

# **A Disequilibrium Macrodynamic Model of Fluctuations**

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**Working Paper No. 81**

**September 2004**

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## **Abstract**

A nonlinear disequilibrium macrodynamic model of fluctuations in the labour and product markets, mediated by variations in factor shares, is developed and the existence of a periodic orbit is proved using the Hopf bifurcation theorem.

JEL Classification Code:C61,E10,E32

Key Words: Disequilibrium Macrodynamics, Macroeconomic Fluctuations, Non-linear Macrodynamics, Hopf Bifurcation

## 1 Introduction and Motivation<sup>1</sup>

"The phenomena [Keynes] described are better regarded as *disequilibrium dynamics*"

Tobin, 1975, p.196; italics added

To analyze aggregate disequilibrium dynamics in the product and labour markets I adopt two mildly unorthodox modelling strategies relative to traditional models in this *genre*: the use of a generalized technical progress function, instead of the conventional production function; and a *q-theory* of investment. The former provides the constraint subject to which cost is minimized (or, equivalently, cost reduction is maximized). The latter allows an explicit consideration of *valuation* issues that link the real and financial side of a macroeconomy and also makes possible the infusion of expectational elements in reasonable ways, even, in the limit, rationally.

Labour market imbalances between supply and demand, for almost four decades, have been modelled as *natural* or *equilibrium* rates of unemployment in almost all conventional macro traditions, especially since Phelps and Friedman introduced the phrase *natural rate of unemployment* and the newclassicals persuaded the macroeconomic profession to adopt equilibrium terminology and strategies for macrodynamic modelling. This, coupled with Muth's felicitous, even if not descriptively accurate, introduction of the predicate *rational* to expectations, has meant that almost any

kind of dynamic macroeconomic modelling exercise is required to adhere to these two concepts if they are to be considered seriously. However, there is a minor historical anomaly that may be worth pointing out so that seemingly unorthodox attempts at modelling *disequilibrium aggregative dynamics* will not be brushed aside without at least been given the chance of a hearing.

While Friedman appealed, although inappropriately, to Wicksell when he invoked and etched in the language of discourse of macroeconomics the predicate *natural* to describe the steady-state imbalance between aggregate supply and demand in the labour market, Phelps, in his earliest contribution to the same subject used the more appropriate Harrodian term *warranted* rate of unemployment. He stated, on reflection:

"My association with that concept goes back to my 1967 essay on optimal inflation control [Phelps, 1967], which gave the idea an algebraic formulation. There I dubbed the concept the 'warranted' rate of unemployment because, in the model there, it is that unemployment level which is called for *if* the public's expectation of the rate of inflation are to be met. Since a characteristic of Roy Harrod's 'warranted rate of growth' was that it might be manipulated if otherwise it would cause harm, I thought I had hit upon a value-free term. But Milton Friedman's *catchy term* for the same idea, though derived from a different model, was the

easy winner. Not that I (nor Friedman) was the first to conceive or utilize the idea: Hayek, Mises, Fellner, and Wallich all talked about and wrote about it in earlier decades, and the latter two taught it to me. It runs in the blood of economists between the Danube and the Rhein."

Phelps, 1979, p.93; second set of italics added.

This curiosum, coupled to the equally little known fact, at least in the standard literature, that Grunberg and Modigliani, used exactly the same Harrodian term<sup>2</sup> *warranted expectations*, also for exactly the same reasons as Phelps, to describe what has come to be known as *rational* expectations, has always signified for me that a disequilibrium between a warranted and a natural rate must lie at the heart of any decent aggregative dynamic model. A rate of unemployment in the labour market is *warranted* by the expectations of profitability on the basis of which optimizing behavior on investments and choice of techniques will imply a particular level of employment and, via the effect on factor shares, effective demand such as to lead to dynamics in the product market. This, in turn, will feedback on the dynamics of the labour market and, by means of a *warranted* rate of expectations of profitability, lead to a next round of impact on investment and choice of techniques as a result of productivity changes and the cycle continues, either towards a stable disequilibrium dynamics in the form of a point attractor or a fluctuating profile in any one of many possible basins of attraction of limit configurations: limit cycles, strange attractors,

etc. The key unorthodox adjustment dynamics in this scenario is provided by freeing factor shares to vary over the cycle. Any use of a conventional production function locks the exponents in such a way that factor shares are prevented from acting as adjustment variables in mediating between the imbalances in aggregate supply and demand in the labour and product markets. However, by using the more general and more flexible technical progress function it is possible to endow the distributive variables a more active role in the overall disequilibrium dynamics of a model of fluctuations in product and labour markets.

The paper is organized as follows. In the next section the basic model is specified and the underlying economic rationale and mathematical assumptions are clearly described and discussed. In sections three and four the workings of the model and the two main theorems of the paper are presented. Finally, in the concluding section, an attempt is made to relate the model presented here to work done by others along similar lines. In addition, in the concluding section and in the brief Mathematical Appendix, some discussion of the mathematical underpinnings and restrictions indicate how to circumvent or tackle them, both in models of the class developed in this paper and in more general cases.

