



Partition functions and chiral algebras.

Title	Partition functions and chiral algebras.
Author(s)	Tuite, Michael P.
Publication Date	2007-05
Publisher	American Mathematical Society

Chiral Algebras and Partition Functions

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In Honor of Jim Lepowsky and Robert Wilson

ABSTRACT We discuss recent work of the authors concerning correlation functions and partition functions for free bosons/fermions and the b-c or ghost system. We compare and contrast the nature of the 1-point functions at genus 1, and explain how one may understand the free boson partition function at genus 2 via vertex operators and sewing complex tori.

1 Introduction

This paper is based on the talk given by one of the authors at the North Carolina State Conference honoring Jim Lepowsky and Robert Wilson. The paper concerns the idea of *partition functions* in the theory of chiral algebras. The genus 1 partition function of a vertex operator algebra - a.k.a. the *graded dimension* - has been studied extensively, but the case when either the genus is *greater* than 1 or else the chiral algebra is *not* a vertex operator algebra

*Partial support provided by NSF DMS-0245225 and the Committee on Research, University of California, Santa Cruz

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has received little attention from mathematicians thus far. Physicists have expended more effort in this direction. The authors have been investigating the higher genus case for some time (cf. [T] for some preliminary results). What has become clear is that in trying to define and study higher genus partition functions attached to chiral algebras, it is imperative to understand the physicist's approach via path integrals.

Let us therefore begin with some well-known ideas in the physics string theory literature on conformal field theory on higher genus Riemann surfaces [MS], [So], [DP], [VV], [ABMNV], [P]. In particular, there has been much progress in recent years in understanding genus two superstring theory [DPI], [DPVI], [DGP]. The g -loop probability amplitude $\mathcal{A}^{(g)}$ for the 26-dimensional bosonic string is given by a heuristic path integral over all metric and string configurations on all two-dimensional compact surfaces with g holes parameterized by σ_1, σ_2 [GSW]:

$$\mathcal{A}^{(g)} = \int Dh(\sigma)DX(\sigma)e^{-S[h,X]},$$

with Polyakov *action*

$$S[h, X] = \int d^2\sigma \sqrt{\det(h)} h^{ab} \partial_a X^\mu \partial_b X_\mu.$$

We are not going to be at all precise about such path integrals here - we are using them to motivate rather than inform. Formally dividing by various infinite group volumes, $\mathcal{A}^{(g)}$ may be re-expressed via the Faddeev-Popov procedure as a path integral over fermionic ghost b, c and bosonic X^μ configurations (loc. cit):

$$\mathcal{A}^{(g)} = \int DbDcDX \exp(-S_0[X] - S_{\text{ghost}}[b, c]),$$

where

$$\begin{aligned} S_0[X] &\equiv S_P[X, h_{\text{conf}}] = \int \partial_z X^\mu \partial_{\bar{z}} X_\mu dz d\bar{z}, \\ S_{\text{ghost}}[b, c] &= \int (\bar{b} \partial_z c + b \partial_{\bar{z}} c) dz d\bar{z}. \end{aligned}$$

with $h_{\text{conf}} = dz \otimes d\bar{z}$ the conformal metric with z a local coordinate on a compact Riemann surface $\mathcal{S}^{(g)}$ of genus g . We let B denote this 26-dimensional boson and ghost system.

One can find a detailed physical discussion of the genus 1 bosonic string in [GSW] or [P], where one finds the factorization property

$$\mathcal{A}^{(1)} = \int d\mu(\tau) \frac{1}{\text{Im}(\tau)^{12}} \left| Z_B^{(1)}(\tau) \right|^2$$

with torus modular parameter τ and $SL(2, \mathbb{Z})$ -invariant measure $d\mu(\tau) = \frac{d^2\tau}{\text{Im}(\tau)^2}$. $Z_B^{(1)}$ is the partition function

$$\begin{aligned} Z_B^{(1)}(\tau) &= \frac{1}{\Delta_{12}^{(1)}} = \eta(q)^{-24} \\ &= q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24}, \end{aligned} \tag{1}$$

i.e. the (inverse of) the familiar cusp-form $\Delta_{12}^{(1)}$ of weight 12 on $SL(2, \mathbb{Z})$. $Z_B^{(1)}(\tau)$ is holomorphic on the complex upper half-plane $\mathbb{H}^{(1)}$.

