrak{K}_v$ and $N_1(\mathfrak{K}_v) = 0$. Thus

$$\mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1-q_v^{-2}t}{(1-t)(1-q_v^4t)}.$$

If $q_v \equiv 1 \mod 4$, then $g_1(X) = X^2 + 1$ has two solutions and $N_1(\mathfrak{K}_v) = 2$. Hence

$$\mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1-q_v^{-2}t}{(1-t)(1-q_v^4t)} + 2\cdot W_1(q_v, t).$$

Chapter 9

Ask zeta functions of relation modules of single matrices

In this chapter, we introduce the notion of admissible matrices of linear forms which is analogue to admissible partial colorings [11, Definition 1.4]. Over a finite field, we compute the number of matrices by rank in relation vector spaces associated with single matrices which allows us to write weakly equivalence classes of these modules for almost all places. Moreover, we give formulas for ask zeta functions of relation modules associated with single matrices, see Theorem E. In Theorem E, we use Notation 4.2.1.

Theorem E. *Let $A \in \mathrm{M}_{d\times e}(K)$ with $r = \mathrm{rank}_K(A)$. Then for almost all places $v \in \mathcal{V}_k$*

$$\mathsf{Z}^{ask}_{\mathrm{Rel}_{d\times e}(AX;\mathcal{O}_v)}(t) = \begin{cases} \frac{(1-q_v^{-e+1}t)^2}{(1-t)(1-q_v^{d-e}t)(1-q_v^{-e+2}t)}, & \text{if } r = 1, \\ \frac{1-q_v^{-e}t}{(1-t)(1-q_v^{d-e}t)}, & \text{otherwise.} \end{cases}$$

9.1 Admissible matrices of linear forms

Here, we introduce admissible matrices of linear forms which are analogous to admissible partial colorings mentioned in [11, Section 1.3].

Recall that $[d] = \{1, \ldots, d\}$. Let $d, e \geq 1$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_\ell$ be pairwise disjoint subsets of $[d] \times [e]$. For a ring R, let $A_1, \ldots, A_\ell \in \mathrm{M}_{d\times e}(R)$ where $A_k = [a_{kij}]$ such that $a_{kij} \in R^\times$ if and only if $(i, j) \in \mathcal{A}_k$ and zero otherwise for $1 \leq k \leq \ell$. Let $A(X) = A_1X_1 + \cdots + A_\ell X_\ell$. Then we say that $A(X)$ is a **matrix of linear forms associated with the pairwise disjoint sets** $\mathcal{A}_1, \ldots, \mathcal{A}_\ell$. Let $J_R \subset [d]$ and $J_C \subset [e]$ such that if $\mathcal{A}_k \cap (J_R \times J_C) \neq \emptyset$, then $\mathcal{A}_k \subset J_R \times J_C$ for all k. Define $B(X)$ to be the submatrix of $A(X)$ obtained by selecting rows from J_R and columns from J_C. Then we say $B(X)$ is a **closed submatrix** of $A(X)$.

Definition 9.1.1. A matrix of linear forms $A(X)$ is **admissible** if:

(i) $A(X)$ is associated with a sequence of pairwise disjoint sets, and

(ii) every non-empty closed submatrix of $A(X)$ contains at least one zero entry.

Example 9.1.2. Consider the matrix pencil $A(X) = A_1X_1 + A_2X_2 \in \mathrm{M}_3(\mathbb{Q}[X])$ where A_1 and A_2 are given by one of the following pairs of matrices.

$$\left(\begin{bmatrix}1&2&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\0&3&0\end{bmatrix}\right) \quad \left(\begin{bmatrix}1&2&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&3&0\end{bmatrix}\right) \quad \left(\begin{bmatrix}1&2&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&2\\0&0&0\end{bmatrix}\right)$$

(a) (b) (c)

Then $A(X)$ is admissible in (a). However, we have the closed submatrices $\begin{bmatrix}X_2\\3X_2\end{bmatrix}$ and $\begin{bmatrix}X_1&2X_1&X_2\\X_1&X_2&2X_2\end{bmatrix}$ in (b) and (c), respectively, which do not contain at least a zero. Hence $A(X)$ is not admissible.

Let $A(X) \in \mathrm{M}_{d\times e}(R)$ be a matrix of linear forms associated with disjoint pairwise subsets $\mathcal{A}_1, \ldots, \mathcal{A}_\ell \subset [d] \times [e]$. We can interpret these sets as sets of partial colorings. Thus, a relation module of a matrix of linear forms associated with these sets is also a relation module of partial colorings mentioned in [11, Section 1.3]. Moreover, relation modules of admissible matrices of linear forms are relation modules associated with admissible partial colorings, see [11, Definition 1.4]. In the following theorem, we use Notation 4.1.2.

Theorem 9.1.3 (Cf. [11, Corollary B])**.** *Let $A(X) \in \mathrm{M}_{d\times e}(\mathfrak{O}[X])$ be an admissible matrix of linear forms. Then $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X);\mathfrak{O})}(t) = \frac{1-q^{-e}t}{(1-t)(1-q^{d-e}t)}$.*

Next, we use Notation 4.2.1.

Corollary 9.1.4. *Let $A(X) \in \mathrm{M}_{d\times e}(K[X])$ be an admissible matrix of linear forms. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X);\mathcal{O}_v)}(t) = \frac{1-q_v^{-e}t}{(1-t)(1-q_v^{d-e}t)}.$$

Proof. Let $A(X) = A_1X_1 + \cdots + A_\ell X_\ell$ where $A_k = [a_{kij}]$ for $1 \le k \le \ell$. Let $S := \{v \in \mathcal{V}_K : v(a_{kij}) \neq 0 \text{ for any } a_{kij} \in A_k, 1 \le k \le \ell\}$. Since S is a finite set, choose any $v \in \mathcal{V}_K \setminus S$. Then $A(X) \in \mathrm{M}_{d\times e}(\mathcal{O}_v[X])$ is an admissible matrix of linear forms. By Proposition 9.1.3, we have

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X);\mathcal{O}_v)}(t) = \frac{1-q_v^{-e}t}{(1-t)(1-q_v^{d-e}t)}$$

for almost all places $v \in \mathcal{V}_K$. □

We use Corollary 9.1.4 and Corollary 4.4.3 to obtain the same formula for relation modules associated with matrices of linear forms which are weakly equivalent to relation modules associated with admissible matrices of linear forms for almost all place $v \in \mathcal{V}_K$.

9.2 Counting the number of matrices in relation vector spaces of single matrices over finite fields

In this section, we compute the number of matrices with rank r in a relation vector space associated with a single matrix over finite fields.

For an arbitrary field K, let $A \in \mathrm{M}_{d\times e}(K)$ be a matrix with $\mathrm{rank}_K(A) = n$. Then there exist $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that $gAh = \mathcal{U}_{n,d,e}$ where $\mathcal{U}_{n,d,e} := \mathrm{diag}(I_n, 0_{(d-n)\times(e-n)})$. Therefore, we have $\mathrm{Rel}(AX; K)$ is equivalent to $\mathrm{Rel}(\mathcal{U}_n X; K)$.

Let $\mathbb{F}_q$ denote a finite field with size q. For a subspace $M \subset \mathrm{M}_{d\times e}(\mathbb{F}_q)$, we denote by $\mathcal{N}_r(M)$ the number of matrices in M with rank r. Landsberg [36] proved that

$$\mathcal{N}_r(M_{d\times e}(\mathbb{F}_q)) = \prod_{k=0}^{r-1} \frac{(q^d - q^k)(q^e - q^k)}{q^r - q^k}. \tag{9.1}$$

Moreover, Bender [7] stated that

$$\mathcal{N}_r(\mathfrak{sl}_n(\mathbb{F}_q)) = q^{-1} \prod_{k=0}^{r-1} \frac{(q^n - q^k)^2}{q^r - q^k} - (1 - q^{-1}) \prod_{k=0}^{r-1} \frac{q^n - q^k}{q^{k+1} - 1}, \tag{9.2}$$

where $\mathfrak{sl}_n(\mathbb{F}_q)$ denotes the traceless $n \times n$ matrices over $\mathbb{F}_q$. Note that

$$\begin{aligned}\mathrm{Rel}(I_n X; \mathbb{F}_q) &= \{x \in \mathrm{M}_n(\mathbb{F}_q) : \langle I_n, x\rangle = 0\} \\ &= \{[x_{ij}] \in \mathrm{M}_n(\mathbb{F}_q) : \sum_{i=1}^{n} x_{ii} = 0\} = \mathfrak{sl}_n(\mathbb{F}_q).\end{aligned}$$

Thus

$$\mathrm{Rel}(\mathcal{U}_{n,d,e} X; \mathbb{F}_q) = \left[\begin{array}{c|c} \mathfrak{sl}_n(\mathbb{F}_q) & \mathrm{M}_{n\times(e-n)}(\mathbb{F}_q) \\ \hline \multicolumn{2}{c}{\mathrm{M}_{(d-n)\times e}(\mathbb{F}_q)} \end{array}\right]. \tag{9.3}$$

In the following, we shall use the formulas in (9.1) and (9.2) to obtain a formula for the number of matrices in (9.3). For a matrix A, we denote by $\mathrm{row}_{\mathbb{F}_q}(A)$ the dimension of the row space of A over $\mathbb{F}_q$. Moreover, define $\mathcal{M}_j(r, d, e) := q^{(r-j)e}\mathcal{N}_j(\mathrm{M}_{(d-r+j)\times e}(\mathbb{F}_q))$.

Lemma 9.2.1. *Let $\bar{M} \subset \mathrm{M}_{m\times n}(\mathbb{F}_q)$ be subspace. Define*

$$M := \left\{[a, b] \in \mathrm{M}_{m\times e}(\mathbb{F}_q) : a \in \bar{M}, b \in \mathrm{M}_{m\times(e-n)}(\mathbb{F}_q)\right\}.$$

Then $\mathcal{N}_r(M) = \sum_{j=0}^{r} \mathcal{M}_j(r, m, e-n)\mathcal{N}_{r-j}(\bar{M})$.

Proof. For $r \geq 0$, fix $a \in \bar{M}$ with rank $r - j$ where $0 \leq j \leq r$. Let $b \in \mathrm{M}_{m\times(e-n)}(\mathbb{F}_q)$ such that $\mathrm{rank}_{\mathbb{F}_q}[a, b] = r$. Then b induces a linear map $\mathbb{F}_q^{e-n} \to \mathbb{F}_q^m$ via $x \mapsto bx$. Let $\psi : \mathbb{F}_q^{e-n} \to \mathbb{F}_q^m/\mathrm{row}_{\mathbb{F}_q}(a)$ be the linear map induced via

$x \mapsto bx + \mathrm{row}_{\mathbb{F}_q}(a) = \bar{b}x$ where $\bar{b}$ is a submatrix of b obtained by removing $r-j$ rows from b with the same indices of $r-j$ independent rows in a. Note that if $\mathrm{rank}_{\mathbb{F}_q}([a,b]) = r$, then $\mathrm{rank}_{\mathbb{F}_q}(\bar{b}) = j$. Moreover, if $\mathrm{rank}_{\mathbb{F}_q}(\bar{b}) = j$, then $\mathrm{rank}_{\mathbb{F}_q}([a,b]) = r$. Hence $\mathrm{rank}_{\mathbb{F}_q}(\bar{b}) = j$ if and only if $\mathrm{rank}_{\mathbb{F}_q}([a,b]) = r$. First, we count all the possible ways to write $\bar{b}$ with $\mathrm{rank}_{\mathbb{F}_q}(\bar{b}) = j$. Since $\mathbb{F}_q^m/\mathrm{row}_{\mathbb{F}_q}(a) \approx \mathbb{F}_q^{m-r+j}$, we count all matrices in $\mathrm{M}_{(m-r+j)\times(e-n)}(\mathbb{F}_q)$ with rank j which is given by $\mathcal{N}_j(\mathrm{M}_{(m-r+j)\times(e-n)}(\mathbb{F}_q))$ in (9.1). Now we count the number of ways to write b such that $\mathrm{rank}_{\mathbb{F}_q}(\bar{b}) = j$. Define $\phi : \mathbb{F}_q^m \to \mathbb{F}_q^m/\mathrm{row}_{\mathbb{F}_q}(a)$ via $y \mapsto y + \mathrm{row}_{\mathbb{F}_q}(a)$. Consider the epimorphism $\mathrm{Hom}(\mathbb{F}_q^{e-n}, \mathbb{F}_q^m) \xrightarrow{\mathrm{Hom}(\mathbb{F}_q^{e-n},\phi)} \mathrm{Hom}(\mathbb{F}_q^{e-n}, \mathbb{F}_q^m/\mathrm{row}_{\mathbb{F}_q}(a))$ where $(\mathbb{F}_q^{e-n} \xrightarrow{b} \mathbb{F}_q^m) \mapsto (\mathbb{F}_q^{e-n} \xrightarrow{b} \mathbb{F}_q^m \xrightarrow{\phi} \mathbb{F}_q^m/\mathrm{row}_{\mathbb{F}_q}(a))$ as in the following diagram:

$$\begin{array}{ccc} \mathbb{F}_q^{e-n} & \xrightarrow{\quad b \quad} & \mathbb{F}_q^m \\ & \searrow^{\bar{b}} & \downarrow \phi \\ & & \mathbb{F}_q^m/\mathrm{row}_{\mathbb{F}_q}(a). \end{array}$$

Then

$$\begin{aligned} \mathcal{N}_r([a, \mathrm{M}_{m\times(e-n)}(\mathbb{F}_q)]) &= |\ker(\mathrm{Hom}(\mathbb{F}_q^{e-n}, \phi)| \cdot \mathcal{N}_j(\mathrm{M}_{(m-r+j)\times(e-n)}(\mathbb{F}_q)) \\ &= |\mathrm{Hom}(\mathbb{F}_q^{e-n}, \ker(\phi))| \cdot \mathcal{N}_j(\mathrm{M}_{(m-r+j)\times(e-n)}(\mathbb{F}_q)) \\ &= |\mathrm{Hom}(\mathbb{F}_q^{e-n}, \mathrm{row}_{\mathbb{F}_q}(a))| \cdot \mathcal{N}_j(\mathrm{M}_{(m-r+j)\times(e-n)}(\mathbb{F}_q)) \\ &= q^{(r-j)(e-n)}\mathcal{N}_j(\mathrm{M}_{(m-r+j)\times(e-n)}(\mathbb{F}_q)). \end{aligned}$$

Therefore, the number of matrices in M is the sum

$$\begin{aligned} \mathcal{N}_r([\bar{M}, \mathrm{M}_{m\times(e-n)}(\mathbb{F}_q)]) &= \sum_{j=0}^{r} q^{(r-j)(e-n)}\mathcal{N}_j(\mathrm{M}_{(m-r+j)\times(e-n)}(\mathbb{F}_q))\mathcal{N}_{r-j}(\bar{M}) \\ &= \sum_{j=0}^{r} \mathcal{M}_j(r, m, e-n)\mathcal{N}_{r-j}(\bar{M}). \end{aligned}$$

□

Remark 9.2.2. Let $\bar{M} \subset \mathrm{M}_{m\times n}(\mathbb{F}_q)$. Define

$$M := \left\{ [a,b]^T \in \mathrm{M}_{d\times n}(\mathbb{F}_q) : a \in \bar{M}, b \in \mathrm{M}_{(d-m)\times n}(\mathbb{F}_q) \right\}.$$

By using the transpose of M^T and Lemma 9.2.1, we obtain the following:

$$\mathcal{N}_r(M^T) = \sum_{j=0}^{r} \mathcal{M}_j(r, n, d-m)\mathcal{N}_{r-j}(\bar{M}) = \mathcal{N}_r(M).$$

Corollary 9.2.3. *Let $\bar{M} \subset \mathrm{M}_{m\times n}(\mathbb{F}_q)$. Define*

$$M := \left\{ \left[\begin{array}{c|c} a & b \\ \hline c & \end{array}\right] \in \mathrm{M}_{d\times e}(\mathbb{F}_q) : a \in \bar{M}, b \in \mathrm{M}_{m\times(e-n)}(\mathbb{F}_q), c \in \mathrm{M}_{(d-m)\times e}(\mathbb{F}_q) \right\}.$$

Then

$$\mathcal{N}_r(M) = \sum_{j=0}^{r} \mathcal{M}_j(r, e, d-m) \sum_{i=0}^{r-j} \mathcal{M}_i(r-j, m, e-n)\mathcal{N}_{r-i-j}(\bar{M}).$$

Proof. Define $\tilde{M} := \{[a, b] \in \mathrm{M}_{m\times e}(\mathbb{F}_q) : a \in \bar{M}, b \in \mathrm{M}_{m\times(e-n)}(\mathbb{F}_q)\}$. By Lemma 9.2.1, we have

$$\mathcal{N}_r(M) = \sum_{j=0}^{r} \mathcal{M}_j(r, e, d-m)\mathcal{N}_{r-j}(\tilde{M}). \tag{9.4}$$

By Remark 9.2.2 and (9.4), we obtain

$$\mathcal{N}_r(M) = \sum_{j=0}^{r} \mathcal{M}_j(r, e, d-m) \sum_{i=0}^{r-i} \mathcal{M}_i(r-j, m, e-n)\mathcal{N}_{r-i-j}(\bar{M}).$$

□

In the following, we use (9.3) to give a formula for counting the number of matrices in $\mathrm{Rel}(AX; \mathbb{F}_q)$ where A.