## 2 The Model

### 2.1 The Product Market

The basic structure of the model considered in this paper is that of a closed, one-good, aggregative economy without an explicit government sector where the aim is to make explicit the disequilibrium dynamics in the product and labour market by using factor shares as an adjustment variable. The underlying theme is the persistence of *equilibrium* fluctuations in the labour and product markets that are in *disequilibrium*. Such a theme implies, therefore, that, except for limit cases, levels of supply and demand in the two markets would be unbalanced but the dynamics should depict equilibrium fluctuations. With these thoughts as a backdrop for the modelling exercise, the following types of definitional ratios and relations are one of the ways of tackling the problem:

$$y = \frac{Y_d}{Y_s} \tag{1}$$

$y$ : ratio of demand to supply in the product market

$Y_d$ : Demand of Output (real);  $Y_s$ : Supply of Output (real)

Taking the time derivative of the log of (1), we get:

$$\frac{\dot{y}}{y} = \frac{\dot{Y}_d}{Y_d} - \frac{\dot{Y}_s}{Y_s} \tag{2}$$

Introducing differential savings propensities out of wages and profits allows the formulation of the following income-expenditure accounting relation:

$$pY_d = [1 - s_w(v, u)] wL + [1 - s_c(u)] (pY_s - wL) \quad (3)$$

where:

$$v = \frac{L}{N} \quad (4)$$

and

$$u = \frac{wL}{pY_s} \quad (5)$$

Taking the log of (5) and differentiating it w.r.t time gives the intrinsic, definitional, adjustment dynamics of factor shares:

$$\frac{\dot{u}}{u} = \left( \frac{\dot{w}}{w} - \frac{\dot{p}}{p} \right) - \left( \frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} \right) \quad (6)$$

$L$ : Labour demand;  $N$ : Labour Supply

$v$ : (un)employment ratio;  $u$ : share of wages

$w$ : money wage rate;  $p$ : price level

$s_w(v, u)$ : savings propensity out of wage income;  $s_c(u)$ : savings propensity out of profits income



The principal modelling strategy is to try to endogenise the natural dynamics of (1) as given by (2), (5) as given by (6) and the corresponding dynamic equation for (4).

Dividing (3) by  $pY_s$  gives:

$$y = 1 + u [s_c(u) - s_w(u, v)] - s_c(u) \quad (7)$$

Differentiating (7) w.r.t time, rearranging and simplifying gives:

$$\frac{dy}{dt} = \{[s_c(u) - s_w(v, u)] + u \left[ \frac{\partial s_c(u)}{\partial u} - \frac{\partial s_w(v, u)}{\partial u} \right] - \frac{\partial s_c(u)}{\partial u}\} \frac{du}{dt} - \left[ u \frac{\partial s_w(v, u)}{\partial v} \right] \frac{dv}{dt} \quad (8)$$

## 2.2 Productivity and the Technical Progress Function

Solow (1979) has persuasively argued that it is not quite sufficient to have *only* a ‘technical’ interpretation of the productive potential of an aggregative economy.<sup>3</sup> From the purely technical or engineering point of view, taking a hint from Johansen (1972, pp. 190-95) it is clear that output per unit of labour could be related to capital per unit of labour to formalize the productive potential of the economy. At the engineering level of the firm such a relation can be given a direct interpretation as Johansen’s ‘technique relation’ (ibid, p.21, equation 2-17) from which a short-run macro production relation can be derived. Putting these two elements together in one formalism gives,

in the notation of this paper, a variation or a generalization of the kind of technical progress function originally suggested by Kaldor (1957, 1961) and Arrow (1962):

$$\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} = \mathfrak{S} \left[ \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right), \left( \frac{\dot{w}}{w} - \frac{\dot{p}}{p} \right) \right] \quad (9)$$

$K$ : capital, and:

$$\mathfrak{S}_1 > 0; \mathfrak{S}_2 > 0 \quad (10)$$

Rewriting (7)<sup>4</sup> as:

$$\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} = \mathfrak{S} \left[ \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} - \frac{\dot{Y}_s}{Y_s} + \frac{\dot{Y}_s}{Y_s} \right), \left( \frac{\dot{w}}{w} - \frac{\dot{p}}{p} \right) \right] \quad (11)$$

and defining:

$$\alpha \equiv \frac{Y_s}{L} \quad (12)$$

$$\beta \equiv \frac{Y_s}{K} \quad (13)$$

We get, therefore, after rearranging, for (12):

$$\frac{\dot{\alpha}}{\alpha} = \mathfrak{S} \left[ \left\{ \left( \frac{\dot{\alpha}}{\alpha} \right) - \left( \frac{\dot{\beta}}{\beta} \right) \right\}, \left( \frac{\dot{w}}{w} - \frac{\dot{p}}{p} \right) \right] \quad (14)$$