For $g > 1$, Belavin and Knizhnik [BK], [Kn] proposed the following remarkable general factorization formula (which is related to the absence of the conformal anomaly)

$$\mathcal{A}^{(g)} = \int dy_i d\bar{y}_i \frac{1}{\det(\text{Im}(\Omega^{(g)}))^{13}} \left| Z_B^{(g)}(y) \right|^2 \tag{2}$$

for $3g - 3$ complex modular parameters y_i , genus g period matrix $\Omega^{(g)}(y)$, and holomorphic partition function $Z_B^{(g)}(y)$. The existence of $Z_B^{(g)}$ is intimately related to Mumford's Theorem concerning the existence of a global section on genus g moduli space for $E = K \otimes \lambda^{-13}$, where K and λ are certain determinant line bundles [Mu]. Physically, K is identified with a chiral ghost determinant and λ^{-1} with a chiral 2-dimensional boson determinant.

For $g = 2$, Knizhnik [Kn] and Moore [Mo] have argued that

$$Z_B^{(2)} = \frac{1}{\Delta_{10}^{(2)}},$$

where $\Delta_{10}^{(2)}$ is the Igusa cusp form of weight 10 on $Sp(4, \mathbb{Z})$, defined on the Siegel upper half-space $\mathbb{H}^{(2)}$ of genus 2. This is a physical result, i.e. not

mathematically rigorous. A complete mathematical explanation remains to be found.

Aside from any details, the partition function of a chiral algebra V at genus g should be a meromorphic function $Z_V^{(g)}$ defined on some domain of several complex variables. In good cases, one expects $Z_V^{(g)}$ to have automorphic properties with respect to an appropriate discrete group. An additional property that one expects is *multiplicativity*:

$$Z_{V_1 \otimes V_2}^{(g)} = Z_{V_1}^{(g)} Z_{V_2}^{(g)}. \quad (3)$$

A general goal is to understand how expressions such as $Z_B^{(1)}$ and $Z_B^{(2)}$ can be interpreted, and (hopefully) calculated, in the language of chiral algebras, i.e. (super) vertex (operator) algebras V . The present paper is devoted to a discussion of a few ideas in this direction. Proofs of some of the results to be described will appear elsewhere.

The paper is organized as follows. In Section 2 we discuss the chiral algebra B which corresponds to the bosonic string, and in Sections 3 and 4 we consider genus 1 (torus) correlation functions, and in particular the partition function and graded dimension of B . One of the main points is that these are *not* the same. In Section 5 we take up a few of the issues that arise at genus 2, limiting the discussion to the vertex operator algebra part of the bosonic string, that is to say the Heisenberg, or free bosonic vertex operator algebra. We refer the reader to [T], [Ma] and [MT1] for additional background.

2 The chiral algebra B

Let M be the Heisenberg vertex operator algebra of rank 1, corresponding to a single free boson. The vertex operator algebra for l free bosons is then just the tensor product $M^{\otimes l}$ of central charge $c = l$.

Let $V_{\mathbb{Z}}$ be the super vertex operator algebra lattice theory based on the integer lattice \mathbb{Z} :

$$V_{\mathbb{Z}} = M \otimes \mathbb{C}[\mathbb{Z}]$$

It has central charge 1. If $\alpha = \alpha(-1)\mathbf{1} \in (V_{\mathbb{Z}})_1$ generates the lattice ($\mathbf{1}$ is the vacuum), the standard conformal vector is $\omega = \frac{1}{2}\alpha(-1)^2\mathbf{1}$. There is a *family*

of conformal vectors

$$\omega_\lambda = \omega - \lambda L(-1)\alpha, \quad \lambda \in \mathbb{C},$$

each defining a chiral algebra

$$(V_{\mathbb{Z}}, Y, \mathbf{1}, \omega_\lambda)$$

with central charge $c_\lambda = 1 - 12\lambda^2$. The *ghost system*, or *bc-system* G , is the chiral algebra which obtains in case $\lambda = 3/2$, when $c_\lambda = -26$. The chiral algebra of the bosonic string is then defined as

$$B = M^{\otimes 26} \otimes G.$$

Evidently, B has central charge zero.