Remark 9.2.4. Let $A \in \mathrm{M}_{d\times e}(K)$ be a matrix with $\mathrm{rank}_K(A) = n$. To compute $\mathcal{N}_r(\mathrm{Rel}(AX; \mathbb{F}_q))$, we compute $\mathrm{Rel}(\mathcal{U}_n X; \mathbb{F}_q)$ by using Corollary 9.2.3 and (9.2). Thus

$$N_r(\mathrm{Rel}(AX; \mathbb{F}_q)) = \sum_{j=0}^{r} \mathcal{M}_j(r, e, d-n) \sum_{i=0}^{r-i} \mathcal{M}_i(r-j, n, e-n)\mathcal{N}_{r-i-j}(\mathfrak{sl}_n(\mathbb{F}_q)).$$

Computing by Sage [51], we obtain a formula for the number of matrices with rank 1 in $\mathrm{Rel}(AX; \mathbb{F}_q)$:

$$\mathcal{N}_1(\mathrm{Rel}(AX; \mathbb{F}_q)) = \frac{q^{d+e-1} + q^{d+e-n-1}(q-1) - q^d - q^e + 1}{q-1}. \tag{9.5}$$

The formula in (9.5) shows that for any two matrices $A, B \in \mathrm{M}_{d\times e}(\mathbb{F}_q)$ with $rank_{\mathbb{F}_q}(A) \neq rank_{\mathbb{F}_q}(B)$, we have

$$\mathcal{N}_1(\mathrm{Rel}(AX; \mathbb{F}_q)) \neq \mathcal{N}_1(\mathrm{Rel}(BX; \mathbb{F}_q)).$$

9.3 Equivalence classes and Proof of Theorem E

In this section, we recall the notion of staircase modules of matrices. For a number field K, we write the number of equivalence classes of relation modules associated with single matrices in $M_{d\times e}(K)$ for almost all places. We also prove Theorem E. Here, we use Notation 4.2.1.

In Remark 9.2.4, we showed that for any two matrices $A, B \in \mathrm{M}_{d\times e}(\mathfrak{K}_v)$ with $\mathrm{rank}_{\mathfrak{K}_v}(A) \neq \mathrm{rank}_{\mathfrak{K}_v}(B)$, we have $\mathcal{N}_1(\mathrm{Rel}(AX; \mathfrak{K}_v)) \neq \mathcal{N}_1(\mathrm{Rel}(BX; \mathfrak{K}_v))$. Hence $\mathrm{Rel}(AX; \mathfrak{K}_v)$ is not equivalent to $\mathrm{Rel}(BX; \mathfrak{K}_v)$.

Lemma 9.3.1. *Let* $A, B \in \mathrm{M}_{d\times e}(K)$ *with* $\mathrm{rank}_K(A) = n_A$ *and* $\mathrm{rank}_K(B) = n_B$. *Then* $\mathrm{Rel}(AX;K)$ *is equivalent to* $\mathrm{Rel}(BX;K)$ *if and only if* $n_A = n_B$.

Proof. If $n_A = n_B$, then A is equivalent to B over K. Corollary 1.2.3(ii) shows that $\mathrm{Rel}(AX;K)$ is equivalent to $\mathrm{Rel}(BX;K)$. Now suppose that $\mathrm{Rel}(AX;K)$ is equivalent to $\mathrm{Rel}(BX;K)$. Then there exist $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that $\mathrm{Rel}(AX;K) = g\,\mathrm{Rel}(BX;K)h$. By Corollary 4.4.1, we have $g \in \mathrm{GL}_d(\mathcal{O}_v)$ and $h \in \mathrm{GL}_e(\mathcal{O}_v)$ for almost all places $v \in \mathcal{V}_K$. Thus $\mathrm{Rel}(AX;\mathcal{O}_v)$ is equivalent to $\mathrm{Rel}(BX;\mathcal{O}_v)$ for almost all places $v \in \mathcal{V}_K$. Let $\mathrm{Rel}(AX;\mathcal{O}_v) \xrightarrow{\theta} \mathrm{Hom}(\mathcal{O}_v^d, \mathcal{O}_v^e)$ and $\mathrm{Rel}(BX;\mathcal{O}_v) \xrightarrow{\bar{\theta}} \mathrm{Hom}(\mathcal{O}_v^d, \mathcal{O}_v^e)$ be the inclusion maps. By Lemma 1.3.4, we have θ and $\bar{\theta}$ are isotopic for almost all places $v \in \mathcal{V}_K$. Let $\mathrm{Rel}(AX;\mathcal{O}_v/\mathfrak{P}_v^n) \xrightarrow{\theta_n} \mathrm{Hom}((\mathcal{O}_v/\mathfrak{P}_v^n)^d, (\mathcal{O}_v/\mathfrak{P}_v^n)^e)$ and $\mathrm{Rel}(BX;\mathcal{O}_v/\mathfrak{P}_v^n) \xrightarrow{\bar{\theta}_n} \mathrm{Hom}((\mathcal{O}_v/\mathfrak{P}_v^n)^d, (\mathcal{O}_v/\mathfrak{P}_v^n)^e)$ be also the inclusion maps. Then θ_n and $\bar{\theta}_n$ are isotopic for almost all places $v \in \mathcal{V}_K$. Thus $\mathrm{Rel}(AX;\mathfrak{K}_v)$ is equivalent to $\mathrm{Rel}(BX;\mathfrak{K}_v)$ for almost all places $v \in \mathcal{V}_K$. By Remark 9.2.4, we have $\mathrm{rank}_{\mathfrak{K}_v}(A) = \mathrm{rank}_{\mathfrak{K}_v}(B)$ for almost all places $v \in \mathcal{V}_K$. Thus $n_A = n_B$. □

Let $m = (m_0, \ldots, m_d) \in \mathbb{N}_0^{d+1}$ and let $e = m_0 + \cdots + m_d$. Define a matrix $\mathcal{S}_m = [s_{ij}]_{\substack{1\le i\le d\\ 1\le j\le e}} \in \mathrm{M}_{d\times e}(\mathbb{Z})$ such that

$$s_{ij} = \begin{cases} 1 & \text{if } j > \sum\limits_{k<i} m_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then we call $\mathcal{S}_m$ the **staircase matrix associated with** m, see [50, Section 5.1.1].

Proposition 9.3.2 ([50, Proposition 5.10])**.** *Let* $M := \{[s_{ij}x_{ij}] \in \mathrm{M}_{d\times e}(\mathfrak{O}) : x_{ij} \in \mathfrak{O}, s_{ij} \in \mathcal{S}_m\}$. *Then*

$$\mathsf{Z}_M^{\mathsf{ask}}(t) = \frac{1}{1-t}\prod_{j=0}^{d-1}\frac{1-q^{-1+d-j-\sum_{k>j}m_k}t}{1-q^{d-j-\sum_{k>j}m_k}t}.$$

A module M, defined as in Proposition 9.3.2, is called a **staircase module associated with** $m \in \mathbb{N}_0^{d+1}$.

Proof of Theorem E. We have A is equivalent to $\mathcal{U}_{r,d,e}$ since $\mathrm{rank}_K(A) = r$. By Corollary 1.2.3(ii) and Lemma 4.4.2, we have $\mathrm{Rel}(AX;\mathcal{O}_v)$ is equivalent to $\mathrm{Rel}(\mathcal{U}_rX;\mathcal{O}_v)$ for almost all places $v \in \mathcal{V}_K$. If $r = 1$, then $\mathrm{Rel}(\mathcal{U}_1X;\mathcal{O}_v)$ is equivalent to a staircase module associated with $(0, \ldots, 0, 1, e-1)$. By Proposition 9.3.2

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathcal{U}_1X;\mathcal{O}_v)}(t) = \frac{(1-q_v^{-e+1}t)^2}{(1-t)(1-q_v^{d-e}t)(1-q_v^{-e+2}t)}.$$

For $r > 1$, we have U_rX is an admissible matrix of linear forms. By Proposition 9.1.3

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathcal{U}_rX;\mathcal{O}_v)}(t) = \frac{1-q_v^{-e}t}{(1-t)(1-q_v^{d-e}t)}.$$

□

For $n = \min\{d, e\}$, Lemma 9.3.1 shows that there are $(n + 1)$ equivalence classes of relation modules associated with single matrices in $M_{d\times e}(K)$. By Corollary 4.4.3, relation modules of each equivalence class of single matrices have the same ask zeta functions for almost all places $v \in \mathcal{V}_K$. However, Theorem B gives us only two formulas for ask zeta functions of these modules. Thus, relation modules of single matrices, that are not equivalent, may have the same ask zeta functions.

Chapter 10

Ask zeta functions of relation modules of matrix pencils

In this chapter, we write all the possible formulas for ask zeta function of relation modules associated with matrix pencils for almost all places. We also prove Theorem F which is analogous to [50, Eq 5.26 & 5.28] for relation modules associated with matrix pencils. Here, we use Notation 4.2.1.

Theorem F. *Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a matrix pencil such that*

$$A(X_1, X_2) = \mathrm{diag}(0_{(d-m)\times(e-n)}, B(X_1, X_2)),$$

where $B(X_1, X_2) \in \mathrm{M}_{m\times n}(K[X_1, X_2])$ is also a matrix pencil. Then for almost all places $v \in \mathcal{V}_K$

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{n-e}t)(1-q_v^{m-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(B(X_1,X_2);\mathcal{O}_v)}(q_v^{n-e}t).$$

Theorem F allows us to reduce a matrix pencil $A(X_1, X_2)$ to a matrix pencil $B(X_1, X_2)$ which does not contain zero rows or columns. Then we use Theorem F to obtain the ask zeta function of the relation module associated with $A(X_1, X_2)$ from the ask zeta of the relation module associated with $B(X_1, X_2)$.

Theorem G. *Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a matrix pencil. Let $B(X_1, X_2)$ be the largest submatrix pencil of $A(X_1, X_2)$ which does not contain zero rows or columns.*

(i) If $B(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil, then for almost all places $v \in \mathcal{V}_K$:

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1-q_v^{-e}t}{(1-t)(1-q_v^{d-e}t)}.$$

(ii) Suppose that $B(X_1, X_2)$ is not weakly equivalent to an admissible matrix pencil. Further suppose that $B(X_1, X_2)$ has one of the following rational forms:

- $\mathrm{R}_B(X_1, X_2) = \mathrm{diag}(L_1, I_{n-1}aX_1 + I_{n-1}X_2)$,

- $\mathrm{R}_B(X_1, X_2) = \mathrm{diag}(L_1^T, I_{n-1}aX_1 + I_{n-1}X_2)$, *or*
- $\mathrm{R}_B(X_1, X_2) = \mathrm{diag}(I_{n-2}a, C)X_1 + I_nX_2$

where C is a companion matrix of size 2, $n \leq d, e$, and $a \in K$. Then for almost all places $v \in \mathcal{V}_K$:

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)^2}{(1-t)(1-q_v^{d-e}t)(1-q_v^{2-e}t)}.$$

(iii) If $B(X_1, X_2)$ has a rational form $\mathrm{R}_B(X_1, X_2) = L_1$ or $\mathrm{R}_B(X_1, X_2) = L_1^T$, then for almost all places $v \in \mathcal{V}_K$:

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)(1-q_v^{2-e}t)}{(1-t)(1-q_v^{d-e}t)(1-q_v^{3-e}t)}.$$

(iv) If $B(X_1, X_2)$ has a rational form $\mathrm{R}_B(X_1, X_2) = CX_1 + I_2X_2$ where C is a companion matrix of size 2, then for almost all places $v \in \mathcal{V}_K$:

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{-e}t)}{(1-t)(1-q_v^{d-e}t)} + \sum_{i=1}^{h} N_i(\mathfrak{K}_v)W_i(q_v, q_v^{2-e}t),$$

where $N_i(\mathfrak{K}_v)$'s and $W_i(q_v, t)$'s are defined as in Theorem D.

In the first two sections, we discuss all the cases of relation modules of non-singular matrix pencils. We start with two cases, in the first section, which are not (always) weakly equivalent to admissible matrix pencils. Then we write the cases which are always weakly equivalent to admissible matrix pencils in Section 10.2. In Section 10.3 and Section 10.4, we discuss all the cases of relation modules of singular matrix pencils that do not contain zero rows or columns. Similarly, we start with four cases, in Section 10.3, which are not (always) weakly equivalent to admissible matrix pencils. In Section 10.4, we write the cases which are always weakly equivalent to admissible matrix pencils. Finally, we prove Theorem F and Theorem G in Section 10.5.

10.1 Relation modules of non-admissible non-singular matrix pencils

In this section, we will show two cases where non-singular matrix pencils are not (always) weakly equivalent to admissible matrix pencils. We will also write formulas for the zeta functions of these two cases.

Recall from Definition 2.2.3 that a non-singular matrix pencil $A(X_1, X_2) \in \mathrm{M}_n(K[X_1, X_2])$ has a rational form $\mathrm{R}_A(X_1, X_2) = \mathrm{R}X_1 + I_nX_2$ where R is a matrix in a rational canonical form. We write $\mathrm{R} = \mathrm{diag}(C_1, \ldots, C_w)$ where C_i is a companion matrix of a monic polynomial $F_i(X)$ for $1 \leq i \leq w$.

Lemma 10.1.1. *Let $C \in \mathrm{M}_2(K)$ be a companion matrix of a monic polynomial $f(X)$ of degree 2. Let $A(X_1, X_2)$ be a non-singular matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = CX_1 + I_2X_2$. Then*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t)$$

for almost all places $v \in \mathcal{V}_K$.

Proof. Let $f(X) = a_0 + a_1X + X^2$ where $a_0, a_1 \in K$. We have

$$\mathrm{R}_A(X_1, X_2) = \begin{bmatrix} X_2 & X_1 \\ -a_0X_1 & X_2 - a_1X_1 \end{bmatrix}.$$

Let $g = \begin{bmatrix} -1 & 0 \\ a_1 & 1 \end{bmatrix}$. Then

$$g\mathrm{R}^T_A(-X_2, X_1) = \begin{bmatrix} -X_1 & a_0X_2 + a_1X_1 \\ X_2 & X_1 \end{bmatrix}.$$

where T is the transpose. By Corollary 1.2.3(ii) and Corollary 1.1.4

$$gM(\mathrm{R}^T_A(X_1, X_2); K) = \left\{ \begin{bmatrix} -x_1 & a_0x_2 + a_1x_1 \\ x_2 & x_1 \end{bmatrix} : x_1, x_2 \in K \right\}.$$

We also have

$$\mathrm{Rel}(\mathrm{R}_A(X_1, X_2); \mathcal{O}_v) = \left\{ \begin{bmatrix} -x_1 & a_0x_2 + a_1x_1 \\ x_2 & x_1 \end{bmatrix} : x_1, x_2 \in \mathcal{O}_v \right\},$$

By Corollary 4.4.1 and Proposition 4.3.3, we obtain

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_A(X_1,X_2);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{gM(\mathrm{R}^T_A(X_1,X_2);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(\mathrm{R}_A(X_1,X_2);\mathcal{O}_v)}(t)$$

for almost all place $v \in \mathcal{V}_K$. □

Lemma 10.1.1 gives us a case that can be computed using Theorem D.

Corollary 10.1.2. *Let $C \in \mathrm{M}_2(K)$ be a companion matrix of a monic polynomial $F(X)$ of degree 2. Write $F(X) = \prod_{i=1}^h g_i^{e_i}$ where g_i's are different monic irreducible polynomials. Let $A(X_1, X_2)$ be a non-singular matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = CX_1 + I_2X_2$. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1 - q_v^{-2}t}{(1-t)^2} + \sum_{i=1}^{h} N_i(\mathfrak{K}_v)W_i(q_v, t),$$

where $W_i(q_v, t) = \frac{q_v^{-1}(1-q_v^{-1})[q_v^{e_i}t^{e_i} - 1 + (1-t)\sum_{m=0}^{e_i-1}(q_vt)^m]}{(1-t)^2(1-q_v^{e_i-1}t^{e_i})}$.

Proof. By Lemma 10.1.1, we have $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t)$ for almost all places $v \in \mathcal{V}_K$. Using Notation 8.1.2, we have $w = 1$ and $k = \ell = 0$. Since $F(X) = \prod_{i=1}^{h} g_i^{e_i}$ where g_i is an irreducible polynomial and $h, e_i \leq 2$, we have $\gamma_i = 1$ and $\pi_i = e_i$. Define $N(\mathfrak{K}_v) = \#\{\alpha \in \mathfrak{K}_v : g_i(\alpha) = 0\}$. By Remark 8.3.1, we have

$$B_y(i) = \frac{q_v^{-1}\sum_{m=0}^{e_i-1}(q_v t)^m}{1 - q_v^{e_i-1}t^{e_i}} \quad \text{and} \quad B_1(i) = \frac{q_v^{e_i-1}t^{e_i}}{(1-t)(1-q_v^{e_i-1}t^{e_i})}.$$

Thus $W_i(q_v, t) = \frac{q_v^{-1}(1-q^{-1})[q^{e_i}t^{e_i}-1+(1-t)\sum_{m=0}^{e_i-1}(q_v t)^m]}{(1-t)^2(1-q_v^{e_i-1}t^{e_i})}$. By Theorem D

$$\mathsf{Z}^{\mathsf{ask}}_{M(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1-q_v^{-2}t}{(1-t)^2} + \sum_{i=1}^{h} N_i(\mathfrak{K}_v)W_i(q_v,t).$$

□

Proposition 10.1.3. *Let $d, e, n \in \mathbb{N}$ where $3 \leq n \leq d, e$ and let $c \in \mathcal{O}_v$. Define $M := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{(n-1)n} = 0\}$, $M_1 = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = 0, x_{(n-1)n} = 0\}$, and $M_2 := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = cx_{n(n-1)}, x_{(n-1)n} = 0\}$. Then $uM = uM_1 = uM_2$ for all $u \in \mathcal{O}_v^d \setminus \{0\}$.*