### 2.3 Investment and Pricing

If investment behaviour is assumed to be a maximization of the (expected) present value of the net profits stream subject to an Uzawa-type installation function (cf. Uzawa (1969), pp. 639-41 and fig. 4, p.640), then it can formally be shown simply to be a function of *Tobin's q ratio*<sup>5</sup> or, equivalently, a function of the *valuation ratio* (cf. Kaldor (1966), appendix, pp. 316-19).<sup>6</sup> Tobin's *q-ratio* or the *valuation ratio* are both relations between the *market* value of securities and the *accounting* value of assets and therefore the investment dynamics set in motion by discrepancies between them encapsulates the central idea of Wicksell on the *cumulative process*. Thus in either case there is, implicit in the derivation, some concept of a *natural* rate of profit compatible with equilibrium in the securities market and, thus, any real side savings-investment identity would be parametrized by that rate of profit. Then, a realized rate of profit, given by  $(1 - u)\beta$  in this model, if incompatible with such a *natural* rate, would have to imply price dynamics on the nominal side or output dynamics on the real side, or both. These Wicksellian themes are the reason for referring to such a rate of profit as *natural*. In any case, the functional postulate for investment behaviour, therefore, is<sup>7</sup>:

$$\frac{I}{K} = \frac{\dot{K}}{K} = \Theta(q) \quad (15)$$

$I$ : real gross investment

$q$ : Tobin's  $q$ -ratio (or, equivalently, the Kaldorian *valuation ration*; in either case, considered an exogenous parameter)

In any macroeconomic model, except those adhering to pure newclassical strictures, it is customary to specify aggregative price dynamics on the basis of a combination of influences from excess demand in the product market and some form of mark-up principle; the latter due, in turn, to an added implicit assumption of imperfect competition in the product market. In a one-good, one primary factor aggregative model, mark-up on unit prime cost is equivalent to introducing the factor share variable as a proxy in the price equation. On the other hand, since investment behaviour is assumed to be parametrized by Tobin's  $q$ -ratio, a modicum of consistency entails that price dynamics is subject to a similar influence; i.e., the influence due to the above mentioned discrepancy between a market rate of interest and an accounting rate of profit. Hence:

$$\frac{\dot{p}}{p} = \Pi(u, y; q) \quad (16)$$

with:

$$\Pi_1 > 0 \text{ and } \Pi_2 > 0 \quad (17)$$

## 2.4 The Labour Market and Wage Dynamics

Given the basic analytical aims of this paper, it is natural that labour demand is endogenous to the dynamics of the model. Labour supply, on the other hand, is used to endow the model with the missing explicit *natural rate of growth* and is assumed *exogenous* so that the various *warranted* rates in the economics of the system are disciplined by this element, too (in addition to that other crucial exogenous element, Tobin's *q-ratio*). Hence:

$$\frac{\dot{v}}{v} = \frac{\dot{L}}{L} - \frac{\dot{N}}{N} \quad (18)$$

$$N = N_0 e^{\gamma t} \quad (19)$$

where:  $\gamma \geq 0$ .

As for wage dynamics, no attempt to go beyond conventional wisdoms is attempted - indeed, perhaps, not even that much. Broadly relying on Solow's enlightening discussion (op.cit), a direct dependence of wage dynamics on disequilibria in the product and labour markets, supplemented by inflation and labour productivity, is postulated. Thus:

$$\frac{\dot{w}}{w} = \Psi \left[ u, v, y, \frac{\dot{p}}{p}, \frac{\dot{\alpha}}{\alpha} \right] \quad (20)$$

where:

$$\Psi_1 \leq 0; \Psi_i > 0, i = 2 \sim 4 \quad (21)$$

## 2.5 The Reduced Form Dynamical System

Deriving the reduced form dynamical system in  $y$ ,  $v$  and  $u$  is fairly straightforward but does require some careful manipulation of the relations thus far postulated. Thus, substituting (16) and (19) in (14) we get:

$$\frac{\dot{\alpha}}{\alpha} = \mathfrak{F} \left[ \left\{ \left( \frac{\dot{\alpha}}{\alpha} \right) - \left( \frac{\dot{\beta}}{\beta} \right) \right\}, \left( \Psi \left[ u, v, y, \Pi(u, y; q), \frac{\dot{\alpha}}{\alpha} \right] - \Pi(u, y; q) \right) \right] \quad (22)$$

The innocuous assumption of the validity of the implicit function theorem w.r.t labour productivity allows (21) to be written, concisely, as:

$$\frac{\dot{\alpha}}{\alpha} = \Phi \left( \frac{\dot{\beta}}{\beta}, u, v, y; q \right) \quad (23)$$

I now return to the previously postponed question of the *optimum choice of techniques*. The assumption here is *a rule that maximizes the reduction in unit costs*. The major part of the problem to be resolved is due to the need to account for the impact of changing relative factor prices on the choice of techniques. The expected benefit from an optimal cost reducing choice is achieved by maximizing the sum of reductions in input requirements weighted by the price of each factor. This, in turn,

amounts to the sum of reductions in labour requirements, weighted by the unit cost of labour, plus the reduction in capital requirements weighted by capital costs. Thus, the *optimizing choice of technique program is to maximize the reduction in unit costs*:

$$u \frac{\dot{\alpha}}{\alpha} + (1 - u) \frac{\dot{\beta}}{\beta} \quad (24)$$

subject to the reduced form of the technical progress function:

$$\frac{\dot{\alpha}}{\alpha} = \Phi \left( \frac{\dot{\beta}}{\beta}, u, v, y; q \right) \quad (25)$$