In addition to the (bosonic) description of G just presented, there is a *fermionic* description which runs as follows. It is generated by states $c \in G_{-1}$ and $b \in G_2$ and relations given by anticommutators:

$$\begin{aligned} \{b(m), c(n)\} &= \delta_{m+n+1,0}I, \\ \{b(m), b(n)\} &= \{c(m), c(n)\} = 0. \end{aligned}$$

A convenient basis of G consists of the states

$$\begin{aligned} b(-m_1)\dots b(-m_r)c(-n_1)\dots c(-n_s)\mathbf{1}, \\ 1 \leq m_1 < \dots < m_r, \quad 1 \leq n_1 < \dots < n_s. \end{aligned} \tag{4}$$

There is a \mathbb{Z} -grading on G given by the *ghost number*, whereby the state (4) has ghost number $r - s$. If $G^{(m)}$ are the states of ghost number m then

$$G = \bigoplus_{m \in \mathbb{Z}} G^{(m)}.$$

$G^{(0)}$ is a subalgebra of G , essentially a rank 1 free bosonic theory with a shifted conformal vector. Each $G^{(m)}$ is a simple $G^{(0)}$ -module. For further background regarding G , see [Ka].

3 Some Graded Characters

The genus 1 partition function of a vertex operator algebra is essentially the same as the *graded dimension* of V . Thus if $V = \bigoplus_{n \in \mathbf{Z}} V_n$ is the usual decomposition of V given by the natural $L(0)$ -grading, then

$$Z_V^{(1)} = \text{Tr}_V q^{L(0)-c/24} = q^{-c/24} \sum_n \dim V_n q^n. \quad (5)$$

where c is the central charge. For example, it is well-known that

$$Z_{M^{\otimes l}}^{(1)} = \eta(q)^{-l}.$$

Now compare this with (3) and (1). We find that

$$\begin{aligned} Z_G^{(1)} &= Z_B^{(1)} / Z_{M^{\otimes 26}}^{(1)} \\ &= \Delta_{12}^{-1} \eta(q)^{26} \\ &= \eta(q)^2 \\ &= q^{1/12} (1 - 2q + \dots) \end{aligned} \quad (6)$$

On the other hand, the graded dimension of G is

$$\begin{aligned} \text{Tr}_G q^{L(0)+26/24} &= 2 \left(\frac{\eta(q^2)}{\eta(q)} \right)^2 \\ &= 2q^{1/12} (1 + 2q + 3q^2 + \dots). \end{aligned}$$

We conclude that $Z_G^{(1)}$ is *not equal* to $\text{Tr}_G q^{L(0)+26/24}$, indeed the two q -expansions are quite different¹. We have not contradicted (5), because G is not a vertex operator algebra. Rather, it arises from the super vertex operator algebra $V_{\mathbb{Z}}$, and this suggests that we might better off to include a ghost number operator - the super version of (5). Formally, this is the *super graded dimension*

$$S\text{Tr}_V q^{L(0)-c/24} = \text{Tr}_V (-1)^F q^{L(0)-c/24},$$

where $(-1)^F$ acts on $G^{(m)}$ as multiplication by $(-1)^m$. But this does not solve the problem (indeed, it appears to make it worse) because it turns out that

$$S\text{Tr}_G q^{L(0)+26/24} = 0.$$

¹The referee points out that this phenomenon is closely related to space-time supersymmetry in string theory.

Thus the correct definition of $Z_G^{(1)}$, which we provide in the next Section, is less straightforward than might have been anticipated.

We complete this Section with a detour to look more closely at $Tr_G q^{L(0)+26/24}$ and $STr_G q^{L(0)+26/24}$. Though neither is the desired partition function of G , nevertheless they have a rôle to play. In order to rescue $STr_G q^{L(0)+26/24}$ from the ignominy of vanishing we introduce yet another operator $e^{2\pi i \alpha(0)z}$. Here, α is the normalized weight 1 state in $G^{(0)}$. Then one has the following:

$$\begin{aligned} Tr_{V_{\mathbb{Z}}} q^{L(0)+26/24} e^{2\pi i z \alpha(0)} &= \Theta_3(q, z)/\eta(q), \\ STr_{V_{\mathbb{Z}}} q^{L(0)+26/24} e^{2\pi i z \alpha(0)} &= \Theta_2(q, z)/\eta(q), \\ Tr_G q^{L(0)+26/24} e^{2\pi i z \alpha(0)} &= \Theta_1(q, z)/\eta(q), \\ STr_G q^{L(0)+26/24} e^{2\pi i z \alpha(0)} &= \Theta_0(q, z)/\eta(q). \end{aligned}$$

Here,

$$\begin{aligned} \Theta_3(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2} q_z)(1 + q^{n-1/2} q_z^{-1}), \\ \Theta_2(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2} q_z)(1 - q^{n-1/2} q_z^{-1}), \\ \Theta_1(\tau, z) &= q^{1/8} (q_z^{1/2} + q_z^{-1/2}) \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n q_z)(1 + q^n q_z^{-1}), \\ \Theta_0(\tau, z) &= q^{1/8} (q_z^{-1/2} - q_z^{1/2}) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n q_z)(1 - q^n q_z^{-1}), \end{aligned}$$

with $q_z = e^{2\pi i z}$. These are the four Jacobi theta functions written out as products. The displayed product formulas arise from the fermionic constructions of $V_{\mathbb{Z}}$ and G . There are also sum formulas arising from the bosonic description ([Ka]).