Proof. Let e_{ij} be the elementary matrix with 1 in position (i,j) and zeros elsewhere. Let $X = [X_{ij}]_{\substack{1\leq i\leq d\\ 1\leq j\leq e}}$. If $X \in M$, then X can be written in terms of the linear combination

$$X = \sum_{J} X_{ij}\mathrm{e}_{ij},$$

where $J = \{(i,j) \in [d]\times[e] : (i,j) \neq (n-1,n)\}$. Thus M is generated by all e_{ij} where $1 \leq i \leq d$ and $1 \leq j \leq e$ with $(i,j) \neq (n-1,n)$. For a fix $s < n$, write $M_1 = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{\substack{k=1\\k\neq s}}^{n} x_{kk} = -x_{ss}, x_{(n-1)n} = 0\}$. If $X \in M_1$, then $X_{ss} = -\sum_{\substack{k=1\\k\neq s}}^{n} X_{kk}$ and X can be expressed as

$$X = \sum_{J_1} X_{ij}\mathrm{e}_{ij} + \sum_{\substack{k=1\\k\neq s}}^{n} X_{kk}\mathrm{e}_{kk} - \sum_{\substack{k=1\\k\neq s}}^{n} X_{kk}\mathrm{e}_{ss} = \sum_{J_1} X_{ij}\mathrm{e}_{ij} + \sum_{\substack{k=1\\k\neq s}}^{n} X_{kk}(\mathrm{e}_{kk} - \mathrm{e}_{ss}),$$

where $J_1 = \{(i,j) \in [d]\times[e] : i \neq j \text{ whenever } i,j \leq n, (i,j) \neq (n-1,n)\}$. Hence M_1 is generated by e_{ij} and $\mathrm{e}_{kk} - \mathrm{e}_{ss}$ (with $k \leq n, i \neq j$ whenever $i,j \leq n$, and $(i,j) \neq (n-1,n)$). Moreover, if $X \in M_2$, then $X_{n(n-1)} = \frac{1}{c}\sum_{k=1}^{n} X_{kk}$ and X can be written as

$$X = \sum_{J_2} X_{ij}\mathrm{e}_{ij} + \sum_{k=1}^{n} X_{kk}(\mathrm{e}_{kk} + \frac{1}{c}\mathrm{e}_{n(n-1)}),$$

where $J_2 = \{(i,j) \in [d]\times[e] : i \neq j \text{ whenever } i,j \leq n, (i,j) \neq (n-1,n) \text{ or } (n,n-1)\}$. Therefore M_2 is generated by e_{ij} and $\frac{1}{c}\mathrm{e}_{n(n-1)} + \mathrm{e}_{kk}$ (with $i \neq j$ whenever $i,j \leq n$, $k \leq n$, and $(i,j) \neq (n-1,n)$ or $(n,n-1)$). We want to show that $uM = uM_1 = uM_2$ for all $u \in \mathcal{O}_v^d \setminus \{0\}$. Since $M_1, M_2 \subset M$, it

suffices to show $uM_1, uM_2 \supset uM$. Let $(\mathrm{e}_1, \ldots, \mathrm{e}_d) \subset \mathcal{O}_v^d$ be the standard basis. Since $u\mathrm{e}_{ij} = u_i\mathrm{e}_j$, we have uM spanned by all $u_i\mathrm{e}_j$ with $(i,j) \neq (n-1,n)$. Similarly, we have uM_1 spanned by all $u_i\mathrm{e}_j$ and $u_k\mathrm{e}_k - u_s\mathrm{e}_s$ (with $i \neq j$ whenever $i,j \leq n$, $k \leq n$, and $(i,j) \neq (n-1,n)$). Moreover, we have uM_2 generated by all $u_i\mathrm{e}_j$ and $\frac{1}{c}u_n\mathrm{e}_{n-1} + u_k\mathrm{e}_k$ (with $i \neq j$ whenever $i,j \leq n$, $k \leq n$, and $(i,j) \neq (n-1,n)$ or $(n,n-1)$). Let $u_\ell = \gcd(u_1, \ldots, u_n)$. Since $n \geq 3$, we may choose $s \neq \ell$ or n. To show that $uM \subset uM_1, uM_2$, we need to show $u_k\mathrm{e}_k \in uM_1 \cap uM_2$ for any $1 \leq k \leq n$ and $u_n\mathrm{e}_{n-1} \in uM_2$. Since $|u_s| \leq |u_\ell|$, we have $u_\ell\mathrm{e}_\ell = (u_\ell\mathrm{e}_\ell - u_s\mathrm{e}_s) + \frac{u_s}{u_\ell}u_\ell\mathrm{e}_s \in uM_1$. Hence $u_k\mathrm{e}_k = (u_k\mathrm{e}_k - u_\ell\mathrm{e}_\ell) + u_\ell\mathrm{e}_\ell \in uM_1$. Therefore, we have $uM \subset uM_1$. Moreover, we have $u_n\mathrm{e}_{n-1} = (u_n\mathrm{e}_{n-1} + cu_s\mathrm{e}_s) - c\frac{u_s}{u_\ell}u_\ell\mathrm{e}_s \in uM_2$. Thus $u_k\mathrm{e}_k = (\frac{1}{c}u_n\mathrm{e}_{n-1} + u_k\mathrm{e}_k) - \frac{1}{c}u_n\mathrm{e}_{n-1} \in uM_2$. □

We will use Proposition 10.1.3 to prove a formula in the following lemma.

Lemma 10.1.4. *For $n \geq 3$, let $\mathrm{R} \in \mathrm{M}_n(K)$ be a rational canonical form such that $\mathrm{R} = \mathrm{diag}(C_1, \ldots, C_1, C_2)$ where C_1 is a companion matrix of a monic polynomial $f_1(X)$ of degree 1, and C_2 is a companion matrix for a monic polynomial $f_2(X)$ of degree 2. Let $A(X_1, X_2)$ be a non-singular matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = \mathrm{R}X_1 + I_nX_2$. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \begin{cases} \frac{(1-q_v^{1-n}t)^2}{(1-t)^2(1-q_v^{2-n}t)} & \text{if } a_0 = -a^2, \\ \frac{1-q_v^{-n}t}{(1-t)^2} & \text{otherwise.} \end{cases}$$

Proof. Let $f_1(X) = a + X$ and $f_2(X) = a_0 + a_1X + X^2$ where $a, a_0, a_1 \in K$ and $f_1(X) | f_2(X)$. If $a_0 \neq -a^2$, then the constant $a \neq 0$ in $f_1(X)$ since $f_1(X) | f_2(X)$. Hence

$$\mathrm{R}(X_1, X_2) = \begin{bmatrix} X_2 + aX_1 & & & & \\ & \ddots & & & \\ & & X_2 + aX_1 & & \\ & & & X_2 & X_1 \\ & & & -a_0X_1 & X_2 - a_1X_1 \end{bmatrix}.$$

Put $X_2' = X_2 + aX_1$. Then we map $\mathrm{R}X_1 + I_nX_2 \longrightarrow \mathrm{R}'X_1 + I_nX_2'$ where $\mathrm{R}' = \mathrm{R} - I_na$. Then

$$\mathrm{R}_A(X_1, X_2') = \begin{bmatrix} X_2' & & & & \\ & \ddots & & & \\ & & X_2' & & \\ & & & X_2' - aX_1 & X_1 \\ & & & -a_0X_1 & X_2' - a_1'X_1 \end{bmatrix}, \text{ where } a_1' = a_1 + a.$$

Let $g_1 = \operatorname{diag}\left(I_{n-3}, \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$ and $h_1 = \operatorname{diag}\left(I_{n-3}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$. Then

$$g_1 \mathrm{R}_A(X_1, X_2')h_1 = \begin{bmatrix} X_2' & & & & \\ & \ddots & & & \\ & & X_2' & X_2' & \\ & & -X_2' & -aX_1 & X_1 \\ & & & -a_0X_1 & X_2' - a_1'X_1 \end{bmatrix}.$$

If $a_1' = 0$, then $g_1\mathrm{R}_A(X_1, X_2')h_1$ is an admissible matrix pencil. However, if $a_1' \neq 0$, let $g_2 = \operatorname{diag}\left(I_{n-2}, \begin{bmatrix} 1 & 0 \\ a_1' & 1 \end{bmatrix}\right)$. Then

$$g_2 g_1 \mathrm{R}_A\left(X_1, X_2'\right) h_1 = \begin{bmatrix} X_2' & & & & \\ & \ddots & & & \\ & & X_2' & X_2' & \\ & & -X_2' & -aX_1 & X_1 \\ & & -a_1'X_2' & -(a_0 + aa_1')X_1 & X_2' \end{bmatrix}. \tag{10.1}$$

Suppose that $a_0 = -aa_1' = -(a^2 + aa_1)$ and write $f_2(X) = (X+a)(X+b)$ where $b \in K$. Since $a_0 = ab$ and $a_1 = a + b$, then $ab = -(a^2 + a(a+b)) = -(2a^2 + ab)$. Thus $a_0 = -a^2$. However, we have $a_0 \neq -a^2$ and $a_0 \neq -(a^2 + aa_1) = -aa_1'$. Then $g_2\left(g_1\mathrm{R}_A\left(X_1, X_2'\right)h_1\right)$ is an admissible matrix pencil. By Corollary 9.1.4 and Corollary 4.4.1

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1 - q_v^{-n}t}{(1-t)^2},$$

for almost all places $v \in \mathcal{V}_K$. For the second part, suppose that $a_0 = -a^2$. Then $a_0 = -aa_1'$ and (10.1) becomes

$$g_2 g_1 \mathrm{R}_A\left(x_1, x_2'\right) h_1 = \begin{bmatrix} X_2' & & & & \\ & \ddots & & & \\ & & X_2' & X_2' & \\ & & -X_2' & -aX_1 & X_1 \\ & & -a_1'X_2' & 0 & X_2' \end{bmatrix}.$$

Let $g_3 = \operatorname{diag}\left(I_{n-3}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_1' & 0 & 1 \end{bmatrix}\right)$ and $h_2 = \operatorname{diag}\left(I_{n-2}, \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\right)$. Then

$$g_3 g_1^{-1} g_2 g_1 \mathrm{R}_A(X_1, X_2') h_1 h_1^{-1} h_2 = \begin{bmatrix} X_2' & & & & \\ & \ddots & & & \\ & & X_2' & & \\ & & & X_2' & X_1 \\ & & & cX_2' & X_2' \end{bmatrix}, \tag{10.2}$$

where $c = a_1' + a$. Put $g = g_3 g_1^{-1} g_2 g_1$ and $h = h_2$. If $c = 0$, define $M_1 := \mathrm{Rel}(g\mathrm{R}_A(X_1, X_2')h; \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_n(\mathcal{O}_v) : \sum_{i=1}^n x_{ii} = 0, x_{(n-1)n} = 0\}$. If $c \neq 0$, define $M_2 := \mathrm{Rel}(g\mathrm{R}_A(X_1, X_2')h; \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_n(\mathcal{O}_v) : \sum_{i=1}^n x_{ii} = cx_{n(n-1)}, x_{(n-1)n} = 0\}$. Moreover, define $M := \{[x_{ij}] \in \mathrm{M}_n(\mathcal{O}_v) : x_{(n-1)n} = 0\}$. By Proposition 10.1.3 (put $d = e = n$), we have $uM = uM_1 = uM_2$ for all $u \in \mathcal{O}_v^d \setminus \{0\}$. Using Corollary 4.1.5, we have $\mathsf{Z}_M^{\mathsf{ask}}(t) = \mathsf{Z}_{M_1}^{\mathsf{ask}}(t) = \mathsf{Z}_{M_2}^{\mathsf{ask}}(t)$. Moreover, we have M is equivalent to a staircase module of $m = (0, \ldots, 0, 1, n-1)$. By Proposition 9.3.2

$$\mathsf{Z}_M^{\mathsf{ask}}(t) = \frac{(1 - q_v^{1-n}t)^2}{(1-t)^2(1-q_v^{2-n}t)}.$$

By Corollary 4.4.1

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1 - q_v^{1-n}t)^2}{(1-t)^2(1-q_v^{2-n}t)}$$

for almost all places $v \in \mathcal{V}_K$. □

10.2 Relation modules associated with admissible non-singular matrix pencils

In this section, we write all cases of non-singular matrix pencils which are always weakly equivalent to admissible matrix pencils. This allows us to use Proposition 9.1.4 to obtain ask zeta functions of modules associated with these matrix pencils for almost all places $v \in \mathcal{V}_K$.

Lemma 10.2.1. *For $\ell \geq 2$, let $\mathrm{R}_{2\ell} = \operatorname{diag}(C, \ldots, C) \in \mathrm{M}_{2\ell}(K)$ where C is a companion matrix of a monic polynomial $f(X)$ of degree 2. Let $A(X_1, X_2)$ be a non-singular matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = \mathrm{R}_{2\ell}X_1 + I_{2\ell}X_2$. Then $A(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil.*

Proof. Let $f(X) = a_0 + a_1X + X^2$ where $a_0, a_1 \in K$. If $\ell = 2$, then

$$\mathrm{R}_A(X_1, X_2) = \begin{bmatrix} X_2 & X_1 & 0 & 0 \\ -a_0X_1 & X_2 - a_1X_1 & 0 & 0 \\ 0 & 0 & X_2 & X_1 \\ 0 & 0 & -a_0X_1 & X_2 - a_1X_1 \end{bmatrix}.$$

Let $g_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and $h_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then

$$g_1 \mathrm{R}_A(X_1, X_2) h_1 = \begin{bmatrix} X_2 & X_1 & 0 & -X_2 \\ -a_0 X_1 & -a_1 X_1 & X_2 & (a_0+1)X_1 \\ 0 & -X_2 & X_2 & X_1 \\ X_2 & (a_0+1)X_1 & -a_0 X_1 & -a_1 X_1 \end{bmatrix}. \tag{10.3}$$

Then $g_1 \mathrm{R}_A(X_1, X_2) h_1 = g_1(\mathrm{R}_4 X_1 + I_4 X_2) h_1$ is an admissible matrix pencil. If $\ell = 3$, then $\mathrm{R}_A(X_1, X_2)$ is of form:

$$\begin{bmatrix} X_2 & X_1 & 0 & 0 & 0 & 0 \\ -a_0 X_1 & X_2 - a_1 X_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_2 & X_1 & 0 & 0 \\ 0 & 0 & -a_0 X_1 & X_2 - a_1 X_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_2 & X_1 \\ 0 & 0 & 0 & 0 & -a_0 X_1 & X_2 - a_1 X_1 \end{bmatrix}.$$

Let $g_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ and $h_2 = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Then $g_2 \mathrm{R}_A(X_1, X_2) h_2$ is of the form:

$$\begin{bmatrix} X_2 & X_1 & 0 & -X_2 & 0 & 0 \\ -a_0 X_1 & -a_1 X_1 & X_2 & (a_0+1)X_1 & 0 & -X_2 \\ 0 & -X_2 & X_2 & X_1 & 0 & -X_2 \\ X_2 & (a_0+1)X_1 & -a_0 X_1 & -a_1 X_1 & 0 & a_0 X_1 \\ 0 & 0 & 0 & 0 & X_2 & X_1 \\ 0 & -X_2 & X_2 & X_1 & -a_0 X_1 & -a_1 X_1 \end{bmatrix}.$$

Then $g_2 \mathrm{R}_A(X_1, X_2) h_2 = \mathrm{R}_6 X_1 + I_6 X_2$ is an admissible matrix pencil. For $\ell > 3$, we have $\mathrm{R}_A(X_1, X_2) = \mathrm{R}_{2\ell} X_1 + I_{2\ell} X_2$. If $2|\ell$, write $g = \mathrm{diag}(g_1, \ldots, g_1)$ and $h = \mathrm{diag}(h_1, \ldots, h_1)$ such that

$$g \mathrm{R}_A(X_1, X_2) h = \mathrm{diag}(g_1(\mathrm{R}_4 X_1 + I_4 X_2) h_1, \ldots, g_1(\mathrm{R}_4 X_1 + I_4 X_2) h_1).$$

If $2 \nmid \ell$, let $g = \mathrm{diag}(g_1, \ldots, g_1, g_2)$ and $h = \mathrm{diag}(h_1, \ldots, h_1, h_2)$ such that $g\mathrm{R}_A(X_1, X_2)h$ is of the form

$$\mathrm{diag}(g_1(\mathrm{R}_4X_1 + I_4X_2)h_1, \ldots, g_1(\mathrm{R}_4X_1 + I_4X_2)h_1, g_2(\mathrm{R}_6X_1 + I_6X_2)h_2).$$

In both cases, we have $g\mathrm{R}_A(X_1, X_2)h$ is an admissible matrix pencil. □

Lemma 10.2.2. *Let $C \in \mathrm{M}_n(K)$ be a companion matrix of a monic polynomial $F(X)$ of degree $n \geq 3$. Let $A(X_1, X_2)$ be a non-singular matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = CX_1 + I_nX_2$. Then $A(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil.*

Proof. Let $F(X) = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + X^n$ where $a_0, \ldots, a_{n-1} \in K$. If $a_{n-1} = 0$, then

$$\mathrm{R}_A(X_1, X_2) = \begin{bmatrix} X_2 & X_1 & & \\ & \ddots & \ddots & \\ & & X_2 & X_1 \\ -a_0X_1 & -a_1X_1 & \cdots & X_2 \end{bmatrix}.$$

Then $\mathrm{R}_A(X_1, X_2)$ is an admissible matrix pencil. If $a_{n-1} \neq 0$ and $a_{n-3} = 0$, let $g = \mathrm{diag}(I_{n-3}, g_1)$ and $h = \mathrm{diag}(I_{n-3}, h_1)$ where $g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $h_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$g\mathrm{R}_A(X_1, X_2)h = \begin{bmatrix} X_2 & X_1 & & & & \\ & \ddots & \ddots & & & \\ & & X_2 & X_1 & 0 & X_1 \\ & & & X_2 & X_1 & X_2 \\ & & & & X_2 & X_1 \\ -a_0X_1 & \ldots & -a_{n-4}X_1 & -X_2 & -(a_{n-2}+1)x_1 & -a_{n-1}x_1 \end{bmatrix}.$$

Then $g\mathrm{R}_A(X_1, X_2)h$ is an admissible matrix pencil. However, if $a_{n-1} \neq 0$ and

$a_{n-3} \neq 0$, let $\bar{h} = \mathrm{diag}(I_{n-3}, h_2)$ where $h_2 = \begin{bmatrix} 1 & 0 & -\frac{a_{n-1}}{a_{n-3}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$\mathrm{R}_A(X_1, X_2)\bar{h} = \begin{bmatrix} X_2 & X_1 & & & & \\ & \ddots & \ddots & & & \\ & & X_2 & X_1 & 0 & -\frac{a_{n-1}}{a_{n-3}}X_1 \\ & & & X_2 & X_1 & -\frac{a_{n-1}}{a_{n-3}}X_2 \\ & & & & X_2 & X_1 \\ -a_0X_1 & -a_1X_1 & \cdots & -a_{n-3}X_1 & -a_{n-2}X_1 & X_2 \end{bmatrix}.$$