This maximization results in an optimum factor intensity given, implicitly, by:

$$\Phi_1 = - \frac{(1 - u)}{u} \quad (26)$$

Assuming an appropriate form of the implicit function theorem, (26) can be rewritten as:

$$\frac{\dot{\beta}}{\beta} = \xi(u, v, y; q) \quad (27)$$

Substituting (27) in (25) gives:

$$\frac{\dot{\alpha}}{\alpha} = \zeta(u, v, y; q) \quad (28)$$

Using (16), (20) and (28) in (6) the reduced form dynamics of the factor share going to wages can be written as:

$$\frac{\dot{u}}{u} = F(u, v, y; q) \quad (29)$$

Equally, substituting (15) in (27) and the result in (28) and then using the postulated given, exogenous, growth rate in labour supply, (19), gives the reduced form disequilibrium dynamics in the labour market:

$$\frac{\dot{v}}{v} = G(u, v, y; q) \quad (30)$$

Thus, from (8), (29) and (30) we get the final dynamic reduced form disequilibrium relation, that in the product market:

$$\frac{\dot{y}}{y} = H(u, v, y; q) \quad (31)$$

The dynamical system given by (29), (30) and (31) is the complete reduced form version for the analysis of macrodynamic disequilibrium in the labour and product markets, mediated by adjustment in factor shares. The formal demonstration of the feasibility, even the inevitability, of equilibrium fluctuations in the dynamics of the labour and product market entails a careful mathematical analysis of the economic underpinnings of the constants and variables of the constituent functions and the way



the latter interact with each other<sup>8</sup>. To these issues I shall turn in the next section, but before that a few general comments are in order.

It will be clear that the way the problem has been formalized and formulated has led to a system of three *non-linear*, ordinary differential equations (ODEs), parametrized by Tobin's *q-ratio*. This particular parametrization has been chosen to keep the economics simple and focused on clear and, eventually, empirically simulable systems of relations that can cast some light on obvious disequilibrium dynamics. From a purely analytical point of view, richer - quantitatively and qualitatively - parametrizations are possible. For example, even at the level of behavioural simplicity with which various functional forms have been chosen, the price equation could have been explicitly parametrized in terms of adjustment and mark-up factors; say  $\lambda$  and  $\pi$ . In addition, or alternatively, if expectational elements are to be explicitly considered in the investment and wage relationships, then the parameters that characterise distribution functions can be used to parametrize the dynamical system. But given the kind of dynamical complications that are possible in even simple three-dimensional non-linear ordinary differential equations, there may well be some virtue in opting for the simplest and most straightforward parametrization, at least in the first instance.

Secondly, in view of the fact that the system is a 3-dimensional, non-linear, ODE, and given the aim to show that the system entails maintained fluctuations in the labour and product markets, and not just stably spiralling solutions in the basin

of attraction of a stable limit point, the usual recourse to the classic theorem of Poincaré-Bendixson and other concepts and theorems of planar dynamics will not be of much use.

Thirdly, the natural and simplest mathematical approach to the study of parametrized (endogenous) non-linear dynamical systems capable of maintained fluctuations seems to me to be *bifurcation theory*. Even though the model in this paper has abstracted away from open economy considerations and ignored an explicit government sector, a simple generalization is fairly easy - except for the fact that it will lead to the disequilibrium of a 5-dimensional non-linear ODE (parametrized appropriately). Once the 3-dimensional, 1-parameter analysis is made canonical, the mathematical complexities become familiar and even slightly mechanical for higher dimensional systems. Hence, even from the point of view of tractable and interesting mathematical analysis, the model presented above could serve as a benchmark for future, more general, analysis - despite the simplicity of its economics.

Fourthly, I do believe that the parametrization with respect to a *valuation ratio* - in this case, of course, Tobin's  $q$  - is also the simplest conceivable within the context of the *genre* of models that can be developed on the basis of variations on themes above. Policy, for example, by influencing yields in the financial spheres and spontaneous exogenous factors - the celebrated '*animal spirits*' - determine the value of  $q$  and in the aggregate there is no need for the conventional restriction of  $q \geq 1$ . All we need

to assume is that investment behaviour and price dynamics depend continuously on  $q$  in *some bounded interval*.

But this last point is a cardinal weakness of almost any kind of formal bifurcation analysis of parametrized non-linear dynamical systems. The yields of such analyses, in the form of theorems about loss of limit point stability and the emergence of maintained fluctuations, are almost without exception *pure existence theorems*. In the context of the above model parametrized by Tobin's  $q$  the results do not give numerical limits to the bounded interval within which the parameter can vary. However, there are two ameliorating features to this melancholy fact. The first feature is that there *is* a remarkable theorem due to Swinnerton-Dyer (1977) that can be used to indicate precise numerical bounds for the parameter in the context of the particular bifurcation analysis of this paper. However, I shall not invoke it simply because the mathematical formalism to make it clear would entail space and technical requirements that are disproportionately heavy. The second ameliorating feature is that any particular specification of the general functional forms specified above can be used, with manageable approximations of the full dynamical system, to derive very specific numerical bounds for the parameter.