4 1-point functions

Vertex operators $Y(v, z)$ can be considered as insertions at a point of the complex plane or sphere with local variable z . The conformal map

$$z \mapsto e^z - 1$$

puts the insertions on a cylinder via

$$Y[v, z] = Y(e^{zL(0)}v, e^z - 1) = \sum_{n \in \mathbf{Z}} v[n]z^{-n-1}.$$

The operators $Y[v, z]$ define a VOA which has Virasoro state $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$ and which is isomorphic to the original VOA. Then $V = \bigoplus_{n \in \mathbf{Z}} V[n]$, where $[n]$ denotes the $L[0]$ -weight and $Y[\tilde{\omega}, z] = \sum_{n \in \mathbf{Z}} L[n]z^{-n-2}$. 1-point functions on the torus may then be defined as

$$Z_V^{(1)}(v, q) = \text{Tr}_V Y[v, z] q^{L(0)-c/24}$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}^{(1)}$ (complex upper half-plane). If $v = \mathbf{1}$ this reduces to $Z_V^{(1)}$. These important ideas go back to Zhu [Z].

In the case of M and G we have the following results. For the free bosonic system,

$$Z_M^{(1)}(v, q) = \frac{Q_M(v)}{\eta(q)} \quad (7)$$

for some $Q_M(v) \in \mathbb{C}[E_2(q), E_4(q), E_6(q)]$ which can be explicitly described. Here,

$$E_{2k}(q) = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n, \quad k \geq 1,$$

is the usual Eisenstein series of weight $2k$ (cf. [Se]). Thus $Q_M(v)$ is a holomorphic *quasi-modular form* on $SL(2, \mathbb{Z})$. On the other hand, for the ghost system we find:

$$STr_G Y[v, z] q^{L(0)+26/24} = -\eta(q)^2 Q_G(v) \quad (8)$$

for some $Q_G(v) \in \mathbb{C}[E_4(q), E_6(q)]$, a holomorphic modular form on $SL(2, \mathbb{Z})$ which can be explicitly described. Thus we find the interesting result that for the ghost system, one-point functions are actually holomorphic modular forms on $SL(2, \mathbb{Z})$, whereas for free bosons they are merely quasi-modular forms.

The results for M are proved in [DMN] and [MT1]. Those for G , due to the authors, are unpublished, and we say a bit more about them here. Taking v in the basis (4), $Q_G(v) = 0$ *unless* the ghost number $r - s$ is zero and $m_1 = n_1 = 1$. In particular,

$$STr_G Y[b[-1]c[-1]\mathbf{1}, z] q^{L(0)+26/24} = -\eta(q)^2.$$

Comparing (7) and (8), we are led by analogy to *define* the partition function of G as the super graded trace of $-b[-1]c[-1]$. This is what we were looking for (cf.(6)), and is also what one gets from physics via the Polyakov path integral and Faddeev-Popov procedure [GSW] and [P], p.212.

5 Genus 2 Bosonic Partition Function

At genus 1, 1-point functions and partition functions (however they are defined) are functions in the complex upper half-plane $\mathbb{H}^{(1)}$. This will no longer be true for $g \geq 2$, when we expect $3g - 3$ variables. For $g = 2$ we thus expect 3 complex parameters.