Then $\mathrm{R}_A(X_1, X_2)\bar{h}$ is an admissible matrix pencil. □

Corollary 10.2.3. *Let* $\mathrm{R} \in \mathrm{M}_n(K)$ *be a matrix in a rational canonical form such that* $\mathrm{R} = \mathrm{diag}(C_1, \ldots, C_w)$ *where* C_i *is a companion matrix of a monic polynomial* $f_i(X)$ *for* $1 \leq i \leq w$. *Let* $A(X_1, X_2)$ *be a non-singular matrix pencil with a rational form* $\mathrm{R}_A(X_1, X_2) = \mathrm{R}X_1 + I_nX_2$. *If there exist at least two companion matrices of polynomials of degree* 2 *among* $C_1, \ldots, C_w$, *then* $A(X_1, X_2)$ *is weakly equivalent to an admissible matrix pencil.*

Proof. Let $s \leq n$ and $S \in \mathrm{M}_{2s}(K)$ be the square submatrix of R which only contains the companion matrices of degree 2. We have S is not an empty matrix since R contains at least two companion matrices of degree 2. Write $S = \mathrm{diag}(\bar{C}_1, \ldots, \bar{C}_s)$ where $\bar{C}_i$'s are the companion matrices of degree 2 for $1 \leq i \leq s$. By Lemma 10.2.1, there exist $g_s, h_s \in \mathrm{GL}_{2s}(K)$ such that $g_s(SX_1 + I_{2s}X_2)h_s$ is an admissible matrix pencil. Let $d < n$ and $D \in \mathrm{M}_d(K)$ be the square submatrix of R which only contains the companion matrices of degree ≥ 3. If D is not an empty matrix, write $D = (\bar{C}_{s+1}, \ldots, \bar{C}_\ell)$ where $\bar{C}_j$ is a companion matrix of degree $d_j \geq 3$ for $s+1 \leq j \leq \ell$. By Lemma 10.2.2, there exist $g_j, h_j \in \mathrm{GL}_{d_j}(K)$ such that $g_j(\bar{C}_jX_1 + I_{d_j}X_2)h_j$ is an admissible matrix pencil. Let $\bar{g}_d, \bar{h}_d \in \mathrm{GL}_d(K)$ such that $\bar{g}_d = \mathrm{diag}(g_{s+1}, \ldots, g_\ell)$ and $\bar{h}_d = \mathrm{diag}(h_{s+1}, \ldots, h_\ell)$. Then

$$\bar{g}_d(DX_1 + I_dX_2)\bar{h}_d = \mathrm{diag}(g_{s+1}(\bar{C}_{s+1}X_1 + I_{d_{s+1}}X_2)h_{s+1}, \ldots, g_\ell(\bar{C}_\ell X_1 + I_{d_\ell}X_2)h_\ell)$$

is an admissible matrix pencil. Let $\bar{\mathrm{R}} = \mathrm{diag}(S, D)$ and let $\bar{g}, \bar{h} \in \mathrm{GL}_{2s+d}(K)$ such that $\bar{g} = \mathrm{diag}(g_s, g_{s+1}, \ldots, g_\ell)$ and $\bar{h} = \mathrm{diag}(h_s, h_{s+1}, \ldots, h_\ell)$. Then

$$\bar{g}(\bar{\mathrm{R}}X_1 + I_{2s+d}X_2)\bar{h} = \mathrm{diag}(g_s(SX_1 + I_{2s}X_2)h_s, \bar{g}_d(DX_1 + I_dX_2)\bar{h}_d)$$

is an admissible matrix pencil. For $e < n$, let e be the size of the squire submatrix of R which only contains the companion matrices of degree 1. If $e > 0$, write $E = \mathrm{diag}(I_ea, C_2, C_2)$ where $a \in K^\times$. By Lemma 10.2.1, there exist $g_1, h_1 \in \mathrm{GL}_4(K)$ such that $g_1\,\mathrm{diag}(C_2, C_2)h_1$ as in (10.3). Let $\mathsf{g}_e \in \mathrm{GL}_{e+4}(K)$

such that $\mathsf{g}_e = \left[\begin{array}{c|cccc} & 1 & 0 & 0 & 0 \\ I_e & \vdots & \vdots & \vdots & \vdots \\ & 1 & 0 & 0 & 0 \\ \hline 0 & \multicolumn{4}{c}{I_4} \end{array}\right]$. Then $\mathsf{g}_e g_1(EX_1 + I_{e+4}X_2)h_1\mathsf{g}_e^{-T}$ is of the form:

$$\left[\begin{array}{cccc|cccc} aX_1 & -X_2 & \dots & -X_2 & X_2 & X_1 & 0 & -X_2 \\ -X_2 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & aX_1 & -X_2 & \vdots & \vdots & \vdots & \vdots \\ -X_2 & \cdots & -X_2 & aX_1 & X_2 & X_1 & 0 & -X_2 \\ \hline -X_2 & \cdots & \cdots & -X_2 & X_2 & X_1 & 0 & -X_2 \\ a_0X_1 & \cdots & \cdots & aX_1 & -a_0X_1 & -a_1X_1 & X_1 & (a_0+1)X_1 \\ 0 & \cdots & \cdots & 0 & 0 & -X_2 & X_2 & X_1 \\ -X_2 & \cdots & \cdots & -X_2 & X_2 & (a_0+1)X_1 & -a_0X_1 & -a_1X_1 \end{array}\right].$$

Thus $\mathsf{g}_e g_1(EX_1 + I_{e+4}X_2)h_1\mathsf{g}_e^{-T}$ is an admissible matrix pencil. Let $\mathsf{g}, \mathsf{h} \in \mathrm{GL}_e(K)$ such that $\mathsf{g} = \mathrm{diag}(\mathsf{g}_e g_1, I_{n-(e+4)})$ and $\mathsf{h} = \mathrm{diag}(\mathsf{g}_e^{-T}h_1, I_{n-(e+4)})$. Let $g, h \in \mathrm{GL}_n(K)$ such that $g = \mathrm{diag}(I_e, \bar{g})$ and $h = (I_e, \bar{h})$. Then

$$\mathsf{g}(g(\mathrm{R}X_1 + I_nX_2)h)\mathsf{h} = \mathsf{g}\,\mathrm{diag}(I_eaX_1 + I_eX_2, \bar{g}(\bar{\mathrm{R}}X_1 + I_{2s+d}X_2)\bar{h})\mathsf{h}$$

is an admissible matrix pencil. □

Corollary 10.2.4. *Let* $\mathrm{R} \in \mathrm{M}_n(K)$ *be a matrix in a rational canonical form such that* $\mathrm{R} = \mathrm{diag}(C_1, \dots, C_w)$ *where* C_i *is a companion matrix of a monic polynomial* $f_i(X)$ *for* $1 \leq i \leq w$. *Let* $A(X_1, X_2)$ *be a matrix pencil with a rational form* $\mathrm{R}_A(X_1, X_2) = \mathrm{R}X_1 + I_nX_2$. *If there exists a companion matrix* C_i *of a monic polynomial* $f_i(X)$ *of degree* ≥ 3, *then* $A(X_1, X_2)$ *is weakly equivalent to an admissible matrix pencil.*

Proof. Let $d \leq n$ and $D \in \mathrm{M}_d(K)$ be the square submatrix of R which only contains the companion matrices of degree ≥ 3. Then D is not an empty matrix since R contains at least one companion matrix of degree ≥ 3. Write $D = \mathrm{diag}(\bar{C}_1, \dots, \bar{C}_\ell)$ where $\bar{C}_j$ is a companion matrix of degree $d_j \geq 3$ for $1 \leq j \leq \ell$. By Lemma 10.2.2, there exist $g_j, h_j \in \mathrm{GL}_{d_j}(K)$ such that $g_j(\bar{C}_jX_1 + I_{d_j}X_2)h_j$ is an admissible matrix pencil. Write $D_j(X_1, X_2) = g_j(\bar{C}_jX_1 + I_{d_j}X_2)h_j$ and let $g_d, h_d \in \mathrm{GL}_d(K)$ such that $g_d = \mathrm{diag}(g_1, \dots, g_\ell)$ and $h_d = \mathrm{diag}(h_1, \dots, h_\ell)$. Then

$$g_d(DX_1 + I_dX_2)h_d = \mathrm{diag}(D_1(X_1, X_2), \dots, D_\ell(X_1, X_2))$$

is an admissible matrix pencil. If R contains more than one companion matrix of degree 2, then Corollary 10.2.3 shows that $A(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil. Hence, suppose that R contains only one companion matrix of degree 2, say $\bar{C}$. Let $S = \mathrm{diag}(\bar{C}, \bar{C}_1)$ where $\bar{C}$ is a

companion matrix of a monic polynomial of the form $f(X) = a_0 + a_1X + X^2$. Let

$$\bar{\mathsf{g}}_1 = \operatorname{diag}\left(\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}, I_{d_1-1}\right) \text{ and } \bar{\mathsf{h}}_1 = \operatorname{diag}\left(\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}, I_{d_1-1}\right).$$

Let $r_1(X_1, X_2)$ and $c_1(X_1, X_2)$ be the first row and column of $D_1(X_1, X_2)$, respectively. Then

$$\mathsf{g}_1(SX_1 + I_{d_1+2}X_2)\mathsf{h}_1 = \begin{bmatrix} X_2 & X_1 & 0 \\ -a_0X_1 & -a_1X_1 & r_1(X_1,X_2) \\ 0 & -c_1(X_1,X_2) & D_1(X_1,X_2)\end{bmatrix},$$

where $\mathsf{g}_1 = \bar{\mathsf{g}}_1 \operatorname{diag}(I_2, g_1)$ and $\mathsf{h}_1 = \operatorname{diag}(I_2, h_1)\bar{\mathsf{h}}_1$. Then $\mathsf{g}_1(SX_1 + I_{d_1+2}X_2)\mathsf{h}_1$ is an admissible matrix pencil. Let $\bar{\mathrm{R}} = \operatorname{diag}(C_2, D)$ and let

$$\bar{\mathsf{g}} = \operatorname{diag}\left(\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}, I_{d-1}\right) \text{ and } \bar{\mathsf{h}} = \operatorname{diag}\left(\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}, I_{d-1}\right).$$

Write $\mathsf{g} = \bar{\mathsf{g}} \operatorname{diag}(I_2, g_d)$ and $\mathsf{h} = \operatorname{diag}(I_2, h_d)\bar{\mathsf{h}}$. Then

$$\mathsf{g}(\bar{\mathrm{R}}X_1 + I_{d+2}X_2)\mathsf{h} = \operatorname{diag}(\mathsf{g}_1(SX_1 + I_{d_1+2}X_2)\mathsf{h}_1, D_2(X_1,X_2), \ldots, D_\ell(X_1,X_2))$$

is an admissible matrix pencil. Let e be the size of the squire submatrix of R which only contains the companion matrices of degree 1. If $e > 0$, then $\mathrm{R} = \operatorname{diag}(I_e a, \bar{\mathrm{R}})$ where $a \in K^\times$. Let $g_0 \in \mathrm{GL}_n(K)$ where

$$g_0 = \left[\begin{array}{c|ccc} & 1 & 0 & \ldots\ 0 \\ I_e & \vdots & \vdots & \quad\ \vdots \\ & 1 & 0 & \ldots\ 0 \\ \hline 0 & \multicolumn{3}{c}{I_{n-e}}\end{array}\right].$$

Then $\bar{g}(\mathrm{R}X_1 + I_nX_2)\bar{h}$ is of the form:

$$\left[\begin{array}{cccc|ccccc} aX_1 & -X_2 & \ldots & -X_2 & X_2 & X_1 & 0 & \cdots & 0 \\ -X_2 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & aX_1 & -X_2 & \vdots & \vdots & \vdots & & \vdots \\ -X_2 & \cdots & -X_2 & aX_1 & X_2 & X_1 & 0 & \cdots & 0 \\ \hline -X_2 & \cdots & \cdots & -X_2 & \multicolumn{5}{c}{} \\ a_0X_1 & \cdots & \cdots & a_0X_1 & \multicolumn{5}{c}{} \\ 0 & \cdots & \cdots & 0 & \multicolumn{5}{c}{\mathsf{g}(\bar{\mathrm{R}}X_1 + I_{d+2}X_2)\mathsf{h}} \\ \vdots & & & \vdots & \multicolumn{5}{c}{} \\ 0 & \cdots & \cdots & 0 & \multicolumn{5}{c}{}\end{array}\right],$$

where $\bar{g} = g_0 \operatorname{diag}(I_e, \mathsf{g})$ and $\bar{h} = \operatorname{diag}(I_e, \mathsf{h}) g_0^{-T}$. Hence $\bar{g}(\mathrm{R}X_1 + I_n X_2)\bar{h}$ is an admissible matrix pencil. If R doesn't contain a companion matrix of order 2, then $\mathrm{R} = \operatorname{diag}(I_e a, D)$. Let $g = g_0 \operatorname{diag}(I_e, g_d)$ and $h = \operatorname{diag}(I_e, h_d) g_0^{-T}$. Then $g(\mathrm{R}X_1 + I_n X_2)h$ is of the form:

$$\left[\begin{array}{cccc|ccc} aX_2 & -X_1 & \dots & -X_1 & r_1(X_1,X_2) & 0 & \cdots \\ -X_1 & \ddots & \ddots & \vdots & \vdots & \vdots & \\ \vdots & \ddots & aX_2 & -X_1 & \vdots & \vdots & \\ -X_1 & \cdots & -X_1 & aX_2 & r_1(X_1,X_2) & 0 & \cdots \\ \hline -c_1(X_1,X_2) & \cdots & \cdots & -c_1(X_1,X_2) & D_1(X_1,X_2) & & \\ 0 & \cdots & \cdots & 0 & & \ddots & \\ \vdots & & & \vdots & & & D_\ell(X_1,X_2) \end{array}\right]$$

Therefore, we have $g(\mathrm{R}X_1 + I_n X_2)h$ is an admissible matrix pencil. □

In this section, we covered all cases of non-singular matrix pencils $A(X_1, X_2)$ with a rational form $\mathrm{R}X_1 + I_n X_2$ except the two cases where $\mathrm{R} = C$ or $\operatorname{diag}(I_{n-2}a, C)$ where C is a companion matrix of a monic polynomial of degree 2 and $a \in K$. Those two cases were dealt with in Section 10.1. Note that the case where $\mathrm{R} = I_n a$ is not a matrix pencil since we can use a change of variables ($X_2' = aX_1 + X_2$) to map the matrix of linear form $A(X_1, X_2)$ into $A(X_2')$ via $I_n a X_1 + I_n X_2 \to I_n X_2'$.

10.3 Relation modules of non-admissible singular matrix pencils

In this section, we show four cases where singular matrix pencils, which do not contain zero rows or columns, are not (always) weakly equivalent to admissible matrix pencils. We also write ask zeta functions of relation modules associated with these cases for almost all places $v \in \mathcal{V}_K$.

Lemma 10.3.1. *Let $A(X_1, X_2) \in \mathrm{M}_{2\times 1}(K[X_1, X_2])$ be a matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = L_1$. Then for almost all places $v \in \mathcal{V}_K$, $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1}{1-q_v^2 t}$.*

Proof. By Definition 2.4.2, we have $L_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. Thus $\mathrm{Rel}(\mathrm{R}_A(X_1, X_2); \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{2\times 1}(K) : x_{11} = 0, x_{21} = 0\} = 0_{2\times 1}$. Then Example 4.1.4 completes the proof. □

The following corollary is obtained by the transpose of Lemma 10.3.1.

Corollary 10.3.2. *Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = L_1^T$. Then for almost all places $v \in \mathcal{V}_K$, $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{1}{1-q_v t}$.*

Proof. This follows immediately from Lemma 10.3.1 and Proposition 4.3.3. □

Proposition 10.3.3. *Let* $2 \leq n \leq d, e$. *Define* $M := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{(n+1)n} = 0\}$ *and* $M_1 = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = 0, x_{(n+1)n} = 0\}$. *Then* $uM = uM_1$ *for all* $u \in \mathcal{O}_v^d \setminus \{0\}$.

Proof. Let e_{ij} be the elementary matrix with 1 in position (i, j) and zeros elsewhere. Let $X = [X_{ij}]_{\substack{1\leq i\leq d\\ 1\leq j\leq e}}$. If $X \in M$, then X can be written in terms of the linear combination

$$X = \sum_{J} X_{ij}\mathrm{e}_{ij},$$

where $J = \{(i, j) \in [d] \times [e] : (i, j) \neq (n + 1, n)\}$. Thus M is generated by all e_{ij} where $1 \leq i \leq d$ and $1 \leq j \leq e$ with $(i, j) \neq (n + 1, n)$. For a fix $s \leq n$, write $M_1 = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{k=1,k\neq s}^{n} x_{kk} = -x_{ss}, x_{(n+1)n=0}\}$. If $X \in M_1$, then $X_{ss} = -\sum_{\substack{k=1\\ k\neq s}}^{n} X_{kk}$ and X can be expressed as

$$\begin{aligned} X &= \sum_{J_1} X_{ij}\mathrm{e}_{ij} + \sum_{\substack{k=1\\ k\neq s}}^{n} X_{kk}\mathrm{e}_{kk} - \sum_{\substack{k=1\\ k\neq s}}^{n} X_{kk}\mathrm{e}_{ss} \\ &= \sum_{J_1} X_{ij}\mathrm{e}_{ij} + \sum_{\substack{k=1\\ k\neq s}}^{n} X_{kk}(\mathrm{e}_{kk} - \mathrm{e}_{ss}), \end{aligned}$$

where $J_1 = \{(i, j) \in [d] \times [e] : i \neq j \text{ whenever } \ i, j \leq n, (i, j) \neq (n + 1, n)\}$. Hence M_1 is generated by e_{ij} and $\mathrm{e}_{kk} - \mathrm{e}_{ss}$ (with $i \neq j$ whenever $i, j \leq n$, $k \leq n$, and $(i, j) \neq (n + 1, n)$). We want to show that $uM = uM_1$ for all $u \in \mathcal{O}_v^d \setminus \{0\}$. Since $M_1 \subset M$, it suffices to show $uM_1 \supset uM$. Let $(\mathrm{e}_1, \ldots, \mathrm{e}_d) \subset \mathcal{O}_v^d$ be the standard basis. Since $u\mathrm{e}_{ij} = u_i\mathrm{e}_j$, we have uM is spanned by all $u_i\mathrm{e}_j$ with $(i, j) \neq (n + 1, n)$. Similarly, we have uM_1 is spanned by all $u_i\mathrm{e}_j$ and $u_k\mathrm{e}_k - u_s\mathrm{e}_s$ (with $i \neq j$ whenever $i, j \leq n$, $k \leq n$, and $(i, j) \neq (n + 1, n)$). Let $u_\ell = \gcd(u_1, \ldots, u_n)$. Since $n \geq 2$, we may choose $s \neq \ell$. To show that $uM \subset uM_1$, we need to show $u_k\mathrm{e}_k \in uM_1$ for any $1 \leq k \leq n$. Since $|u_s| \leq |u_\ell|$, $u_\ell\mathrm{e}_\ell = (u_\ell\mathrm{e}_\ell - u_s\mathrm{e}_s) + \frac{u_s}{u_\ell}u_\ell\mathrm{e}_s \in uM_1$. Also $u_k\mathrm{e}_k = (u_k\mathrm{e}_k - u_\ell\mathrm{e}_\ell) + u_\ell\mathrm{e}_\ell \in uM_1$. Therefore $uM \subset uM_1$. □

The following Proposition is obtained by a similar argument as in the proof of Proposition 10.3.3.