### 3 Local Stability of an Equilibrium

Direct computation of the partial derivatives of the Jacobian of the linearized system for (29)~(31)<sup>9</sup> shows that most of the effects of the partial derivatives on the local dynamics are well determined. There are, however, as can be expected in a 3-dimensional dynamical system including endogenous parameters (for example,  $s_w(v, u)$  and  $s_c(u)$ ), some ambiguous elements in the Jacobian. For example, the assumptions of the previous section do not unambiguously determine the sign of  $\frac{\partial H}{\partial u}$  although they are sufficient to guarantee  $\frac{\partial H}{\partial v} > 0$  and  $\frac{\partial H}{\partial y} > 0$ .<sup>10</sup> I shall assume that  $\frac{\partial H}{\partial u} < 0$  (but almost zero in magnitude<sup>11</sup>) because it is a compound influence of long-term factors on an ultra short-term variable.

Define as follows: *for*  $A_i$ ,  $i = 1, 2, 3$  :

$$A_1 \equiv u \frac{\partial F}{\partial u} + v \frac{\partial G}{\partial v} + y \frac{\partial H}{\partial y} \quad (32)$$

$$A_2 \equiv \begin{vmatrix} u \frac{\partial F}{\partial u} & u \frac{\partial F}{\partial v} \\ v \frac{\partial G}{\partial u} & v \frac{\partial G}{\partial v} \end{vmatrix} + \begin{vmatrix} v \frac{\partial G}{\partial v} & v \frac{\partial G}{\partial y} \\ y \frac{\partial H}{\partial v} & y \frac{\partial H}{\partial y} \end{vmatrix} + \begin{vmatrix} u \frac{\partial F}{\partial u} & u \frac{\partial F}{\partial y} \\ y \frac{\partial H}{\partial u} & y \frac{\partial H}{\partial y} \end{vmatrix} \quad (33)$$

$$A_3 \equiv \begin{vmatrix} u \frac{\partial F}{\partial u} & u \frac{\partial F}{\partial v} & u \frac{\partial F}{\partial y} \\ v \frac{\partial G}{\partial u} & v \frac{\partial G}{\partial v} & v \frac{\partial G}{\partial y} \\ y \frac{\partial H}{\partial u} & y \frac{\partial H}{\partial v} & y \frac{\partial H}{\partial y} \end{vmatrix} \quad (34)$$

On the basis of the assumptions of the previous section and straightforward, brute-force, calculations it is clear that  $-A_3 > 0$ . We do not have to add any new explicit assumptions<sup>12</sup> to ensure the existence of a (locally) unique equilibrium configuration  $[u^*, v^*, y^*]$  for (29)~(31).<sup>13</sup> Therefore, invoking the inverse function theorem and using the usual Routh-Hurwitz criterion, the local stability of this singular point requires:

$$-A_1 > 0; A_2 > 0; -A_1 A_2 + A_3 > 0; -A_3(-A_1 A_2 + A_3) > 0 \quad (35)$$

From the assumptions and discussions of the previous section and paragraphs it is clear that these conditions are satisfied. Thus, from the last two of the above relations we know that the unique equilibrium is locally stable and, hence, implying eigenvalues with negative real parts. To study the possible local oscillatory characteristics of this singular point, as it loses its stability via bifurcation, a more detailed analysis of the structure of its eigenvalues is necessary. To these issues and to the two main propositions of this paper, I now turn.

#### 4 Disequilibrium Dynamic Oscillations

I emphasise the idea of *disequilibrium dynamic oscillations* in  $u, v$  and  $y$  to encapsulate the central aim of this paper as proposed in the opening paragraph: an endogenous model to interpret the simultaneous *equilibrium dynamics* in the labour and product

markets, mediated by variations in factor shares, under conditions of *disequilibria* in the balance between the *levels of supply and demand* in the two markets. This may sound somewhat paradoxical or even a slightly contorted description of the simple idea of a dynamically maintained disequilibrium, but that is all that is being suggested. Simultaneously setting the dynamical variation of proportional growth in  $u$ ,  $v$  and  $y$  to zero gives those values of the variables for which the *rates of growth* of supply and demand in the two markets are equal while maintaining - possibly - imbalances in the value of the *levels* between the two blades of the Marshallian scissors.

In 2-dimensional dynamical systems, as mentioned earlier, the powerful Poincaré-Bendixson theorem can be invoked to suggest conditions under which stably maintained oscillations can be shown to exist. This beautiful and versatile theorem has no analogue in higher dimensions. I invoke, therefore, an equally famous theorem that is, in a purely numerical sense, almost more useful and in its higher dimensional possibilities definitely much more versatile: the *Hopf bifurcation theorem*.<sup>14</sup>

**Proposition 1** *Given the technical assumptions of §2 and §3, there exist values of  $q = q^*$  such that the Jacobian of the dynamical system defined by (29)~(31) has a pair of purely imaginary eigenvalues.*

**Proof.** *From the results and discussion in §3 we know that the eigenvalues have, at most, negative real parts. On the other hand, from the characteristic equation for a third order system we know that the criterion for purely imaginary roots is:*

$$B^2 + 4E^3 > 0 \quad (36)$$

where:

$$B = A_3 - \frac{A_1 A_2}{3} + \frac{2A_1^3}{q} \quad (37)$$

and,

$$E = \frac{A_2}{3} - \frac{A_1^3}{q} \quad (38)$$

Given the order of magnitude of the elements of  $A_1^3$  and  $A_2$  and the assumption that investment behaviour depends continuously on Tobin's  $q$ -ratio, with no restriction for the feasible range of  $q$  in the aggregate economy, it is straightforward to verify that (36) is satisfied for some  $q = q^*$ . ■

It is now a matter of direct computation to show that the conditions of Hopf's theorem are satisfied for the dynamical system (29)~(31) parametrized by  $q$ .