Consider the genus two Riemann surface, which we denote by $T_1 \# T_2$, formed by sewing together two tori T_1 and T_2 as follows. For $a = 1, 2$, each torus T_a has a modulus $\tau_a \in \mathbb{H}^{(1)}$ which determines a lattice $L_a \subseteq \mathbb{C}$ spanned by τ_a and 1 with $T_a = \mathbb{C}/L_a$. Let $z_a \in T_a$ be a local coordinate and let $D_a > 0$ be the *minimum distance* of L_a . We introduce a complex sewing parameter ϵ where $|\epsilon| < \frac{1}{4}D_1D_2$ and identify the annuli $\{|\epsilon|/r_2 \leq |z_1| \leq r_1\} \subset T_1$ and $\{|\epsilon|/r_1 \leq |z_2| \leq r_2\} \subset T_2$ for $r_a < \frac{1}{2}D_a$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{9}$$

The domain in which sewing T_1 and T_2 is possible is then

$$D^\epsilon = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}^{(1)} \times \mathbb{H}^{(1)} \times \mathbb{C} \mid |\epsilon| < \frac{1}{4}D_1D_2\}. \tag{10}$$

This provides a natural parameterization of genus two Riemann surfaces, at least in the neighborhood of the degeneration point where $\epsilon = 0$.

Important for this is work of [Y] on sewing Riemann surfaces. Define infinite matrices $(A_a(k, l)), a = 1, 2$, as follows:

$$A_a(k, l) = A_a(k, l, \tau_a, \epsilon) = (-1)^{k+1} \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} \frac{(k+l-1)!}{(k-1)!(l-1)!} E_{k+l}(\tau_a).$$

This should be construed as a *weighted moment* of the normalized differential of the second kind on T_a . The normalized differential itself is

$$P_2(\tau_a, x - y) dx dy,$$

where $P_2(\tau, z) = \wp_{L_a}(z) + E_2(\tau)$ and $\wp_{L_a}(z)$ is the *Weierstrass \wp -function*. Note that

$$A_1 A_2 = A_N + O(\epsilon^{N+1})$$

where A_N is the $(2N - 1) \times (2N - 1)$ principal submatrix. Then we define

$$\det(I - A_1 A_2) = \lim_{N \rightarrow \infty} \det(I - A_N)$$

where I denotes the infinite identity matrix. Then one finds

Theorem 5.1 ([MT2]) (a) *The infinite matrix*

$$(I - A_1 A_2)^{-1} = \sum_{n \geq 0} (A_1 A_2)^n,$$

is convergent for $(\tau_1, \tau_2, \epsilon) \in D^\epsilon$.

(b) $\det(I - A_1 A_2)$ is non-vanishing and holomorphic on D^ϵ .

We can assign to $T_1 \# T_2$ its *normalized period matrix*

$$\Omega = \Omega(\tau_1, \tau_2, \epsilon) \in \mathbb{H}^{(2)},$$

which defines a map

$$\begin{aligned} D^\epsilon & \xrightarrow{F^\epsilon} \mathbb{H}^{(2)} \\ (\tau_1, \tau_2, \epsilon) & \longmapsto \Omega(\tau_1, \tau_2, \epsilon) \end{aligned} \tag{11}$$

The entries of Ω can be found explicitly in terms of the moment matrices A_1, A_2 by applying the methods of [Y] to find ([T], [MT2])

$$\begin{aligned} 2\pi i \Omega_{11} &= 2\pi i \tau_1 + \epsilon(A_2(I - A_1 A_2)^{-1})(1, 1), \\ 2\pi i \Omega_{22} &= 2\pi i \tau_2 + \epsilon(A_1(I - A_2 A_1)^{-1})(1, 1), \\ 2\pi i \Omega_{12} &= -\epsilon(I - A_1 A_2)^{-1}(1, 1). \end{aligned}$$

Here $(1, 1)$ refers to the $(1, 1)$ -entry of a matrix. Furthermore

Theorem 5.2 ([MT2]) *The map F^ϵ is holomorphic on D^ϵ .*

We now consider the definition of the genus two partition function $Z_M^{(2)}$ for M . Firstly recall that M admits a unique *Li-Zamolochikov metric*, i.e. a non-degenerate M -invariant symmetric bilinear form

$$\langle \cdot, \cdot \rangle : M \otimes M \rightarrow \mathbb{C}$$

where

$$\langle Y(u, z)v, w \rangle = \langle v, Y(u, z)^\dagger w \rangle$$

and $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. ([B], [Li], [FHL]). There is a corresponding orthogonal direct sum decomposition

$$M = \perp_{n \geq 0} M_n,$$

so the Gram matrix on M_n for basis $\{u_i^n\}$, that is

$$(G_{ij}^n) = (\langle u_i^n, u_j^n \rangle),$$

is *symmetric* and *invertible*. We can thus define a dual basis $\{\bar{u}_i^n = (G_{ij}^n)^{-1}u_j^n\}$ for M_n such that $\langle u_i^n, \bar{u}_j^n \rangle = \delta_{ij}$.