Corollary 10.3.4. *Let* $2 \leq n \leq d, e$. *Define* $M := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{n(n+1)} = 0\}$ *and* $M_1 = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = 0, x_{n(n+1)} = 0\}$. *Then* $uM = uM_1$ *for all* $u \in \mathcal{O}_v^d \setminus \{0\}$.

Recall from [28, Section 0.9.5] that a **reversal matrix** is a permutation matrix of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

In the following lemma, we denote by $\mathfrak{I}_n$ the reversal matrix in $\mathrm{GL}_n(K)$.

Lemma 10.3.5. *For $n \geq 2$, let $A(X_1, X_2) \in \mathrm{M}_{(n+1)n}(K[X_1, X_2])$ be a singular matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_1, \mathrm{R}_{\bar{A}}(X_1, X_2))$ where the non-singular core has a rational form $\mathrm{R}_{\bar{A}}(X_1, X_2) = \mathrm{R}X_1 + I_{n-1}X_2$. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2),\mathcal{O}_v)}(t) = \begin{cases} \frac{(1-q_v^{1-n}t)^2}{(1-t)(1-q_v^{2-n}t)(1-q_vt)} & \text{if } \mathrm{R} = I_{n-1}a \text{ for } \ a \in K, \\ \frac{1-q_v^{-n}t}{(1-t)(1-q_vt)} & \text{otherwise.} \end{cases}$$

Proof. If $\mathrm{R} = I_{n-1}a$, put $X_2' = aX_1 + X_2$. Then we map $I_{n-1}aX_1 + I_{n-1}X_2 \longrightarrow I_{n-1}X_2'$. Then

$$\mathrm{R}_A(X_1, X_2') = \begin{bmatrix} X_1 & & & & \\ X_2' - aX_1 & & & & \\ & X_2' & & & \\ & & \ddots & & \\ & & & & X_2' \end{bmatrix}.$$

Let $g_1 = \mathrm{diag}(\bar{g}_1, I_{n-1})$ where $\bar{g}_1 = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$. Then

$$g_1\mathrm{R}_A(X_1, X_2') = \begin{bmatrix} X_1 & & & & \\ X_2' & & & & \\ & X_2' & & & \\ & & \ddots & & \\ & & & & X_2' \end{bmatrix}.$$

Let $\mathfrak{I}_{n+1} \in \mathrm{GL}_{n+1}(K)$ and $\mathfrak{I}_n \in \mathrm{GL}_n(K)$ be the reversal matrices. Then

$$\mathfrak{I}_{n+1}g_1\mathrm{R}_A(X_1, X_2')\mathfrak{I}_n = \begin{bmatrix} X_2' & & & \\ & \ddots & & \\ & & X_2' & \\ & & X_1 & \end{bmatrix}. \tag{10.4}$$

Let $g = \mathfrak{I}_{n+1}g_1$ and $h = \mathfrak{I}_n$. Define $M_1 = \mathrm{Rel}(g\mathrm{R}_A(X_1, X_2')h; \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{(n+1)n}(\mathcal{O}_v) : \sum_i^n x_{ii} = 0, x_{(n+1)n} = 0\}$ and define $M = \{[x_{ij}] \in \mathrm{M}_{(n+1)n}(\mathcal{O}_v) : x_{(n+1)n} = 0\}$. By Proposition 10.3.3 (put $d = n+1, e = n$), we have $uM = uM_1$. By Corollary 4.1.5, we have $\mathsf{Z}^{\mathsf{ask}}_M(t) = \mathsf{Z}^{\mathsf{ask}}_{M_1}(t)$. We have M is equivalent to a staircase module associated with $m = (0, \ldots, 0, 1, n-1)$. Using Proposition 9.3.2, we obtain

$$\mathsf{Z}^{\mathsf{ask}}_M(t) = \frac{(1-q_v^{1-n}t)^2}{(1-t)(1-q_v^{2-n}t)(1-q_vt)}.$$

Suppose now that $\mathrm{R} \neq I_{n-1}a$. If $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil, then there exist $g_1, h_1 \in \mathrm{GL}_{n-1}(K)$ such that $g_1\mathrm{R}_{\bar{A}}(X_1, X_2)h_1$ is an admissible matrix pencil. Let $g = \mathrm{diag}(I_2, g_1)$ and $h = \mathrm{diag}(1, h_1)$. Then

$$g\mathrm{R}_A(X_1, X_2)h = \begin{bmatrix} X_1 & 0 \\ X_2 & 0 \\ 0 & g_1\mathrm{R}_{\bar{A}}(X_1, X_2)h_1 \end{bmatrix}$$

is also an admissible matrix pencil. If $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is not weakly equivalent to an admissible matrix pencil, then Section 10.1 shows that either $\mathrm{R} = C$ or $\mathrm{R} = \mathrm{diag}(I_{n-3}a, C)$ where C is a companion matrix of a monic polynomial of the form $f(x) = a_0 + a_1X + X^2$. If $\mathrm{R} = C$, then

$$\mathrm{R}_A(X_1, X_2) = \begin{bmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \\ 0 & X_2 & X_1 \\ 0 & -a_0X_1 & X_2 - a_1X_1 \end{bmatrix}.$$

Let $g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ and $h = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$g\mathrm{R}_A(X_1, X_2)h = \begin{bmatrix} X_1 & 0 & X_1 \\ X_2 & 0 & X_2 \\ 0 & X_2 & X_1 \\ -X_2 & -a_0X_1 & -a_1X_1 \end{bmatrix}.$$

By Definition 9.1.1, we have $g\mathrm{R}_A(X_1, X_2)h$ is an admissible matrix pencil. If $\mathrm{R} = \mathrm{diag}(I_{n-3}a, C)$, put $X_2' = aX_1 + X_2$. Then we map $I_{n-3}aX_1 + I_{n-3}X_2 \longrightarrow I_{n-3}X_2'$. Hence

$$\mathrm{R}_A(X_1, X_2') = \begin{bmatrix} X_1 & & & & & & \\ X_2' - aX_1 & & & & & & \\ & X_2' & & & & & \\ & & \ddots & & & & \\ & & & X_2' & & & \\ & & & & X_2' - aX_1 & X_1 & \\ & & & & -a_0X_1 & X_2' - a_1'X_1 \end{bmatrix},$$

where $a_1' = a_1 + a$. Let g_1, g_2 and h_1 be as in proof of Lemma 10.1.4. Let $g = \mathrm{diag}\left(\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, g_1g_2\right)$ and $h = \mathrm{diag}(1, h_1)$. Then

$$g\mathrm{R}_A(X_1, X_2')h = \begin{bmatrix} X_1 & & & & & \\ X_2' & & & & & \\ & X_2' & & & & \\ & & \ddots & & & \\ & & & X_2' & X_2' & \\ & & & -X_2' & -aX_1 & X_1 \\ & & & -a_1'X_2' & -(a_0+aa_1')X_1 & X_2' \end{bmatrix}.$$

Thus, we have $g\mathrm{R}_A(X_1, X_2')h$ is an admissible matrix pencil. By Proposition 9.1.3

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(g\mathrm{R}_A(X_1,X_2')h;\mathcal{O}_v)}(t) = \frac{1-q_v^{-n}t}{(1-t)(1-q_vt)}.$$

□

The following lemma is obtained by a similar argument as in Lemma 10.3.5.

Lemma 10.3.6. *For $n \geq 2$, let $A(X_1, X_2) \in \mathrm{M}_{n(n+1)}(K[X_1, X_2])$ be a singular matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_1^T, \mathrm{R}_{\bar{A}}(X_1, X_2))$ where $\mathrm{R}_A(X_1, X_2) = \mathrm{R}X_1 + I_{n-1}X_2$ is a rational form of the non-singular core. Then for almost all places $v \in \mathcal{V}_k$*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_A(X_1,X_2),\mathcal{O}_v)}(t) = \begin{cases} \frac{(1-q_v^{-n}t)^2}{(1-t)(1-q_v^{-n+1}t)(1-q_v^{-1}t)} & \text{if } R = I_{n-1}a \text{ for } a \in K, \\ \frac{1-q_v^{-n-1}t}{(1-t)(1-q_v^{-1}t)} & \text{otherwise.} \end{cases}$$

Proof. For the first part, let g and h be as in the same case in proof of Lemma 10.3.5. Therefore, Corollary 10.3.4 and Corollary 4.1.5 show that $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(h^T\mathrm{R}_A(X_1,X_2)g^T,\mathcal{O}_v)}(t) = \mathsf{Z}^{\mathsf{ask}}_M(t)$ where $M := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{n(n+1)} = 0\}$. Since M is equivalent to a staircase module associated with $m = (0, \ldots, 0, 1, n)$, Proposition 9.3.2 proves the first formula. For the second part, we also use the same arguments in the second parts of proof of Lemma 10.3.5. However, if $\mathrm{R} = C$ where C is a companion matrix of degree 2, let g and h be as in the same case in proof of Lemma 10.3.5. Then $h^T\mathrm{R}_A(X_1, X_2)g^T$ is an admissible matrix pencil. If $\mathrm{R} = \mathrm{diag}(I_{n-3}, C)$, we follow the same steps in the same case in proof of Lemma 10.3.5. However, we put $g = \mathrm{diag}(1, g_1, g_2)$ and $h = \mathrm{diag}\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, h_1\right)$ where g_1, g_2, h_1 are the same as in proof Lemma 10.1.4. Then $g\mathrm{R}_A(X_1, X_2')h$ is an admissible matrix pencil where $X_2' = aX_1 + X_2$. Hence, Proposition 9.1.3 proves the second formula. □

10.4 Relation modules of admissible singular matrix pencils

In this section, we write all the cases where singular matrix pencils are always weakly equivalent to admissible matrix pencils except the case when their

rational forms contain L_0 or L_0^T. The latter case will be dealt with in Section 10.5.

Recall from (2.14) that a singular matrix pencil $A(X_1, X_2)$ has a rational form

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, \mathrm{R}_{\bar{A}}(X_1, X_2)).$$

where $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is a rational form of the non-singular core. In this section, we always assume that $m_s > 0$ for $1 \leq s \leq k + \ell$. For a minimal canonical submatrix pencil L_m, Definition 9.1.1 shows that L_1 and L_1^T are not admissible submatrix pencil. However, a matrix pencil of the form

$$\mathrm{diag}(L_1, L_1) = \begin{bmatrix} X_1 & 0 \\ X_2 & 0 \\ 0 & X_1 \\ 0 & X_2 \end{bmatrix} \text{ or } \mathrm{diag}(L_1, L_1^T) = \begin{bmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \\ 0 & X_1 & X_2 \end{bmatrix}$$

is an admissible matrix pencil. Moreover, if $m \geq 2$, then Definition 9.1.1 also shows that

$$L_m = \begin{bmatrix} X_1 & & & \\ X_2 & X_1 & & \\ & \ddots & \ddots & \\ & & X_2 & X_1 \\ & & & X_2 \end{bmatrix}$$

is admissible matrix pencil. Similarly, the transpose of an admissible matrix pencil is also an admissible matrix pencil. Hence, we obtain the following remark.

Remark 10.4.1. Let $A(X_1, X_2)$ be a singular matrix pencil with a rational form

$$\mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}).$$

If $k + \ell$ or $m_i \geq 2$ where $1 \leq i \leq k + \ell$, then $A(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil.

The following two lemmas will generalize Remark 10.4.1.

Lemma 10.4.2. *Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a singular matrix pencil with a rational form*

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, \mathrm{R}_{\bar{A}}(X_1, X_2)),$$

where $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is the non-singular core. If there exists $m_i \geq 2$ for $1 \leq i \leq k + \ell$, then $A(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil.

Proof. If there exists $m_i \geq 2$ where $1 \leq i \leq k+\ell$, then Remark 10.4.1 shows

$$\mathrm{diag}(L_{m_1}, \ldots, L_{m_\ell}, L^T_{m_{\ell+1}}, \ldots, L^T_{m_{\ell+k}})$$

is an admissible matrix pencil. If $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is an empty matrix, then $\mathrm{R}_A(X_1, X_2)$ is an admissible matrix pencil. Hence, suppose that $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is not an empty matrix. Let $n = \mathrm{rank}_{K(X_1,X_2)}(\mathrm{R}_{\bar{A}}(X_1, X_2))$. If the non-singular core $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is weakly equivalent to an admissible matrix of linear forms, there exist $\bar{g}, \bar{h} \in \mathrm{GL}_n(K)$ such that $\bar{g}\mathrm{R}_{\bar{A}}(X_1, X_2)\bar{h}$ is an admissible matrix of linear forms. Let $g = \mathrm{diag}(I_{d-n}, \bar{g})$ and $h = \mathrm{diag}(I_{e-n}, \bar{h})$. Then

$$g\mathrm{R}_A(X_1, X_2)h = \mathrm{diag}(L_{m_1}, \ldots, L_{m_\ell}, L^T_{m_{\ell+1}}, \ldots, L^T_{m_{\ell+k}}, \bar{g}\mathrm{R}_{\bar{A}}(X_1, X_2)\bar{h})$$

is an admissible matrix pencil. Suppose that $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is not weakly equivalent to an admissible matrix of linear forms. Since $m_1 \leq \cdots \leq m_k$ and $m_{k+1} \leq \cdots \leq m_{k+\ell}$, then either m_k or $m_{k+\ell} \geq 2$. If $m_k \geq 2$, we multiply $\mathrm{R}_A(X_1, X_2)$ by permutation matrices $P_1 \in \mathrm{GL}_d(K)$ and $P_2 \in \mathrm{GL}_e(K)$ such that

$$P_1\mathrm{R}_A(X_1, X_2)P_2 = \mathrm{diag}(L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, L_{m_1}, \ldots, L_{m_{k-1}}, S(X_1, X_2)),$$

where $S(X_1, X_2) = \mathrm{diag}(L_{m_k}, \mathrm{R}_{\bar{A}}(X_1, X_2))$. We have $\mathrm{R}_{\bar{A}}(X_1, X_2) = \mathrm{R}X_1 + I_nX_2$. If $R = I_na$, let

$$\bar{g} = \left[\begin{array}{cccc|c} & & I_{m_k+1} & & 0 \\ \hline a & 0 & \cdots & 0 & \\ \vdots & \vdots & & \vdots & I_n \\ a & 0 & \cdots & 0 & \end{array}\right] \text{ and } \bar{h} = \left[\begin{array}{c|ccc} & -1 & \cdots & -1 \\ I_{m_k} & 0 & \cdots & 0 \\ & \vdots & & \vdots \\ & 0 & \cdots & 0 \\ \hline 0 & I_n & & \end{array}\right].$$

Then

$$\bar{g}S(X_1, X_2)\bar{h} = \left[\begin{array}{cccc|cccc} & & & & -X_1 & \cdots & \cdots & -X_1 \\ & & & & -X_2 & \cdots & \cdots & -X_2 \\ & & L_{m_k} & & 0 & \cdots & \cdots & 0 \\ & & & & \vdots & & & \vdots \\ & & & & 0 & \cdots & \cdots & 0 \\ \hline aX_1 & 0 & \cdots & 0 & X_2 & -aX_1 & \cdots & -aX_1 \\ \vdots & \vdots & & \vdots & -aX_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots & X_2 & -aX_1 \\ aX_1 & 0 & \cdots & 0 & -aX_1 & \cdots & -aX_1 & X_2 \end{array}\right]$$

is an admissible matrix pencil. Let $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that $g = \mathrm{diag}(I_{d-m_k-n-1}, \bar{g})$ and $h = \mathrm{diag}(I_{e-m_k-n}, \bar{h})$. Then

$$gP_1\mathrm{R}_A(X_1, X_2)P_2h = \mathrm{diag}(L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, L_{m_1}, \ldots, L_{m_{k-1}}, \bar{g}S(X_1, X_2)\bar{h})$$

is an admissible matrix pencil. If $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is a non-singular matrix pencil, then Section 10.1 shows that either $\mathrm{R} = C$ or $\mathrm{diag}(I_{n-2}, C)$ where C is a companion matrix of a monic polynomial of the form $f(X) = a_0 + a_1X + X^2$. If $\mathrm{R} = C$, let

$$\bar{g} = \left[\begin{array}{cccc|c} & & & & 0 \\ & I_{m_k+1} & & & \vdots \\ & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & I_2 \\ a_1 & 0 & \cdots & 0 & \end{array}\right] \text{ and } \bar{h} = \left[\begin{array}{c|cc} & 0 & 1 \\ I_{m_k} & 0 & 0 \\ & \vdots & \vdots \\ & 0 & 0 \\ \hline 0 & I_2 & \end{array}\right].$$