**Proposition 2** *Given the assumptions guaranteeing the validity of Proposition 1, the dynamical system (29)~(31), parametrized by  $q$  exhibits a Hopf bifurcation from a limit point to a non-trivial periodic orbit.*

**Proof.** *To verify that the hypotheses of the Hopf bifurcation theorem are satisfied requires only one extra complication, in addition to all previous conditions, to be considered. From the assumptions in §2 and §3 we know that all the constituent functions underlying (29)~(31) are sufficiently smooth and, therefore, the dynamical system is*

$C'$ . From the discussion in §3 we know that the singular point under consideration is locally stable. From Proposition 1, for  $q = q^*$ , the Jacobian of the dynamical system (29)~(31), at the singular point, has a pair of purely imaginary eigenvalues. It is now necessary only to show that the eigenvalues cross the imaginary axis with non-zero speed. In other words, we have to show that for the eigenvalues at the origin where:

$$\alpha(q) \pm i\beta(q) \text{ with } \alpha(q^*) = 0 \text{ and } \beta(0) \neq 0 \quad (39)$$

that:

$$\frac{d\alpha(q^*)}{dq} \neq 0 \quad (40)$$

Now, using any one of the standard formulas relating the roots of an equation and the coefficients characterising them, it is a direct computational task to verify that (40) is satisfied. Then all the conditions of the Hopf theorem are satisfied and, therefore, in any neighbourhood  $IJ$  of  $q^*$ , and for any given  $\bar{q} > 0$  there exists a  $\hat{q}$  with  $|\hat{q}| < \bar{q}$  such that (29)~(31) has a non-trivial periodic orbit in  $IJ$ . ■

## 5 Concluding Notes

The approach taken in the paper is closely related, in overall aims, underlying concepts and mathematical underpinnings, to the class of models developed by Asada (1989), Franke and Asada (1994), Flascel et.al (1997), Chiarella and Flaschel (2000), Asada et.al., 2003 and Asada et.al., 2004. From a broad economic theoretic point



of view, the similarities are not surprising since our starting points have much in common: Goodwin (1967), Kaldor (1957), Keynes (1936, especially ch.12) and Tobin (op.cit). However, the main economic difference between the impressive line of work reported in the above papers and the model developed here lies in the crucial role played by the technical progress function in mediating between the behavioral basis of investment and its optimizing, choice of technique, aspects. Another difference, although it may well be less than decisive, is my attempt to make explicit distributional considerations on the basis of Kaldorian mechanisms<sup>15</sup> (Kaldor, 1955-6) in a way that would make it underpin the classical framework suggested in Goodwin (op.cit). My aim here was to build the foundations of a model that might, eventually, link the adjustment dynamics of personal and functional income distribution in such a way that long-run constancy in relative shares could be underpinned by steady state distributions in individual incomes. On the other hand, from a purely technical point of view, the similarity, for example, with the elegant paper by Franke and Asada (op.cit), is most evident in that they, too, use local stability analysis and the Hopf bifurcation theorem to establish crucial propositions. Their work and those others mentioned above have applied such techniques - and more general ones - to investigate aggregate disequilibria in higher dimensional dynamic economic systems. To that extent I find it reassuring that my occasional hints, in the main part of the paper, on the feasibility of going beyond the three dimensions to which I have

confined my analysis, can be substantiated by their impressive successes.

A few notes on the economic underpinning for the oscillatory behaviour branching off from a stable singular point are in order. As the value of Tobin's *q-ratio* varies away from its equilibrium defining value, the system loses the characteristics of locally stable behaviour and enters the basin of attraction of an oscillating attractor. For some range of values of  $q$ , for any deviation away from  $q = q^*$ , the dynamical system sets up self-correcting forces so that it seeks, as an auto-pilot, a return to those values of  $u$ ,  $v$  and  $y$  characterizing the dynamic equilibrium between *rates of growth* (but not necessarily balancing *levels*) of supply and demand in the two markets.<sup>16</sup>

In a very definite economic sense, this type of analysis substantiates the Marshallian basis of Leijonhufvud's important *corridor hypothesis* (cf. Leijonhufvud, 1981, Ch.6, p.109); i.e., for some *corridor of values of  $q$*  market behaviour is stable in traditional senses. The problem, however, is that the analytical apparatus and theoretical technologies that I have harnessed in this paper belong to that class of mathematics, even in an area bristling with dynamics, simulations and computations, that is replete with pure existence proofs. Thus, the traditional version of the Hopf theorem does not provide enough structure to determine precise *numerical bounds* to the value of  $q$  where transition from one basin of attraction to another occur. Indeed, if the theorem can be enriched in a way that allows the determination of such numerical bounds, then the framework of analysis can be used, with some care, for policy analysis and

for the analysis of transition economies.