For a VOA with unique Li-Zamolochikov metric we then define the genus 2 partition function in terms of genus 1 data as follows ([T], [MT3]):

$$Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V_{[n]}} Z_V^{(1)}(u, q_1) Z_V^{(1)}(\bar{u}, q_2). \quad (12)$$

Here, $q_a = e^{2\pi i \tau_a}$ and the inner sum is taken over any basis for $V_{[n]}$. Note that (12) automatically satisfies the multiplicativity requirement (3). Expressions such as (12) might be expected from physical considerations concerning CFT and sewing punctured Riemann surfaces e.g. [So], [P], [MS]. (Indeed, the very axioms for vertex algebras can be couched in this language for the Riemann sphere [H]).

Using (7) and some involved but natural combinatorics, we can calculate $Z_M^{(2)}$ for the bosonic VOA M to find:

Theorem 5.3 ([MT3]) (a) *We have*

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = \frac{Z_M^{(1)}(q_1) Z_M^{(1)}(q_2)}{\sqrt{\det(I - A_1 A_2)}}.$$

(b) $Z_M^{(2)}$ is holomorphic on D^ϵ .

This result seems to be consistent with the physico/algebraic geometry idea of viewing partition functions as sections of determinant line bundles. In particular, the appearance of the inverse square root of a bosonic determinant concurs with the remarks concerning the determinant line bundle λ following (2), i.e. $\det(I - A_1 A_2)$ appears to be associated with a local section for λ .

Note that $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ is not an automorphic form on $Sp(4, \mathbb{Z})$. Indeed, it is not even a function on the Siegel half-space $\mathbb{H}^{(2)}$. The best one can do along these lines is as follows (cf. [MT2], [MT3] for details). Let $G = SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. There is an embedding $G \hookrightarrow Sp(4, \mathbb{Z})$ in which G maps to a 'diagonal' subgroup, and there is also a natural action of G on the domain \mathcal{D}^ϵ (10). In this way, G acts on both domain and codomain in (11), and it can be shown that the map F^ϵ is G -equivariant. Moreover, $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ transforms under G as an automorphic form of weight $-1/2$.

We briefly discuss an alternative construction for a genus two Riemann surface obtained by self-sewing a single torus T with modulus τ . We may identify two annuli with relative position w on T via a sewing relation analogous to (9) and with sewing parameter ρ . The sewing scheme is defined for $(\tau, \rho, w) \in D^\rho$, an appropriately defined domain. Explicit formulas for the period matrix Ω can be found in terms of an appropriate infinite moment matrix R . An *alternative* definition for the genus two partition function $\tilde{Z}_M^{(2)}(\tau, \rho, w)$ in terms of genus one 2-point functions can then be formulated. This can be calculated to find

Theorem 5.4 ([MT3]) (a) *We have*

$$\tilde{Z}_M^{(2)}(\tau, \rho, w) = \frac{Z_M^{(1)}(q)}{\sqrt{\det(I - R)}}.$$

(b) $\tilde{Z}_M^{(2)}$ is holomorphic on D^ρ .

Crucially, $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ and $\tilde{Z}_M^{(2)}(\tau, \rho, w)$ can be compared in a neighborhood of the two-torus degeneration point. One finds that $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ and $\tilde{Z}_M^{(2)}(\tau, \rho, w)$ *do not* agree. This again concurs with the results of physical string theory since λ *does not* have a global section.

The challenge remains to carry out this programme for the ghost super VOA system at genus two. As outlined in Section 4, even the definition of

$Z_G^{(1)}$ is subtle. However, once properly defined, our ultimate goal is to compute $Z_G^{(2)}(\tau_1, \tau_2, \epsilon)$ and $\tilde{Z}_G^{(2)}(\tau, \rho, w)$ and hence show that $Z_B^{(2)} = Z_G^{(2)} Z_{M^{\otimes 26}}^{(2)}$ is globally defined with

$$\begin{aligned} Z_B^{(2)} &= Z_G^{(2)}(\tau_1, \tau_2, \epsilon) Z_{M^{\otimes 26}}^{(2)}(\tau_1, \tau_2, \epsilon) \\ &= \tilde{Z}_G^{(2)}(\tau, \rho, w) \tilde{Z}_{M^{\otimes 26}}^{(2)}(\tau, \rho, w) \\ &= \frac{1}{\Delta_{10}^{(2)}} \end{aligned}$$

where $\Delta_{10}^{(2)}$ is the Igusa cusp form of weight 10.

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