Then

$$\bar{g}S(X_1, X_2)\bar{h} = \left[\begin{array}{cccc|cc} & & & & 0 & X_1 \\ & & & & 0 & X_2 \\ & L_{m_k} & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & & & & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & X_2 & X_1 \\ a_1X_1 & 0 & \cdots & 0 & -a_0X_1 & X_2 \end{array}\right]$$

is an admissible matrix pencil. Let $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that $g = \mathrm{diag}(I_{d-m_k-3}, \bar{g})$ and $h = \mathrm{diag}(I_{e-m_k-2}, \bar{h})$. Then

$$gP_1\mathrm{R}_A(X_1, X_2)P_2h = \mathrm{diag}(L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, L_{m_1}, \ldots, L_{m_{k-1}}, \bar{g}S(X_1, X_2)\bar{h})$$

is an admissible matrix pencil. Similarly, if $\mathrm{R} = \mathrm{diag}(I_{n-2}a, C_2)$, let

$$\bar{g} = \left[\begin{array}{cccc|c} & I_{m_k+1} & & & 0 \\ \hline a & 0 & \cdots & 0 & \\ \vdots & \vdots & & \vdots & \\ a & 0 & \cdots & 0 & I_n \\ 0 & 0 & \cdots & 0 & \\ a_1 & 0 & \cdots & 0 & \end{array}\right] \text{ and } \bar{h} = \left[\begin{array}{c|cccc} & -1 & \cdots & -1 & 0 \quad 1 \\ I_{m_k} & 0 & \cdots & 0 & 0 \quad 0 \\ & \vdots & & \vdots & \vdots \quad \vdots \\ & 0 & \cdots & 0 & 0 \quad 0 \\ \hline 0 & \multicolumn{4}{c}{I_n} \end{array}\right].$$

Then $\bar{g}S(X_1, X_2)\bar{h}$ is of the form:

$$\left[\begin{array}{cccc|cccc|cc} & & & & -X_1 & \cdots & \cdots & -X_1 & 0 & X_1 \\ & & & & -X_2 & \cdots & \cdots & -X_2 & 0 & X_2 \\ & L_{m_k} & & & 0 & \cdots & \cdots & 0 & 0 & 0 \\ & & & & \vdots & & & \vdots & \vdots & \vdots \\ & & & & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \hline aX_1 & 0 & \cdots & 0 & X_2 & -aX_1 & \cdots & -aX_1 & 0 & aX_1 \\ \vdots & \vdots & & \vdots & -aX_1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots & X_2 & -aX_1 & \vdots & \vdots \\ aX_1 & 0 & \cdots & 0 & -aX_1 & \cdots & -aX_1 & X_2 & 0 & aX_1 \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & X_2 & X_1 \\ a_1X_1 & 0 & \cdots & 0 & -a_1X_1 & \cdots & \cdots & -a_1X_1 & -a_0X_1 & X_2 \end{array}\right].$$

By Definition 9.1.1, we have $\bar{g}S(X_1, X_2)\bar{g}$ is an admissible matrix pencil. Let $g = \mathrm{diag}(I_{d-m_k-n-1}, \bar{g})$ and $h = \mathrm{diag}(I_{e-m_k-n}, \bar{h})$. Then

$$gP_1\mathrm{R}_A(X_1, X_2)P_2h = \mathrm{diag}(L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, L_{m_1}, \ldots, L_{m_{k-1}}, \bar{g}S(X_1, X_2)\bar{h})$$

is an admissible matrix pencil. If $m_k < 2$, then $m_{k+\ell} \geq 2$. Let $S(X_1, X_2) = \mathrm{diag}(L^T_{m_{k+\ell}}, R_{\bar{A}}(X_1, X_2))$. Following the same argument in the previous cases but with transpose fashion, we have $\bar{h}^T S(X_1, X_2)\bar{g}^T$ is an admissible matrix pencil. By writing $g = \mathrm{diag}(I_{d-m_{k+\ell}-n-1}, \bar{g}^T)$ and $h = \mathrm{diag}(I_{e-m_{k+\ell}-n}, \bar{h}^T)$, we have

$$h\mathrm{R}_A(X_1, X_2)g = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell-1}}, \bar{h}^T S(X_1, X_2)\bar{g}^T)$$

is an admissible matrix pencil.

□

Lemma 10.4.3. *Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a singular matrix pencil with rational form*

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, \mathrm{R}_{\bar{A}}(X_1, X_2)),$$

where $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is the non-singular core. If $k + \ell \geq 2$, then $A(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil.

Proof. If there exist m_i such that $m_i \geq 2$ where $1 \leq i \leq k + \ell$, then Lemma 10.4.2 shows that $A(X_1, X_2)$ is weakly equivalent to an admissible matrix pencil. Hence, we assume $m_i = 1$ for all $1 \leq i \leq k + \ell$. If $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is an empty matrix, then Remark 10.4.1 shows that

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}})$$

is an admissible matrix pencil. Hence, suppose that $\mathrm{R}_{\bar{A}}(X_1, X_2) = \mathrm{R}X_1 + I_n X_2$ is not an empty matrix. If $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is weakly equivalent to an admissible matrix of linear forms, there exist $\bar{g}, \bar{h} \in \mathrm{GL}_n(K)$ such that $\bar{g}\mathrm{R}_{\bar{A}}(X_1, X_2)\bar{h}$ is an admissible matrix of linear forms. Let $g = \mathrm{diag}(I_{d-n}, \bar{g})$ and $h = \mathrm{diag}(I_{e-n}, \bar{h})$ then

$$g\mathrm{R}_A(X_1, X_2)h = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, \bar{g}\mathrm{R}_{\bar{A}}(X_1, X_2)\bar{h})$$

is an admissible matrix pencil. Suppose that $\mathrm{R}_{\bar{A}}(X_1, X_2)$ is not weakly equivalent to an admissible matrix of linear forms. If $\ell \geq 2$, we have $L_{m_{k+\ell-1}} = L_{m_{k+\ell}} = L_1$. Write

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell-2}}, S(X_1, X_2)),$$

where $S(X_1, X_2) = \mathrm{diag}(L_1^T, L_1^T, \mathrm{R}_{\bar{A}}(X_1, X_2))$. In the proof of Lemma 10.3.6, we have $\mathrm{diag}(L_1, \mathrm{R}_{\bar{A}}(X_1, X_2))$ is weakly equivalent to an admissible matrix pencil if $\mathrm{R} \neq I_n a$ where $a \in K$. Hence, suppose that $\mathrm{R} = I_n a$. Let

$$\bar{g} = \left[\begin{array}{cc|c} I_2 & & 0_{2\times n} \\ \hline a & 0 & \\ \vdots & \vdots & I_n \\ a & 0 & \end{array}\right] \text{ and } \bar{h} = \left[\begin{array}{c|ccc} & -1 & \cdots & -1 \\ I_4 & 0 & \cdots & 0 \\ & 0 & \cdots & 0 \\ & 0 & \cdots & 0 \\ \hline 0_{n\times 4} & \multicolumn{3}{c}{I_n} \end{array}\right].$$

Then

$$\bar{g}S(X_1, X_2)\bar{h} = \left[\begin{array}{cccc|cccc} X_1 & X_2 & 0 & 0 & -X_1 & \cdots & \cdots & -X_1 \\ 0 & 0 & X_1 & X_2 & 0 & \cdots & \cdots & 0 \\ \hline aX_1 & aX_2 & 0 & 0 & X_2 & -aX_1 & \cdots & -aX_1 \\ \vdots & \vdots & \vdots & \vdots & -aX_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & X_2 & -aX_1 \\ aX_1 & aX_2 & 0 & 0 & -aX_1 & \cdots & -aX_1 & X_2 \end{array}\right]$$

is an admissible matrix pencil. Let $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that $g = \mathrm{diag}(I_{d-2-n}, \bar{g})$ and $h = \mathrm{diag}(I_{e-4-n}, \bar{h})$. Then

$$g\mathrm{R}_A(X_1, X_2)h = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell-2}}, \bar{g}S(X_1, X_2)\bar{h})$$

is an admissible matrix pencil. If $\ell = 1$, we have $L_{m_k} = L_{m_{k+1}} = L_1$. Write

$$\mathrm{R}_A(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_{k-1}}, S(X_1, X_2)),$$

where $S(X_1, X_2) = \mathrm{diag}(L_1, L_1^T, \mathrm{R}_{\bar{A}}(X_1, X_2))$. Similarly, suppose that $\mathrm{R} = I_n a$. Let

$$\bar{g} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{I_3} & 0_{3\times n} \\ \hline a & 0 & 0 & \\ \vdots & \vdots & \vdots & I_n \\ a & 0 & 0 & \end{array}\right] \text{ and } \bar{h} = \left[\begin{array}{c|ccc} & -1 & \cdots & -1 \\ I_3 & 0 & \cdots & 0 \\ & 0 & \cdots & 0 \\ \hline 0_{n\times 3} & \multicolumn{3}{c}{I_n} \end{array}\right].$$

Then

$$\bar{g}S(X_1,X_2)\bar{h} = \left[\begin{array}{ccc|cccc} X_1 & 0 & 0 & -X_1 & \cdots & \cdots & -X_1 \\ X_2 & 0 & 0 & -X_2 & \cdots & \cdots & -X_2 \\ 0 & X_1 & X_2 & 0 & \cdots & \cdots & 0 \\ \hline aX_1 & 0 & 0 & X_2 & -aX_1 & \cdots & -aX_1 \\ \vdots & \vdots & \vdots & -aX_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & X_2 & -aX_1 \\ aX_1 & 0 & 0 & -aX_1 & \cdots & -aX_1 & X_2 \end{array}\right]$$

is an admissible matrix pencil. Let $g = \mathrm{diag}(I_{d-3-n}, \bar{g})$ and $h = \mathrm{diag}(I_{e-3-n}, \bar{h})$. Then

$$g\mathrm{R}_A(X_1,X_2)h = \mathrm{diag}(L_{m_1}, \ldots, L_{m_{k-1}}, \bar{g}S(X_1,X_2)\bar{h})$$

is an admissible matrix pencil. If $\ell = 0$, we have $L_{m_{k-1}} = L_{m_k} = L_1$. Write

$$\mathrm{R}_A(X_1,X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_{k-2}}, S(X_1,X_2)),$$

where $S(X_1,X_2) = \mathrm{diag}(L_1, L_1, \mathrm{R}_{\bar{A}}(X_1,X_2))$. From the proof of Lemma 10.3.5, we have $\mathrm{diag}(L_1, \mathrm{R}_{\bar{A}}(X_1,X_2))$ is weakly equivalent to an admissible matrix pencil if $\mathrm{R} \neq I_n a$ where $a \in K$. Hence, suppose that $\mathrm{R} = I_n a$. Let

$$\bar{g} = \left[\begin{array}{cccc|c} & & I_4 & & 0_{4\times n} \\ \hline a & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & I_n \\ a & 0 & 0 & 0 & \end{array}\right] \text{ and } \bar{h} = \left[\begin{array}{cc|ccc} 1 & 0 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ \hline \multicolumn{2}{c|}{0_{n\times 2}} & \multicolumn{3}{c}{I_n} \end{array}\right].$$

Then

$$\bar{g}S(X_1,X_2)\bar{h} = \left[\begin{array}{cc|cccc} X_1 & 0 & -X_1 & \cdots & \cdots & -X_1 \\ X_2 & 0 & -X_2 & \cdots & \cdots & -X_2 \\ 0 & X_1 & 0 & \cdots & \cdots & 0 \\ 0 & X_2 & 0 & \cdots & \cdots & 0 \\ \hline aX_1 & 0 & X_2 & -aX_1 & \cdots & -aX_1 \\ \vdots & \vdots & -aX_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & X_2 & -aX_1 \\ aX_1 & 0 & -aX_1 & \cdots & -aX_1 & X_2 \end{array}\right].$$

is an admissible matrix pencil. Let $g \in \mathrm{GL}_d(K)$ and $h \in \mathrm{GL}_e(K)$ such that $g = \mathrm{diag}(I_{d-4-n}, \bar{g})$ and $h = \mathrm{diag}(I_{e-2-n}, \bar{h})$. Then

$$gP_1\mathrm{R}_A(X_1,X_2)P_2h = \mathrm{diag}(L_{m_{k+1}}^T, \ldots, L_{m_{k+\ell}}^T, L_{m_1}, \ldots, L_{m_{k-2}}, \bar{g}S(X_1,X_2)\bar{h})$$

is an admissible matrix pencil. □

Here, we covered all cases of relation modules of singular matrix pencils, with rational forms that do not contain any L_0 or L_0^T, except the four cases which mentioned in Section 10.3. In the proof of Theorem F, we will cover relation modules of singular matrix pencils which contain L_0's and L_0^T's.

10.5 Effects of generic rows and columns and proof of Theorem F and Theorem G

Let $A(X_1, X_2)$ and $B(X_1, X_2)$ be defined as in Theorem F. In this section, we start by proving the formula in Theorem F for the cases where $B(X_1, X_2)$ does not contain zero rows or columns. Those are the same cases mentioned in the previous sections. Then we prove Theorem F and Theorem G.

By Definition 9.1.1, the zero matrix $\mathrm{diag}(L_0, \ldots, L_0, L_0^T, \ldots, L_0^T)$ is an admissible matrix. Let $A(X_1, X_2) \in \mathrm{M}_{d\times e}(K[X_1, X_2])$ be a matrix pencil with a rational form $\mathrm{R}_A(X_1, X_2)$ which contains L_0's and L_0^T's. When multiplying $\mathrm{R}_A(X_1, X_2)$ by some permutation matrices, we can rearrange the canonical minimal submatrices L_0's and L_0^T's to be in first rows and columns of $\mathrm{R}_A(X_1, X_2)$, respectively. We denote such a form by $\bar{\mathrm{R}}_A(X_1, X_2)$. Hence

$$\bar{\mathrm{R}}_A(X_1, X_2) = \mathrm{diag}(0_{(d-m)\times(e-n)}, L_{m_1}, \ldots, L_{m_k}, L_{m_{k+1}}^T, \ldots, L_{m_{k+\ell}}^T, \mathrm{R}_{\bar{A}}(X_1, X_2)),$$

where $m_i > 0$ for $1 \le i \le k+\ell$. Define $B(X_1, X_2) \in \mathrm{M}_{m\times n}(K[X_1, X_2])$ to be a matrix pencil with a rational form

$$\mathrm{R}_B(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L_{m_{k+1}}^T, \ldots, L_{m_{k+\ell}}^T, \mathrm{R}_{\bar{A}}(X_1, X_2)).$$

Note that the non-singular core of $A(X_1, X_2)$ is also the non-singular core of $B(X_1, X_2)$ (i.e. $\mathrm{R}_{\bar{A}}(X_1, X_2) = \mathrm{R}_{\bar{B}}(X_1, X_2)$). Thus

$$\bar{\mathrm{R}}_A(X_1, X_2) = \left[\begin{array}{c|c} 0_{(d-m)\times(e-n)} & 0_{(d-m)\times n} \\ \hline 0_{m\times(e-n)} & \mathrm{R}_B(X_1, X_2) \end{array}\right]. \tag{10.5}$$

Since $R_B(X_1, X_2)$ does not contains L_0's and L_0^T's, we discussed ask zeta functions of relation modules of all cases of $B(X_1, X_2)$ through Sections 10.1–10.4. In this section, we will go through them once again to compute the ask zeta function of $A(X_1, X_2)$.

In the followings, we always assume that $\bar{\mathrm{R}}_A(X_1, X_2)$ and $\mathrm{R}_B(X_1, X_2)$ as in (10.5).

Lemma 10.5.1. *Let $\mathrm{R}_B(X_1, X_2)$ be equivalent to an admissible matrix pencil. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{n-e}t)(1-q_v^{m-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{n-e}t).$$

Proof. Since $\mathrm{R}_B(X_1, X_2)$ is an admissible matrix pencil, then $\bar{\mathrm{R}}_A(X_1, X_2)$ is also admissible matrix pencil. By Proposition 9.1.3

$$\begin{aligned}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) &= \frac{(1-q_v^{-e}t)}{(1-t)(1-q_v^{d-e}t)} \\ &= \frac{(1-q_v^{n-e}t)(1-q_v^{m-e})}{(1-t)(1-q_v^{d-e}t)}\frac{1-q_v^{-e}t}{(1-q_v^{n-e}t)(1-q_v^{m-e}t)} \\ &= \frac{(1-q_v^{n-e}t)(1-q_v^{m-e})}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{n-e}t)\end{aligned}$$

for almost all places $v \in \mathcal{V}_K$. □

Next, we prove the formula for all cases of matrix pencils with rational forms equivalent to non-admissible matrix pencils.

When $B(X_1, X_2)$ is a non-singular matrix pencil:

Here, we will go through all cases of $B(X_1, X_2)$ mentioned in Section 10.1.