Clearly the most immediate economic shortcomings of the framework developed here relate to the abstractions from the complications due to government activity and behaviour and open economy considerations. Essential features of such issues can be incorporated but only at the cost of increasing the reduced form dynamics to at least a 5-dimensional dynamical system. From a purely technical, mathematical, point of view only more complex computations will be involved. For example, Franke and Asada (op.cit) investigate a four-dimensional macrodynamic system using an identical mathematical methodology quite elegantly. But to keep track of the vast interdependencies that arise with the usual combinatorial explosion of various cross- and direct-effects that are inherent in the Jacobian of the linearised part of a non-linear dynamical system can also be mind-boggling. The advantages of qualitative dynamics will be lost, even if more determinate numerical possibilities are introduced. The only alternative would be to give up, to some extent, the reliance on general specifications for the constituent functional forms. Choosing a formalism with definite functional forms allows numerical analysis and makes approximations feasible in precise ways so that simulations to investigate bounds, emergence of a sequence of equilibria, the structure of basins of attractions and so on can be analysed in an experimental way.

Finally, although I have concentrated on analyzing bifurcations with respect to

just Tobin's  $q$  to retain analytical simplicity, an immediate generalization, preserving the 3-dimensional dynamics in  $u, v$  and  $y$ , would be to study, in the space of the three parameters,  $\lambda$ ,  $\pi$  and  $q$ , the bifurcating behaviour of the three differential equations, (8), (29) and (30). Such a generalization would almost immediately suggest, if anything like completeness is the goal, that the kind of *local* bifurcation analysis with which I have proceeded in this paper will have to be abandoned in favour of the much more thorny domain of *global* bifurcation analysis. On the other hand, even without abandoning the simplifying framework of local bifurcation analysis and using nothing much more than the techniques utilized in this paper plus some ingenious elementary geometry, it is easy to show the emergence and existence of homoclinic trajectories in the 3-dimensional dynamics in  $u, v$  and  $y$  for perfectly plausible parameter ranges of  $\lambda$ ,  $\pi$  and  $q$ .

The main objective, methodologically, however, was to develop a model integrating, from the very outset, *disequilibrium* **and** *dynamic* elements in a way that would allow the coexistence of imbalances between levels of supply and demand in two crucial aggregate markets, whilst maintaining equilibrium in their dynamics. To anyone who feels uncomfortable with microfoundations for such unorthodoxies I can only paraphrase Paul Samuelson's *Nobel* wisdom (Samuelson, 1970, p.73; italics added): the above model 'provides a typical example of a dynamic system that can *in no useful sense* be related to a maximum problem'.

## A Mathematical Appendix<sup>17</sup>

The purpose of this brief appendix is simply to give a rigorously formal version of the *Hopf bifurcation theorem* in a form that facilitates its use in the main part of the paper. The comments and remarks appended to it are to clarify a few technical points, some of which were mentioned also in the main body of the paper, at appropriate places.

### Theorem 3 *The Hopf Bifurcation Theorem*

*Consider the system of  $k$  real first-order  $C^k$  differential equations*

$$\dot{\mathbf{x}} = \mathbf{A}(\mu)\mathbf{x} + \mathbf{B}(\mathbf{x}, \mu) \quad (41)$$

*where  $\mathbf{x}, \mathbf{B}$  are column vectors,  $\mathbf{A}$  is a  $k$ -th order square matrix and  $\mu$  is a scalar parameter such that  $\mathbf{B}(\mathbf{x}^*, \mu) = 0$  and  $D_x \mathbf{B}(\mathbf{x}^*, \mu) = 0$  for all sufficiently small  $|\mu|$ . Assume that the linear part,  $\mathbf{A}(\mu)$ , at  $\mathbf{x}^*$ , has the eigenvalues:*

$$\alpha(\mu) \pm i\beta(\mu) \text{ with } \alpha(\mu^*) = 0 \text{ and } \beta(\mu^*) \neq 0 \quad (42)$$

*Assume also that the eigenvalues cross the imaginary axis with non-zero speed; i.e.,*

$$\frac{d\alpha(\mu)}{d\mu} \neq 0 \quad (43)$$

*Then, in any neighbourhood  $I$  of  $\mathbf{x}^*$  and for any given  $\bar{\mu} > 0$ , there exists a  $\hat{\mu}$ , with  $|\hat{\mu}| < \bar{\mu}$  such that (41) has a non-trivial periodic orbit in  $I$ .*

**Remark 4** *The most significant fact about this remarkable theorem is, of course, that all the essential hypotheses are confined to the linear part of the system of differential equations. But the price one pays for this rich linear harvest in a non-linear system of high-dimensions is that one is only guaranteed the **existence** of a periodic orbit. Not only do we remain uninformed about various numerical bounds for the limits of variation of the parameter for which various transitions occur, from one basin of attraction to another; we are also deprived of any detailed knowledge of the stability of the periodic orbit. For this latter point - and also for information on numerical bounds - it is necessary to investigate the effects of the part given by  $\mathbf{B}(\mathbf{x}, \mu)$ .*