Lemma 10.5.2. *Let $\mathrm{R}_B(X_1, X_2) = CX_1 + I_2X_2$ be a non-singular rational form where C is a companion matrix of a monic polynomial $F(X)$ of degree 2. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{2-e}t)^2}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{2-e}t).$$

Proof. Let $F(X) = a_0 + a_1X + X^2$ where $a_0, a_1 \in K$. Then

$$\bar{\mathrm{R}}_A(X_1, X_2) = \left[\begin{array}{c|cc} 0_{(d-2)\times(e-2)} & 0_{(d-2)\times 2} & \\ \hline & X_2 & X_1 \\ 0_{2\times(e-2)} & -a_0X_1 & X_2 - a_1X_1 \end{array}\right].$$

Define $M := \mathrm{Rel}(\bar{\mathrm{R}}_A(X_1, X_2); \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{(d-1)(e-1)} + x_{de} = x_{(d-1)e} - a_0x_{d(e-1)} - a_1x_{de} = 0\}$. Also, define $M^\circ := \mathrm{Rel}(\bar{\mathrm{R}}^\circ_A(X_1, X_2); \mathcal{O}_v)$ where

$\bar{\mathrm{R}}_A^{\circ}(X_1, X_2)$ is the $\circ$ Knuth dual of $\bar{\mathrm{R}}_A(X_1, X_2)$. Then

$$M^{\circ} = \left\{ \begin{bmatrix} x_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d-2} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_{d-2} \\ x_{d-1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & x_{d-1} & 0 & 0 \\ \vdots & \cdots & 0 & -x_{d-1} & a_1 x_{d-1} + x_d \\ 0 & \cdots & 0 & x_d & a_0 x_{d-1} \\ x_d & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & x_d & 0 & 0 \end{bmatrix} : x_1, \ldots, x_d \in \mathcal{O}_v \right\}. \quad (10.6)$$

By Proposition 4.5.2

$$\mathsf{Z}_M^{\mathsf{ask}}(q_v^{-s}) = \frac{1}{1 - q_v^{-1}} \int\limits_{\mathfrak{O}^{d+1}} |y|^{s+e-d-1} \cdot \prod_{i=1}^{e} \frac{||f_{i-1}(x)||}{||f_i(x) \cup f_{i-1}(x) y||} \, \mathrm{d}\mu(x, y), \quad (10.7)$$

where $x = (x_1, \ldots, x_d) \in \mathcal{O}_v^d$ and $f_i(x)$ is the set of all nonzero $i \times i$ minors of M°. Define $\mathbf{U}(\cdot) = \{(y, x_1, \ldots, x_d) \in \mathcal{O}_v^{d+1} : (v_y, v_1, \ldots, v_d) \in \cdot\}$ where $v_y = v(y)$ and $v_j = v(x_j)$ for $1 \leq j \leq d$. Then we divide the domain of the integral in (10.7) into $\mathbf{U}(\mathsf{C}_y), \mathbf{U}(\mathsf{C}_1), \ldots, \mathbf{U}(\mathsf{C}_d)$ where

$$\begin{aligned} \mathsf{C}_y &= \{(v_y, v_1, \ldots, v_d) \in \mathbb{R}_{\geq 0}^{d+1} : v_y < v_1, \ldots, v_d\}, \\ \mathsf{C}_1 &= \{(v_y, v_1, \ldots, v_d) \in \mathbb{R}_{\geq 0}^{d+1} : v_1 \leq v_y, v_1 < v_2, \ldots, v_d\}, \\ &\vdots \\ \mathsf{C}_d &= \{(v_y, v_1, \ldots, v_d) \in \mathbb{R}_{\geq 0}^{d+1} : v_d \leq v_y, v_1, \ldots, v_{d-1}\}. \end{aligned}$$

In $\mathbf{U}(\mathsf{C}_y)$, we obtain from Lemma 3.6.2 (put $\ell = d$)

$$\int_{\mathbf{U}(\mathsf{C}_y)} |y|^{s+e-d-1} \cdot \prod_{i=1}^{e} \frac{||f_{i-1}(x)||}{||f_i(x) \cup f_{i-1}(x)y||} \,\mathrm{d}\mu(x,y) = \frac{q_v^{-d}(1-q_v^{-1})}{1-t}, \qquad (10.8)$$

where $t = q_v^{-s}$. In $\mathbf{U}(\mathsf{C}_j)$, we have $|x_j| \geq ||x_1, \ldots, x_d, y||$ for $1 \leq j \leq d$. By Lemma 3.6.1, we have $||f_i(x_1, \ldots, x_d)|| \leq |x_j|^i$ for $1 \leq i \leq e$. In (10.6), we have $x_j^i \in f_i(x_1, \ldots, x_d)$ for $1 \leq i \leq e$ and $j < d-2$. Hence $|f_i(x_1, \ldots, x_d)| = |x_j|^i$ for $1 \leq i \leq e$ and $j < d-2$. Moreover, we have $x_j^i \in f_i(x_1, \ldots, x_d)$ for $1 \leq i \leq e-1$ and $j = d-1, d$. Thus $||f_i(x_1, \ldots, x_d)|| = |x_j|^i$ for $1 \leq i \leq e-1$. From (10.6)

$$\begin{aligned} ||f_e(x_1, \ldots, x_d)|| &= |x_j|^{e-2} ||x_j x_1, \ldots, x_j x_{d-2}, -a_0 x_{d-1}^2 - a_1 x_{d-1} x_d - x_d^2|| \\ &\leq |x_j|^e \end{aligned}$$

for $j = d-1, d$. In $\mathbf{U}(\mathsf{C}_{d-1})$, we have $|x_{d-1}| > |x_d|$. Hence

$$\begin{aligned} ||f_e(x_1, \ldots, x_d)|| &= |x_{d-1}|^{e-2} | - a_0 x_{d-1}^2 - a_1 x_{d-1} x_d - x_d^2| \\ &= |x_{d-1}|^e | - a_0 - a_1 \frac{x_d}{x_{d-1}} - (\frac{x_d}{x_{d-1}})^2| = |x_{d-1}|^e \end{aligned}$$

for almost all places $v \in \mathcal{V}_K$ since $| - a_0 - a_1 \frac{x_d}{x_{d-1}} - (\frac{x_d}{x_{d-1}})^2| = 1$. Thus $||f_e(x_{d-1})|| = |x_{d-1}|^e$. By Lemma 3.6.3(put $\ell = d$)

$$\int_{\mathbf{U}(\mathsf{C}_j)} |y|^{s+e-d-1} \cdot \prod_{i=1}^{e} \frac{|x_j|^{i-1}}{|x_j|^i} \,\mathrm{d}\mu(x,y) = \frac{q_v^{-d+j}(1-q_v^{-1})^2}{(1-t)(1-q_v^{d-e}t)},$$

for $j \leq d-1$.

By computing the integral in (10.8) over $\mathbf{U}(\mathsf{C}_1), \ldots, \mathbf{U}(\mathsf{C}_{d-1})$, we obtain

$$\frac{q_v^{-d+1}(1-q_v^{-1})^2 + \ldots + q_v^{-1}(1-q_v^{-1})^2}{(1-t)(1-q_v^{d-e}t)} = \frac{(q_v^{-1} - q_v^{-d})(1-q_v^{-1})}{(1-t)(1-q_v^{d-e}t)}. \qquad (10.9)$$

In $\mathbf{U}(\mathsf{C}_d)$, we have $|x_d| \geq ||x_1, \ldots, x_{d-1}, y||$. Put $y = y'x_d$ and $x_j = x'_j x_d$ for $j \leq d-1$. Then we map $J : \mathcal{O}_v^{d+1} \to U(C_d)$ via $J(x'_1, \ldots, x'_{d-1}, x_d, y') \mapsto (x'_1 x_d, \ldots, x'_{d-1} x_d, x_d, x_d y')$. Then

$$\begin{aligned} ||f_e(x'_1, \ldots, x'_{d-1}, x_d)|| &= |x_d|^e ||x'_1, \ldots, x'_{d-2}, -a_0 (x'_{d-1})^2 - a_1 x'_{d-1} - 1|| \\ &= |x_d|^e ||x'_1, \ldots, x'_{d-2}, F^*(x'_{d-1})||, \end{aligned}$$

where F^* is the reciprocal polynomial of F. Moreover, the determinant of Jacobian matrix $\det \left[\frac{\partial J(x'_1, \ldots, x'_{d-1}, x_d, y')}{\partial (x'_1, \ldots, x'_{d-1}, x_d, y')} \right] = x_d^d$. Hence

$$\begin{aligned} &\int_{\mathbf{U}(\mathsf{C}_d)} |y|^{s+e-d-1} ||f_e(x_1, \ldots, x_d), y x_d^{e-1}||^{-1} \,\mathrm{d}\mu(x_1, \ldots, x_d, y) \\ &= \int_{\mathcal{O}_v^{d+1}} |y'|^{s+e-d-1} |x_d|^{s-1} ||x'_1, \ldots, x'_{d-2}, F^*(x'_{d-1}), y'||^{-1} \,\mathrm{d}\mu(x'_1, \ldots, x'_{d-1}, x_d, y') \\ &= \frac{1-q_v^{-1}}{1-t} \int_{\mathcal{O}_v^d} |y'|^{s+e-d-1} ||x'_1, \ldots, x'_{d-2}, F^*(x'_{d-1}), y'||^{-1} \,\mathrm{d}\mu(x'_1, \ldots, x'_{d-1}, y'). \end{aligned} \qquad (10.10)$$

Let $\bar{\alpha} \in \mathfrak{K}_v$ such that $F^*(\bar{\alpha}) = 0$. By Corollary 8.1.5, there exists $\alpha \in \mathcal{O}_v$ such that for all $x'_{d-1} \equiv \bar{\alpha} \bmod \mathfrak{P}_v$

$$|F^*(x'_{d-1})| = |x'_{d-1} - \alpha|^{e_i},$$

where $e_i, i \leq 2$. We can decompose the domain of x'_{d-1} in (10.10) into sets of the form $\alpha + \mathfrak{P}_v$ where $\alpha \in \mathcal{O}_v$. If $F^*(\alpha) \not\equiv 0 \bmod \mathfrak{P}_v$, then $|F^*(x'_{d-1})| = 1$ for all $x'_{d-1} \in \alpha + \mathfrak{P}_v$. Hence $||x'_1, \ldots, x'_{d-1}, F^*(x'_{d-1}), y'|| = 1$. Put $\mathsf{p}_v \bar{x} = x'_{d-1} - \alpha$ then we map $J_0 : \mathcal{O}_v \to \alpha + \mathfrak{P}_v$ via $J_0(\bar{x}) = \mathsf{p}_v \bar{x} + \alpha$ where $\left|\frac{\partial J_0(\bar{x})}{\partial \bar{x}}\right| = q_v^{-1}$. Then (10.10) becomes

$$\frac{q_v^{-1}(1-q_v^{-1})}{1-t} \int\limits_{\mathcal{O}_v^d} |y'|^{s+e-d-1} \, \mathrm{d}\mu(x'_1, \ldots, x'_{d-2}, \bar{x}, y') = \frac{q_v^{-1}(1-q_v^{-1})^2}{(1-t)(1-q_v^{d-e}t)}. \tag{10.11}$$

However, if $F^*(\alpha) \equiv 0 \bmod \mathfrak{P}_v$, we write $|F^*(x'_{d-1})| = |x'_{d-1} - \alpha|^{e_i}$. Then (10.10) becomes

$$\frac{q_v^{-1}(1-q_v^{-1})}{1-t} \int\limits_{\mathcal{O}_v^d} |y'|^{s+e-d-1} ||x'_1, \ldots, x'_{d-2}, (\mathsf{p}_v \bar{x})^{e_i}, y'||^{-1} \, \mathrm{d}\mu(x'_1, \ldots, x'_{d-2}, \bar{x}, y'). \tag{10.12}$$

Divide the domain of the integral in (10.12) into $\mathbf{U}(\mathsf{C}'_y), \mathbf{U}(\mathsf{C}'_1), \ldots, \mathbf{U}(\mathsf{C}'_{d-2})$, and $\mathbf{U}(\bar{\mathsf{C}})$, where

$$\begin{aligned}
\mathsf{C}'_y &= \{(v'_1 \ldots, v'_{d-2}, \bar{v}, v'_y) \in \mathbb{R}^d_{\geq 0} : v'_y < v'_1, \ldots, v'_{d-2}, e_i(\bar{v}+1)\},\\
\mathsf{C}'_1 &= \{(v'_1, \ldots, v'_{d-2}, \bar{v}, v'_y) \in \mathbb{R}^d_{\geq 0} : v'_1 \leq v'_y, v'_1 < v'_2, \ldots, v'_{d-2}, e_i(\bar{v}+1)\},\\
&\vdots\\
\mathsf{C}'_{d-2} &= \{(v'_1, \ldots, v'_{d-2}, \bar{v}, v_y) \in \mathbb{R}^d_{\geq 0} : v'_{d-2} \leq v'_y, v'_1, \ldots, v'_{d-3}, v'_{d-2} < e_i(\bar{v}+1)\},\\
\bar{\mathsf{C}} &= \{(v'_1, \ldots, v'_{d-2}, \bar{v}, v'_y) \in \mathbb{R}^d_{\geq 0} : e_i(\bar{v}+1) \leq v'_y, v'_1, \ldots, v'_{d-2}\},
\end{aligned}$$

where $v'_y = v(y'), \bar{v} = v(\bar{x})$ and $v'_i = v(x'_i)$ for $1 \leq i \leq d-2$. The generating functions of $\mathsf{C}'_y, \mathsf{C}'_1, \ldots, \mathsf{C}'_{d-2}, \bar{\mathsf{C}}$ are

$$\begin{aligned}
[\mathsf{C}'_y](X_1, \ldots, X_{d-1}, Y) &= \sum_{\bar{v}=0}^{\infty} \sum_{v'_y=0}^{e(\bar{v}+1)-1} \sum_{v'_1=v'_y+1}^{\infty} \cdots \sum_{v'_{d-2}=v'_y+1}^{\infty} X_1^{v'_1} \cdots X_{d-2}^{v'_{d-2}} X_{d-1}^{\bar{v}} Y^{v'_y}\\
&= \frac{\prod_{j=1}^{d-2} X_j \sum_{m=1}^{e_i-1} (X_1 \cdots X_{d-2} y)^m}{\prod_{j=1}^{d-1}(1-X_j)(1 - X_{d-1}(X_1 \cdots X_{d-2}Y)^{e_i})}, \qquad (10.13)\\
[\mathsf{C}'_r](X_1, \ldots, X_{d-1}, Y) &= \sum_{\bar{v}=0}^{\infty} \sum_{v'_r=0}^{e(\bar{v}+1)-1} \prod_{j=1}^{r-1} \sum_{v'_j=v'_r}^{\infty} \prod_{j=r+1}^{d-2} \sum_{v'_j=v'_r+1}^{\infty} X_1^{v'_1} \cdots X_{d-2}^{v'_{d-2}} X_{d-1}^{\bar{v}} Y^{v'_y}\\
&= \frac{\prod_{j=r+1}^{d-2} X_j \sum_{m=0}^{e_r-1} (X_1 \cdots X_{d-2}Y)^m}{(1-Y)\prod_{\substack{j=1\\ j\neq r}}^{d-1}(1-X_j)(1 - X_{d-1}(X_1 \cdots X_{d-2} y)^{e_i})},
\end{aligned} \tag{10.14}$$

for $1 \leq r \leq d-2$, and

$$[\bar{C}](X_1,\ldots,X_{d-1},Y) = \sum_{\bar{v}=0}^{\infty} \sum_{v'_y=e_i(\bar{v}+1)}^{\infty} \prod_{j=1}^{d-2} \sum_{v'_j=e_i(\bar{v}+1)}^{\infty} X_1^{v'_1}\cdots X_{d-2}^{v'_{d-2}} X_{d-1}^{\bar{v}} Y^{v'_y}$$
$$= \frac{(X_1\cdots X_{d-2}Y)^{e_i}}{(1-Y)\prod_{j=1}^{d-2}(1-X_k)(1-x_{d-1}(X_1\cdots X_{d-2}Y)^{e_i})}. \tag{10.15}$$

Then (10.12) becomes

$$\frac{q_v^{-1}(1-q_v^{-1})}{1-t}\left[\int_{\mathbf{U}(\mathsf{C}'_y)} |y'|^{s+e-d-2}\,\mathrm{d}\mu(u) + \int_{\mathbf{U}(\mathsf{C}'_1)} |y'|^{s+e-d-1}|x'_1|^{-1}\,\mathrm{d}\mu(u)\right.$$
$$\left. +\cdots+ \int_{\mathbf{U}(\mathsf{C}'_{d-2})} |y'|^{s+e-d-1}|x'_{d-2}|^{-1}\,\mathrm{d}\mu(u) + q_v^{e_i}\int_{\mathbf{U}(\bar{\mathsf{C}})} |y'|^{s+e-d-1}|\bar{x}|^{-e_i}\,\mathrm{d}\mu(u)\right],$$

where $u = (x'_1,\ldots,x'_{d-2},\bar{x},y')$. By Corollary 3.5.6

$$\frac{q_v^{-1}(1-q_v^{-1})^{d+1}}{1-t}\left[[\mathsf{C}'_y](q_v^{-1},\ldots,q_v^{-1},q_v^{d-e+1}t) + [\mathsf{C}'_1](1,q_v^{-1},\ldots,q_v^{-1},q_v^{d-e}t)\right.$$
$$\left. +\cdots+ [\mathsf{C}'_{d-2}](q_v^{-1},\ldots,q_v^{-1},1,q_v^{-1},q_v^{d-e}t) + q_v^{e_i}[\bar{\mathsf{C}}](q_v^{-1},\ldots,q_v^{-1},q_v^{e_i-1},q_v^{d-e}t,)\right].$$

From (10.13), (10.14), and (10.15), we obtain

$$\frac{q_v^{-1}(1-q_v^{-1})^2}{(1-t)(1-q_v^{d-e}t)(1-q_v^{(3-e)e_i-1}t^{e_i})}\left((1-q_v^{-1})q_v^{(3-e)e_i}t^{e_i} + \left[(1-q_v^{d-e}t)q_v^{2-d}\right.\right.$$
$$\left.\left. +(1-q_v^{-1})\sum_{j=3}^{d} q_v^{j-d}\right]\sum_{m=0}^{e_i-1}(q_v^{3-e}t)^n\right)$$
$$= \frac{q_v^{-1}(1-q_v^{-1})^2\left[(1-q_v^{-1})q_v^{(3-e)e_i}t^{e_i} + (1-q_v^{2-e}t)\sum_{m=0}^{e_i-1}(q_v^{3-e}t)^n\right]}{(1-t)(1-q_v^{d-e}t)(1-q_v^{(3-e)e_i-1}t^{e_i})}. \tag{10.16}$$

Define $H_1(q_v,t)$ and $H_2(q_v,t)$ as in the equations (10.11) and (10.16), respectively. Since the roots of $F^*(X)$ are the same roots of $F(X)$ over $\mathfrak{K}_v$, let g_i's to be irreducible polynomials such that $F(X) = \prod_{i=1}^{h} g_i^{e_i}$. Define $N(\mathfrak{K}_v) = \#\{\bar{\alpha} \in \mathfrak{K}_v : g_i(\bar{\alpha}) = 0\}$. Then the integral in (10.10) becomes

$$\begin{aligned}
&(q_v - N(\mathfrak{K}_v))H_1(q_v,t) + N(\mathfrak{K}_v)H_2(q_v,t)\\
&= q_vH_1(q_v,t) + N(\mathfrak{K}_v)(H_2(q_v,t) - H_1(q_v,t))\\
&= \frac{(1-q_v^{-1})^2}{(1-t)(1-q_v^{d-e}t)} + (1-q_v^{-1})\sum_{i=0}^{h} N_i(\mathfrak{K}_v)W_i(q_v,t),
\end{aligned} \tag{10.17}$$

where

$$W_i(q_v,t) = \frac{q_v^{-1}(1-q_v^{-1})[q_v^{(3-e)e_i}t^{e_i} - 1 + (1-q_v^{2-e}t)\sum_{m=0}^{e_i-1}(q_v^{3-e}t)^n]}{(1-t)(1-q_v^{d-e})(1-q_v^{(3-e)e_i-1}t^{e_i})}.$$