**Remark 5** *Denote the non-trivial periodic orbit generated by the loss of stability and the Hopf bifurcation in the above theorem by  $\Gamma(\mu)$ . Suppose it is possible to have an explicit description - say a formula - for  $\Gamma(\mu)$ . Then, along lines exactly similar to the one followed in the basic Hopf bifurcation theorem, it is possible to test for the first bifurcation  $\Gamma(\mu)$ . So far as I know, there is nothing in the literature of economic dynamics that has tried to address this problem - i.e., the problem of determining an explicit description for  $\Gamma(\mu)$  - analytically, and very seldom even numerically.*

## Notes

<sup>1</sup>I am deeply indebted to my friend Serena Sordi who helped me, a long time ago, with the details of some of the tedious mathematical calculations with enormous patience and tremendous cheerfulness. Geoff Harcourt's appreciative comments on an earlier version were also very helpful. An embryonic version of this paper, in an early incarnation, had the benefit of the late Richard Goodwin's wise and constructively critical comments. The usual disclaimer applies.

<sup>2</sup>"There exists, then, at least one correct public prediction, provided that the supply and demand curves intersect once in the positive quadrant. Note that in our example public prediction prevents possible error of expectation on the part of suppliers. As suppliers fully accept the public prediction - which turns out to be correct - they act on the basis of *warranted expectation*."

Grunberg and Modigliani (1954, p.469; italics added)

<sup>3</sup>cf. also Solow (1979a) where Keynes's crucial insight on the downward stickiness of nominal wages, underpinned by convictions on the fairness of relative wages, was given a lucid interpretation by the suggestion that 'the unconventional device of including the wage as an argument in the firm's production function [could] represent the morale, productivity and quality effects in a summary way' (pp.347-8). In other words, by including the wage rate in the formalism for productive potential a crucial Keynesian behavioural insight can be encapsulated.

<sup>4</sup>Note that (7) is more general than a conventional production function, not only because  $\mathfrak{F}$  is considered nonlinear, but also because it encapsulates productive potential in the economy and, hence, the relevant output variable is  $Y_s$ . However, that does not imply that  $Y_s$  is *capacity* output.

<sup>5</sup>For example, as shown quite elegantly in Hayashi (1982).

<sup>6</sup>Both derivations amount to equilibrium behavior in the securities market, which remains exogenous in this work. Hence, either one of them, i.e., Tobin's  $q$  or the *valuation ratio*, can be chosen to parametrize investment behavior; the former is chosen, purely for convenience and familiarity, in this paper.

<sup>7</sup>This part of the postulate for investment behavior is about the level of capacity creation; there is the second part, not necessarily sequentially determined, where the problem of choice of technique has to be confronted. I defer a discussion of that part of the problem to the last subsection of this part.

<sup>8</sup>The analysis, although straightforward, is tedious. The interested reader can obtain, on request, all the details of the calculations from the author.

<sup>9</sup>Anyone interested in the full details of the computations can obtain them, as mentioned earlier, on written request, from the author.

<sup>10</sup>The former indicates that, as the disequilibrium in the labour market diminishes, product market disequilibria are also reduced. In the latter case the heuristics are immediate.



<sup>11</sup>In fact, for reasons of mathematical rigour and numerical purposes, I assume that it is an *infinitesimal* in the strict technical sense of *non-standard analysis*.

<sup>12</sup>Although this is not strictly correct I shall simply assume, for expository simplicity, that it is a true assertion. Correcting it is quite a simple, but tedious matter and diverts from the main aims.

<sup>13</sup>I am not stating the results on the existence of an equilibrium and its local stability as formal propositions due to their almost trivial nature.

<sup>14</sup>A precise statement of this theorem is given in the brief mathematical appendix, below. It is sometimes and more accurately called the *Poincaré-Andronov-Hopf theorem*. Hopf, in his classic paper, did make generous references to Poincaré's priority for the main ideas and methods for the theorem in all essential aspects (cf. Hopf, 1942, p.168 and footnote on that same page). Andronov's priority, by almost a decade and a half, was always acknowledged in the Russian literature. However, by now the theorem is almost indelibly linked, in the mind of the dynamical system theorist, as the *Hopf bifurcation theorem* and I shall bow to practice and violate ethics, however regrettably.

<sup>15</sup>See, above, equations (3), (5) and (6), and the way they form the basis for the derivation of (7) and (8).

<sup>16</sup>Further variations in the value of  $q$ , beyond some well-defined neighbourhood of  $q^*$ , could induce the dynamical system to enter the basin of attraction of wholly

different attractors in exotic ways. I shall not enter into the details of such an analysis in the interests of keeping the analysis of this paper as simple as possible.

<sup>17</sup>An excellent exposition of the theorem can be found in Hale and Koçak (1991). I have used this reference together with the important but neglected paper by Swinnerton-Dyer (op.cit) for the purposes of this paper.

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