By (10.8), (10.9), and (10.17), we obtain

$$\mathsf{Z}^{\mathsf{ask}}_{M}(t) = \frac{1-q_v^{-e}t}{(1-t)(1-q_v^{d-e}t)} + \sum_{i=0}^{h} N_i(\mathfrak{K}_v)W_i(q_v,t).$$

By Corollary 10.1.2, we obtain

$$\mathsf{Z}^{\mathsf{ask}}_{M}(t) = \frac{(1-q_v^{2-e}t)^2}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{2-e}t)$$

for almost all place $v \in \mathcal{V}_K$. □

Lemma 10.5.3. *Let* $\mathrm{R}_B(X_1,X_2) = \mathrm{R}X_1 + I_nX_2 \in \mathrm{M}_n(K[X_1,X_2])$ *be a rational form of non-singular matrix pencil where* $\mathrm{R} = \mathrm{diag}(I_{n-2}a, C)$ *and* C *is a companion matrix of a monic polynomial* $F(X) = a_0 + a_1X + X^2$*. Then for almost all places* $v \in \mathcal{V}_K$

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{n-e}t)^2}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{n-e}t).$$

Proof. From the proof of Lemma 10.1.4, we have $\mathrm{R}_B(X_1,X_2)$ is an non-admissible matrix pencil when $a_0 = -a^2$. In this case, let $P_1 \in \mathrm{GL}_d(K)$ and $P_1 \in \mathrm{GL}_e(K)$ be permutation matrices such that

$$P_1 = \begin{bmatrix} 0_{n\times(d-n)} & I_n \\ I_{d-n} & 0_{(d-n)\times n} \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0_{(e-n)\times n} & I_{e-n} \\ I_n & 0_{n\times(e-n)} \end{bmatrix}.$$

Then

$$P_1\bar{\mathrm{R}}_A(X_1,X_2)P_2 = \begin{bmatrix} \mathrm{R}_B(X_1,X_2) & 0_{n\times(e-n)} \\ 0_{(d-n)\times n} & 0_{(d-n)\times(e-n)} \end{bmatrix}.$$

Put $X_2' = aX_1 + X_2$. Then we map $\mathrm{R}X_1 + I_nX_2 \to \mathrm{R}'X_1' + I_nX_2$ where $R' = R - I_na$. By (10.2) in Lemma 10.1.4, there exist $\bar{g} \in \mathrm{GL}_n(K)$ and $\bar{h} \in \mathrm{GL}_n(K)$ such that

$$\bar{g}\mathrm{R}_B(X_1,X_2')\bar{h} = \begin{bmatrix} X_2' & & & & & \\ & \ddots & & & & \\ & & X_2' & & & \\ & & & & X_2' & X_1 \\ & & & & cX_2' & X_2' \end{bmatrix},$$

where $c = a_1 + 2a$. Let $g = \mathrm{diag}(\bar{g}, I_{d-n})$ and $h = \mathrm{diag}(\bar{h}, I_{e-n})$. Then

$$gP_1\bar{\mathrm{R}}_A(X_1,X_2')P_2h = \left[\begin{array}{ccccc|c} X_2' & & & & & \\ & \ddots & & & & \\ & & X_2' & & & 0_{n\times(e-n)} \\ & & & & X_2' & X_1 \\ & & & & cX_2' & X_2' \\ \hline \multicolumn{5}{c|}{0_{(d-n)\times n}} & 0_{(d-n)\times(e-n)} \end{array}\right].$$

Define $M := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{(n-1)n} = 0\}$. If $c = 0$, then define $M_1 = \mathrm{Rel}(gP_1\bar{\mathrm{R}}_A(X_1, X_2)P_2h; \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = 0, x_{(n-1)n} = 0\}$. If $c \neq 0$, then define $M_2 = \mathrm{Rel}(gP_1\bar{\mathrm{R}}_A(X_1, X_2)P_2h; \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = cx_{n(n-1)}, x_{(n-1)n} = 0\}$. By Proposition 10.1.3, we have $uM = uM_1 = uM_2$ for all $u \in \mathcal{O}_v^d \setminus \{0\}$. Using Corollary 4.1.5

$$\mathsf{Z}_M^{\mathsf{ask}}(t) = \mathsf{Z}_{M_1}^{\mathsf{ask}}(t) = \mathsf{Z}_{M_2}^{\mathsf{ask}}(t).$$

Moreover, We have M is equivalent to a staircase module associated with $m = (0, \ldots, 0, 1, e-1)$. By Proposition 9.3.2 and Lemma 10.1.4

$$\begin{aligned}\mathsf{Z}_M^{\mathsf{ask}}(t) &= \frac{(1-q_v^{1-e}t)^2}{(1-t)(1-q_v^{d-e}t)(1-q_v^{2-e}t)}\\ &= \frac{(1-q_v^{n-e}t)^2}{(1-t)(1-q_v^{d-e}t)}\frac{(1-q_v^{1-e}t)^2}{(1-q_v^{n-2}t)^2(1-q_v^{2-e}t)}\\ &= \frac{(1-q_v^{n-e}t)^2}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{1-e}t).\end{aligned}$$

If $a \neq -a^2$, the proof of Lemma 10.1.4 shows that $\mathrm{R}_B(X_1, X_2)$ is an admissible matrix pencil. Therefore, we have $\bar{\mathrm{R}}_A(X_1, X_2)$ is also an admissible matrix pencil. Hence, Lemma 10.5.1 completes the proof. □

When $B(X_1, X_2)$ is a singular matrix pencil:

Here, we will go through all cases of $B(X_1, X_2)$ mentioned in Section 10.3.

Lemma 10.5.4. *Let $B(X_1, X_2)$ be a singular matrix pencil with rational form $\mathrm{R}_B(X_1, X_2) = L_1$. Then*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)(1-q_v^{2-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{1-e}t).$$

Proof. We have $\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1, X_2); \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{(d-1)e} = x_{de} = 0\}$ which is equivalent to a staircase module associated with $m = (0, \ldots, 0, 1, 0, e-1)$. By Proposition 9.3.2

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)(1-q_v^{2-e}t)}{(1-t)(1-q_v^{d-e}t)(1-q_v^{3-e}t)}.$$

By Lemma 10.3.1

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)(1-q_v^{2-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{1-e}t).$$

□

Lemma 10.5.5. *Let $B(X_1, X_2)$ be a singular matrix pencil with rational form $\mathrm{R}_B(X_1, X_2) = L_1^T$. Then*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)(1-q_v^{2-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{2-e}t).$$

Proof. We have $\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{d(e-1)} = x_{de} = 0\}$ which is equivalent to a staircase module associated with $m = (0,\ldots,0,2,e-2)$. By Proposition 9.3.2

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)(1-q_v^{2-e}t)}{(1-t)(1-q_v^{d-e}t)(1-q_v^{3-e}t)}.$$

By Corollary 10.3.2

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)(1-q_v^{2-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,A_2);\mathcal{O}_v)}(q_v^{2-e}t).$$

□

For $2 \le n \le d,e$, note that if $\mathrm{R}_B(X_1,X_2) = \mathrm{diag}(L_1, \mathrm{R}_{\bar{B}}(X_1,X_2))$ or $\mathrm{diag}(L_1^T, \mathrm{R}_{\bar{B}}(X_1,X_2))$ where $\mathrm{R}_{\bar{B}}(X_1,X_2) = \mathrm{R}X_1 + I_{n-1}X_2$ is the non-singular core, proof of Lemma 10.3.5 and Lemma 10.3.6 show that $\mathrm{R}_B(X_1,X_2)$ is equivalent to an admissible matrix pencil if $\mathrm{R} \neq I_{n-1}a$ for $a \in K$. The following two lemmas will deal with the case of $\mathrm{R} = I_{n-1}a$.

Lemma 10.5.6. *For $2 \le n \le d,e$, let $B(X_1,X_2)$ be a singular matrix pencil with a rational form $\mathrm{R}_B(X_1,X_2) = \mathrm{diag}(L_1, \mathrm{R}_{\bar{B}}(X_1,X_2))$ where $\mathrm{R}_{\bar{B}}(X_1,X_2) = I_{n-1}aX_1 + I_{n-1}X_2$ is the non-singular core where $a \in K$. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{n-e}t)(1-q_v^{n+1-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q^{n-e}t).$$

Proof. Let

$$P_1 = \begin{bmatrix} 0_{(n+1)\times(d-n-1)} & I_{n+1} \\ I_{d-n-1} & 0_{(d-n-1)\times(n+1)} \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0_{(e-n)\times n} & I_{e-n} \\ I_n & 0_{n\times(e-n)} \end{bmatrix}.$$

Then

$$P_1\bar{\mathrm{R}}_A(X_1,X_2)P_2 = \begin{bmatrix} \mathrm{R}_B(X_1,X_2) & 0_{(n+1)\times(e-n)} \\ 0_{(d-n-1)\times n} & 0_{(d-n-1)\times(e-n)} \end{bmatrix}.$$

Put $X_2' = aX_1 + X_2$. Then we map $I_{n-1}aX_1 + I_{n-1}X_2 \to I_{n-1}X_2'$. By (10.4) in Lemma 10.3.5, there exist $\bar{g} \in \mathrm{GL}_{n+1}(K)$ and $\bar{h} \in \mathrm{GL}_n(K)$ such that

$$\bar{g}\mathrm{R}_B(X_1,X_2')\bar{h} = \begin{bmatrix} X_2' & & \\ & \ddots & \\ & & X_2' \\ & & X_1 \end{bmatrix}.$$

Let $g = \mathrm{diag}(\bar{g}, I_{d-n-1})$ and $h = \mathrm{diag}(\bar{h}, I_{e-n})$. Then

$$gP_1\bar{\mathrm{R}}_A(X_1,X_2')P_2h = \left[\begin{array}{ccc|c} X_2' & & & \\ & \ddots & & 0_{(n+1)\times(e-n)} \\ & & X_2' & \\ & & X_1 & \\ \hline \multicolumn{3}{c|}{0_{(d-n-1)\times n}} & 0_{(d-n-1)\times(e-n)} \end{array}\right].$$

Define $M_1 = \mathrm{Rel}(gP_1\bar{\mathrm{R}}_A(X_1, X_2')P_2h; \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = 0, x_{(n+1)n} = 0\}$ and define $M := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{(n+1)n} = 0\}$. By Proposition 10.3.3, we have $uM = uM_1$ for all $u \in \mathcal{O}_v^d \setminus \{0\}$. Using Corollary 4.1.5

$$\mathsf{Z}_M^{\mathsf{ask}}(t) = \mathsf{Z}_{M_1}^{\mathsf{ask}}(t).$$

Then M is equivalent to a staircase module associated with $m = (0, \ldots, 0, 1, e-1)$. Proposition 9.3.2 shows that

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)^2}{(1-t)(1-q_v^{d-e}t)(1-q_v^{2-e}t)}$$

for almost all places $v \in \mathcal{V}_K$. By Lemma 10.3.5

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{n-e}t)(1-q_v^{n+1-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_B(X_1,X_2);\mathcal{O}_v)}(q_v^{n-e}t)$$

for almost all places $v \in \mathcal{V}_K$. □

Lemma 10.5.7. *For $2 \le n \le d, e$, let $B(X_1, X_2)$ be a singular matrix pencil with a rational form $\mathrm{R}_B(X_1, X_2) = \mathrm{diag}(L_1^T, \mathrm{R}_{\bar{B}}(X_1, X_2))$ where $\mathrm{R}_{\bar{B}}(X_1, X_2) = \mathrm{R}X_1 + I_{n-1}X_2$ is the non-singular core. Then for almost all places $v \in \mathcal{V}_K$*

$$\mathsf{Z}_M^{\mathsf{ask}}(t) = \frac{(1-q_v^{n+1-e}t)(1-q_v^{n-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(q^{n+1-e}t).$$

Proof. Let P_1 and P_2 be defined as in the proof of Lemma 10.5.6. Then

$$P_2\bar{\mathrm{R}}_A(X_1, X_2)P_1 = \begin{bmatrix} \mathrm{R}_B(X_1, X_2) & 0_{n\times(e-n-1)} \\ 0_{(d-n)\times(n+1)} & 0_{(d-n)\times(e-n-1)} \end{bmatrix}.$$

Let g and h be as in Lemma 10.5.6. Then

$$h^TP_2\bar{\mathrm{R}}_A(X_1, X_2')P_1g^T = \left[\begin{array}{ccc|c} X_2' & & & \\ & \ddots & & 0_{n\times(e-n-1)} \\ & & X_2' \quad X_1 & \\ \hline & 0_{(d-n)\times(n+1)} & & 0_{(d-n)\times(e-n-1)} \end{array}\right],$$

where $X_2' = aX_1 + X_2$. Therefore, we have $\mathrm{Rel}(h^TP_2\bar{\mathrm{R}}_A(X_1, X_2')P_1g^T; \mathcal{O}_v) = \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : \sum_{i=1}^{n} x_{ii} = 0, x_{n(n+1)} = 0\}$. By Corollary 10.3.4 and Corollary 4.1.5

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\mathrm{R}_A(X_1,X_2);\mathcal{O}_v)}(t) = \mathsf{Z}_M^{\mathsf{ask}}(t),$$

where $M := \{[x_{ij}] \in \mathrm{M}_{d\times e}(\mathcal{O}_v) : x_{n(n+1)} = 0\}$ for almost all places $v \in \mathcal{V}_K$. Since M is equivalent to a staircase module associated with $m = (0, \ldots, 0, 1, e-1)$, Proposition 9.3.2 shows that

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{1-e}t)^2}{(1-t)(1-q_v^{d-e}t)(1-q_v^{2-e}t)}$$

for almost all places $v \in \mathcal{V}_K$. By Lemma (10.3.6)

$$\mathsf{Z}_M^{\mathsf{ask}}(t) = \frac{(1-q_v^{n+1-e}t)(1-q_v^{n-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(\bar{\mathrm{R}}_A(X_1,X_2);\mathcal{O}_v)}(q^{n+1-e}t)$$

for almost all places $v \in \mathcal{V}_K$. □

Proof of Theorem F

Let $A(X_1, X_2)$ and $B(X_1, X_2)$ be defined as in Theorem F. Let $\mathrm{R}_A(X_1, X_2)$ and $\mathrm{R}_B(X_1, X_2)$ be rational forms of the matrix pencils $A(X_1, X_2)$ and $B(X_1, X_2)$, respectively. Then

$$\bar{\mathrm{R}}_A(X_1, X_2) = \left[\begin{array}{c|c} 0_{(d-m)\times(e-n)} & 0_{(d-m)\times n} \\ \hline 0_{m\times(e-n)} & \mathrm{R}_B(X_1, X_2) \end{array}\right].$$

Let $C(X_1, X_2) \in \mathrm{M}_{m_0\times n_0}(K)$ be a matrix pencil such that $\bar{\mathrm{R}}_B(X_1, X_2) = \mathrm{diag}(0_{(m-m_0)\times(n-n_0)}, \mathrm{R}_C(X_1, X_2))$, where

$$\mathrm{R}_C(X_1, X_2) = \mathrm{diag}(L_{m_1}, \ldots, L_{m_k}, L^T_{m_{k+1}}, \ldots, L^T_{m_{k+\ell}}, \mathrm{R}_C(X_1, X_2))$$

and $m_i > 0$ for $1 \leq i \leq k + \ell$. Since $\mathrm{R}_C(X_1, X_2)$ is a matrix pencil that does not contain zero rows or zero columns, Lemmas 10.5.1–10.5.7 show that

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(B(X_1,X_2);\mathcal{O}_v)}(t) = \frac{(1-q_v^{n_0-n}t)(1-q_v^{m_0-n}t)}{(1-t)(1-q_v^{m-n}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(C(X_1,X_2);\mathcal{O}_v)}(q_v^{n_0-n}t) \tag{10.18}$$

for almost all places $v \in \mathcal{V}_K$. By substituting t for $q_v^{n-e}t$ in (10.18), We have

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(B(X_1,X_2);\mathcal{O}_v)}(q_v^{n_0-e}t) = \frac{(1-q_v^{n_0-e}t)(1-q_v^{m_0-e}t)}{(1-q_v^{n-e}t)(1-q_v^{m-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(C(X_1,X_2);\mathcal{O}_v)}(q_v^{n_0-e}t) \tag{10.19}$$

for almost all places $v \in \mathcal{V}_K$. Moreover, Lemmas 10.5.1–10.5.7 and (10.19) show that

$$\begin{aligned}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(A(X_1,X_2);\mathcal{O}_v)}(t) &= \frac{(1-q_v^{n_0-e}t)(1-q_v^{m_0-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(C(X_1,X_2);\mathcal{O}_v)}(q_v^{n_0-e}t) \\ &= \frac{(1-q_v^{n-e}t)(1-q_v^{m-e}t)}{(1-t)(1-q_v^{d-e}t)}\mathsf{Z}^{\mathsf{ask}}_{\mathrm{Rel}(B(X_1,X_2);\mathcal{O}_v)}(q_v^{n-e}t)\end{aligned}$$

for almost all places $v \in \mathcal{V}_K$.

Proof of Theorem G

Theorem G summarizes all formulas for ask zeta functions of relation modules of matrix pencils in Chapter 10. The formula in Theorem G(i) is the formula of ask zeta functions of relation modules of matrix pencils which are weakly equivalent to admissible matrix pencils, see Corollary 9.1.4. The formula in Theorem G(ii) covers the non-admissible part in Lemma 10.1.4, Lemma 10.3.5, Lemma 10.3.6, Lemma 10.5.3, Lemma 10.5.6, and Lemma 10.5.7. The formula in Theorem G(iii) is the formula for Lemma 10.3.1, Corollary 10.3.2, Lemma 10.5.4, and Lemma 10.5.5. All that remain are the cases in Corollary 10.1.2 and Lemma 10.5.2 which are given by the formula in Theorem G(iv).

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